

## Final Exam Answers

## Part I

$$\textcircled{1} \text{ a) } e^{2x-y} \left( 2 - \frac{dy}{dx} \right) + 1 = 0$$

$$\frac{dy}{dx} = \frac{2e^{2x-y} + 1}{e^{2x-y}} = 2 + e^{y-2x}$$

$$2 + e^{y-2x} \stackrel{?}{=} 2 + e^{y-2x} \quad \checkmark$$

$$\text{b) } e^{2 \cdot 1 - 2} + 1 = C$$

$$1 + 1 = C$$

$$\textcircled{C=2}$$

$$\textcircled{2} \text{ a) } \int (2y-1)^{-\frac{2}{3}} dy = \int 3x^2 dx$$

$$\frac{3}{2} (2y-1)^{\frac{1}{3}} = x^3 + C$$

$$(2y-1)^{\frac{1}{3}} = \frac{2}{3} x^3 + K$$

$$2y-1 = \left( \frac{2}{3} x^3 + K \right)^3$$

$$2y = 1 + \left( \frac{2}{3} x^3 + K \right)^3$$

$$y = \frac{1}{2} \left[ 1 + \left( \frac{2}{3} x^3 + K \right)^3 \right]$$

b) No.  $f(x,y) = 3x^2 (2y-1)^{\frac{2}{3}}$  is cont. every where, but  
 $\frac{\partial f}{\partial y} = 3x^2 \cdot \frac{2}{3} (2y-1)^{-\frac{1}{3}} \cdot 2$  is not cont. whenever  $y = \frac{1}{2}$ .

$$\textcircled{3} \frac{dy}{dt} + \frac{6}{3t-1} y = \frac{1}{(3t-1)^2}$$

$$\int \frac{6}{3t-1} dt = 2 \ln(3t-1) \quad e^{2 \ln(3t-1)} = (3t-1)^2$$

$$(3t-1)^2 \frac{dy}{dt} + 6(3t-1)y = 1$$

$$\frac{d}{dt} [(3t-1)^2 y] = 1$$

$$(3t-1)^2 y = t + C$$

$$y = \frac{t}{(3t-1)^2} + \frac{C}{(3t-1)^2}$$

$$\textcircled{4} y'' + 9y = 0$$

$$r^2 + 9 = 0$$

$$r = \pm 3i$$

$$y_c = C_1 \cos 3t + C_2 \sin 3t$$

$$y = (At+B)e^{2t}$$

$$y' = Ae^{2t} + (At+B)e^{2t} = \cancel{2Ae^{2t}} + \cancel{Ae^{2t}} + \cancel{2B}e^{2t} = (2At+A+2B)e^{2t}$$

$$y'' = 2Ae^{2t} + (2At+A+2B) \cdot 2e^{2t} = (4At+4A+4B)e^{2t}$$

$$(4At+4A+4B)e^{2t} + 9(At+B)e^{2t} = 26te^{2t}$$

$$\left. \begin{array}{l} 3A = 26 \\ 4A + 3B = 0 \end{array} \right\} \Rightarrow A=2, B = -\frac{8}{3}$$

$$y = C_1 \cos 3t + C_2 \sin 3t + \left(2t - \frac{8}{3}\right)e^{2t}$$

5

a)  $r^2 + 2r + 5 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm 4i}{2} = \cancel{-1} - 1 \pm 2i$$

$$y_c = c_1 e^{2t} \cos 2t + c_2 e^{2t} \sin 2t$$

$$y_p = A \cos 2t + B \sin 2t$$

b)  $r^2 - r - 6 = 0$

$$(r-3)(r+2) = 0$$

$$y = c_1 e^{3t} + c_2 e^{-2t}$$

$$y_p = t(A t^2 + B t + C) e^{-2t} = (A t^3 + B t^2 + C t) e^{-2t}$$

6)  $(s^2 Y - 2s + 1) + 2(sY - 2) + 4Y = \frac{3!}{(s+2)^2} 4$

$$(s^2 + 2s + 4)Y = \frac{3!}{(s+2)^2} 4 + 2s - 1 + 4$$

$$Y = \frac{3!}{(s+2)^2 (s^2 + 2s + 4)} + \frac{2s + 3}{s^2 + 2s + 4}$$

$$\textcircled{7} \text{ a) } \frac{3s+6}{s^2+8s+25} = \frac{3s+6}{(s+4)^2+9} = \frac{A(s+4) + B \cdot 3}{(s+4)^2+9}$$

$$As + (4A+3B) = 3s+6 \Rightarrow A=3$$

$$4A+3B=6 \Rightarrow 12+3B=6$$

$$3B=-6$$

$$B=-2$$

$$\mathcal{L}^{-1} \left\{ \frac{3s+6}{s^2+8s+25} \right\} = \mathcal{L}^{-1} \left\{ 3 \frac{s+4}{(s+4)^2+9} - 2 \frac{3}{(s+4)^2+9} \right\} = 3e^{-4t} \cos 3t - 2e^{-4t} \sin 3t$$

$$\text{b) } \frac{2}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} = \frac{As(s+1) + B(s+1) + Cs^2}{s^2(s+1)}$$

$$(A+B)s^2 + (A+B)s + B = 2$$

$$\Rightarrow B=2, A=-2, C=2$$

$$\mathcal{L}^{-1} \left\{ \frac{2}{s^2(s+1)} \right\} = \mathcal{L}^{-1} \left\{ -\frac{2}{s} + \frac{2}{s^2} + \frac{2}{s+1} \right\} = -2 + 2t + 2e^{-t}$$

$$\text{c) } \mathcal{L}^{-1} \left\{ \frac{1}{(s-5)^3} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{2!} \frac{2!}{(s-5)^3} \right\} = \frac{1}{2} e^{5t} t^2$$

$$\mathcal{L}^{-1} \left\{ \frac{e^{-6s}}{(s-5)^3} \right\} = \frac{1}{2} e^{5(t-6)} (t-6)^2 u(t-6)$$

### Points

① a) 8	② a) 10	③ 10	④ $\gamma_c$ 4	⑤ a) $\gamma_c$ 4	⑥ 9
b) 3	b) 6		$\gamma_p$ 10	$\gamma_p$ 2	⑦ a) 10
			$\gamma$ 2	b) $\gamma_c$ 4	b) 10
				$\gamma_p$ 4	c) 8

Total = 104

## Part II



b)  $y=0$  attracts,  $y=4$  repels

c)  $y=0$

② a)  $\vec{x}'(t) = \begin{bmatrix} e^{-t} \cos 2t + 2e^{-t} \sin 2t \\ -e^{-t} \sin 2t + 1e^{-t} \cos 2t \end{bmatrix}$

$$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -e^{-t} \cos 2t \\ e^{-t} \sin 2t \end{bmatrix} = \vec{\phantom{x}} \quad \checkmark$$

$$b) \begin{vmatrix} -e^{-t} \cos 2t & e^{-t} \sin 2t \\ e^{-t} \sin 2t & e^{-t} \cos 2t \end{vmatrix} = -e^{-2t} (\cos^2 2t + \sin^2 2t) = -e^{-2t} \neq 0$$

$$c) \underline{x}(t) = c_1 \begin{bmatrix} -e^{-t} \cos 2t \\ e^{-t} \sin 2t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \sin 2t \\ e^{-t} \cos 2t \end{bmatrix}$$

$$d) c_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix} \Rightarrow \begin{matrix} -c_1 = 6 \Rightarrow c_1 = -6 \\ c_2 = 5 \end{matrix}$$

$$-6 \begin{bmatrix} -e^{-t} \cos 2t \\ e^{-t} \sin 2t \end{bmatrix} + 5 \begin{bmatrix} e^{-t} \sin 2t \\ e^{-t} \cos 2t \end{bmatrix}$$

$$\textcircled{3} \begin{vmatrix} 2-r & -1 & 1 \\ 0 & 3-r & 0 \\ -2 & -1 & 4-r \end{vmatrix} = (3-r) \begin{vmatrix} 2-r & 1 \\ -2 & 4-r \end{vmatrix} = (3-r)(r^2 - 6r + 8 + 2)$$

$$= (3-r)(r^2 - 6r + 10) = 0 \quad r=3, \quad \frac{6 \pm \sqrt{36-40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$$

$$\textcircled{4} \left[ \begin{array}{cc|c} -2-(2+3i) & -3 & 0 \\ 3 & -2-(2+3i) & 0 \end{array} \right] = \left[ \begin{array}{cc|c} -3i & -3 & 0 \\ 3 & -3i & 0 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \begin{matrix} u_1 = i \\ u_2 = 3 \end{matrix} \rightarrow \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$e^{-2t} (\cos 3t + i \sin 3t) \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \sin 3t \\ e^{-2t} \cos 3t \end{bmatrix} + i \begin{bmatrix} e^{-2t} \cos 3t \\ e^{-2t} \sin 3t \end{bmatrix}$$

$$y = c_1 \begin{bmatrix} e^{-2t} \sin 3t \\ e^{-2t} \cos 3t \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t} \cos 3t \\ e^{-2t} \sin 3t \end{bmatrix}$$

$$\textcircled{5} \text{ a) } \Gamma = -1: \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 1 & 2 & 0 & | & 0 \\ -1 & 0 & 2 & | & 0 \end{bmatrix} \begin{array}{l} R_2 - R_1 \\ R_3 + R_1 \end{array} \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}$$

$$\begin{array}{l} R_1 - R_2 \\ R_3 - R_2 \end{array} \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{array}{l} u_1 = 2s \\ u_2 = -s \\ u_3 = s \end{array} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{b) } \Gamma = 1: \begin{bmatrix} -1 & 1 & -1 & | & 0 \\ 1 & 0 & 0 & | & 0 \\ -1 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ -1 & 1 & -1 & | & 0 \\ -1 & 0 & 0 & | & 0 \end{bmatrix} \begin{array}{l} R_2 + R_1 \\ R_3 + R_1 \end{array} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

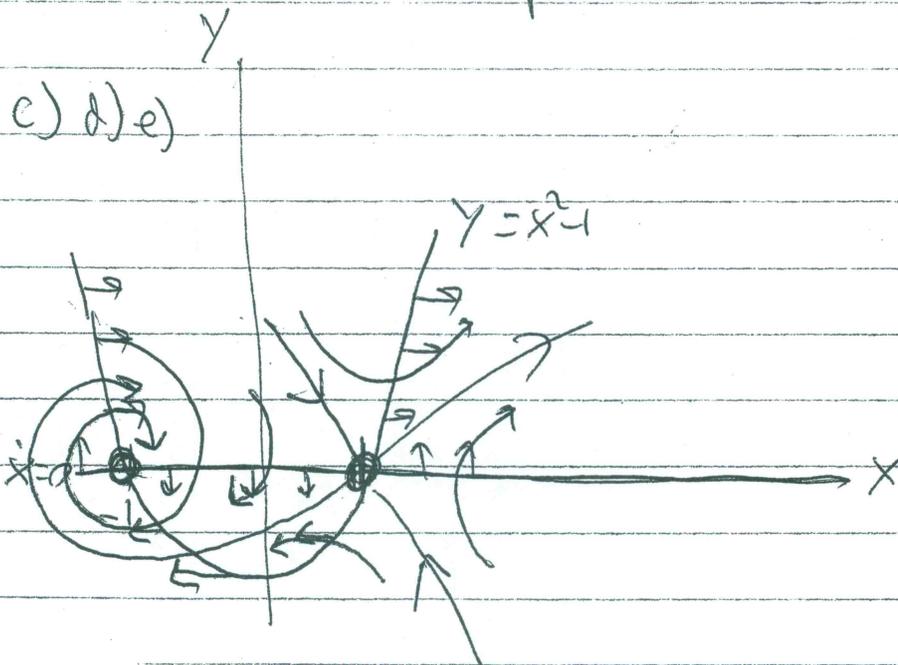
$$\begin{array}{l} u_1 = 0 \\ u_2 = s \\ u_3 = s \end{array} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{c) } c_1 e^{-t} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

⑥ a)  $y=0, x=\pm 1: (1,0), (-1,0)$

b)  $\begin{bmatrix} 0 & 1 \\ 2x & -1 \end{bmatrix}$      $(1,0): \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$      $-\Gamma(-1-\Gamma)-2=0$   
 $\Gamma^2+\Gamma-2=0$   
 $(\Gamma+2)(\Gamma-1)=0$   
 $\Gamma=-2, 1: \text{saddle}$

$(-1,0): \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$      $-\Gamma(-1-\Gamma)+2=0$   
 $\Gamma^2+\Gamma+2=0$   
 $\Gamma = \frac{-1 \pm \sqrt{1-4 \cdot 2}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$   
 Spiral attractor



Points

① a) 4	② a) 8	③ 10	④ 12	⑤ a) 9	⑥ a) 3	Total = 95
b) 2	b) 4			b) 9	b) 12	
c) 2	c) 2			c) 3	c) 4	
	d) 6				d) 4	
					e) 2	