MA 341-005 Final Exam

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Use your own paper to work the problems. On all problems, **show your work**.

This test is in two parts. *Use separate sheets of paper to work on on each part.* When you finish, *write your name on each part in the space provided*, fold the Part I test sheet together with your Part I work (with your name showing outside), do the same for Part II, and turn in.

Part I

1. Consider the differential equation

$$\frac{dy}{dx} = 2 + (y - 2x)^{\frac{2}{3}}$$

   (a) Show that the equation

$$y = 2x + \frac{1}{27}(x + C)^3$$

   defines a family of solutions. (Just substitute and check that it works.)

   (b) Find the value of C for which this equation gives a solution to the initial value problem

$$\frac{dy}{dx} = 2 + (y - 2x)^{\frac{2}{3}}, \quad y(1) = 2$$

   (c) Does the Existence-Uniqueness Theorem guarantee that the initial value problem in part (b) has a unique solution? Explain very briefly.

2. Find the general solution of the separable differential equation

$$\frac{dy}{dx} = \frac{2y + 1}{(x + 1)^2}$$

   Give your answer with y an explicit function of x if possible.

3. Consider the differential equation $\frac{dy}{dt} = e^y(y+1)(y-1)$. (Do not try to solve this differential equation.)

   (a) Sketch the phase line. Be sure to show the equilibria. (Notice that e^y is always positive.)

   (b) Which equilibria are attractors and which are repellers?

   (c) Consider the solution with $y(0) = \frac{1}{2}$. What does this solution approach as t increases?

4. Find the general solution.

   (a) $y'' + 6y' + 9y = 0$

   (b) $y'' + 4y' + 13y = 0$

5. Find the general solution using the method of undetermined coefficients.

$$y'' - 4y = 9te^{-t}$$

6. Solve using Laplace transforms.

$$y'' + 4y' + 4y = 12e^{-2t}$$
$$y(0) = 1, \quad y'(0) = -2$$

7. Find the inverse Laplace transform of the following functions.

   (a) $\frac{2s+4}{s^2+6s+25}$

   (b) $\frac{2s^2+3s+4}{(s+1)(s+2)^2}$

8. Find the Laplace transform of

$$h(t) = \begin{cases} 2t & t < 4, \\ 8 & t \geq 4. \end{cases}$$

Part II

1. Consider the differential equation

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & -9 \\ 1 & -1 \end{bmatrix} \mathbf{x}(t).$$

Let

$$\mathbf{x}_1(t) = \begin{bmatrix} 3e^{-t}\cos 3t \\ e^{-t}\sin 3t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} -3e^{-t}\sin 3t \\ e^{-t}\cos 3t \end{bmatrix}.$$

(a) Check that $\mathbf{x}_1(t)$ is a solution. (Just substitute and check that it works.)

(b) $\mathbf{x}_2(t)$ is also a solution. (You don't need to check this.) Show that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent.

(c) Give the general solution.

(d) Give the solution that satisfies the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} 6 \\ -5 \end{bmatrix}.$$

(e) Check that one solution of

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & -9 \\ 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 7e^{-2t} \\ -3e^{-2t} \end{bmatrix}$$

is

$$\mathbf{x}(t) = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}.$$

(f) Give the general solution.

2. Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & -5 \\ 2 & 4 & 0 \end{bmatrix}.$$

3. The matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

has the eigenvalues $r = -1, 1, 2$. An eigenvector for the eigenvalue -1 is

$$\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find an eigenvector for the eigenvalue 1.

(b) Find an eigenvector for the eigenvalue 2.

(c) Give the general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

4. The matrix

$$\mathbf{A} = \begin{bmatrix} -1 & -8 \\ 2 & -1 \end{bmatrix}$$

has the eigenvalues $r = -1 \pm 4i$. Find the general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

5. Consider the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= x^2 \\ \frac{dy}{dt} &= y \end{aligned}$$

It has one equilibrium, at $(x, y) = (0, 0)$.

(a) In a sketch of the xy-plane, show the equilibrium and the direction vectors at the eight points $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$, $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$. (You can make the direction vectors shorter than their correct length.)

(b) Think a little about the direction of the vectors you have not sketched. Then sketch some representative trajectories, with arrows indicating their direction.

(c) Determine formulas for the trajectories by solving the phase plane equation (i.e., the equation for $\frac{dy}{dx}$).

(d) Check your work to make sure your answers to parts (b) and (c) are consistent. (No answer is needed.)

6. Consider the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= 4 - xy \\ \frac{dy}{dt} &= 4x - y \end{aligned}$$

(a) Find the equilibria. (There are two.)

(b) Use the matrix of the linearization at the equilibria to determine their types (attracting or repelling node, attracting or repelling spiral, saddle).