MA 341-001 Final Exam

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Use your own paper to work the problems. On all problems, you must **show your work** to receive credit.

This test is in two parts. *Use separate sheets of paper to work on on each part.* When you finish, *write your name on each part*, fold the Part I test sheet together with your Part I work (with your name showing outside), do the same for Part II, and turn in.

Part I

1. Consider the differential equation
$$\frac{dy}{dx} = 2 + e^{y-2x}.$$
   (a) Show that the equation
$$e^{2x-y} + x = C$$
   implicitly defines a family of solutions. (Just check that it works.)

   (b) Find the value of C for which this equation gives a solution to the initial value problem
$$\frac{dy}{dx} = 2 + e^{y-2x}, \quad y(1) = 2.$$

2. Consider the separable differential equation
$$\frac{dy}{dx} = 3x^2(2y-1)^{\frac{2}{3}}.$$
   (a) Find a family of solutions. Give your answer with y an explicit function of x if possible.

   (b) Does the Existence-Uniqueness Theorem guarantee that the initial value problem
$$\frac{dy}{dx} = 3x^2(2y-1)^{\frac{2}{3}}, \quad y(0) = \frac{1}{2}$$
   has a unique solution? Explain briefly.

3. Find the general solution of the following linear differential equation:

$$(3t-1)\frac{dy}{dt}+6y=\frac{1}{3t-1}$$

Give your answer with y an explicit function of t if possible.

4. Find the general solution using the method of undetermined coefficients:

$$y''+9y=26te^{2t}$$

5. For each of the following problems, give (i) the complementary solution (i.e., the general solution of the associated homogeneous equation) and (ii) the form you would use to find a particular solution using the method of undetermined coefficients.

   (a) $y''+2y'+5y=\cos 2t$

   (b) $y''-y'-6y=t^2e^{-2t}$.

6. Find $Y(s)$, the Laplace transform of the solution $y(t)$ of the following initial value problem. Do not simplify $Y(s)$, and do not find $y(t)$.

$$y''+2y'+4y=t^3e^{-2t}$$
$$y(0)=2, \quad y'(0)=-1$$

7. Find the inverse Laplace transform of the following functions.

   (a) $\frac{3s+6}{s^2+8s+25}$

   (b) $\frac{2}{s^2(s+1)}$

   (c) $\frac{e^{-6s}}{(s-5)^3}$

Part II

1. Consider the differential equation $\frac{dy}{dt} = ye^{-y}(y-4)$. (Do not try to solve this differential equation.)

   (a) Sketch the phase line. Be sure to show the equilibria. (Notice that e^{-y} is always positive.)

   (b) Which equilibria are attractors and which are repellers?

   (c) Consider the solution with $y(0) = 3$. What does this solution approach as t increases?

2. Consider the differential equation

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \mathbf{x}(t).$$

   Let

$$\mathbf{x}_1(t) = \begin{bmatrix} -e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} e^{-t}\sin 2t \\ e^{-t}\cos 2t \end{bmatrix}.$$

   (a) Check that $\mathbf{x}_1(t)$ is a solution. (Just check that it works.)

   (b) $\mathbf{x}_2(t)$ is also a solution. (You don't need to check this.) Show that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent.

   (c) Give the general solution.

   (d) Give the solution that satisfies the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} 6 \\ 5 \end{bmatrix}.$$

3. Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 0 \\ -2 & -1 & 4 \end{bmatrix}.$$

4. The matrix

$$\mathbf{A} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$$

   has the eigenvalues $r = -2 \pm 3i$. Find the general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

5. The matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

   has the eigenvalues $r = -1, 1, 2$. An eigenvector for the eigenvalue 2 is

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

(a) Find an eigenvector for the eigenvalue -1.

(b) Find an eigenvector for the eigenvalue 1.

(c) Give the general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

6. Consider the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x^2 - 1 - y \end{aligned}$$

(a) Find the equilibria. (There are two.)

(b) Use the matrix of the linearization at the equilibria to determine their types (attracting or repelling node, attracting or repelling spiral, saddle).

(c) Draw the nullclines (the curves in the xy-plane where $\frac{dx}{dt} = 0$ and where $\frac{dy}{dt} = 0$). In your sketch, indicate the two equilibria.

(d) Add to your sketch the vector field on the nullclines.

(e) Add to your sketch some typical trajectories. Use all the information from parts (a) to (d) to help you do this.