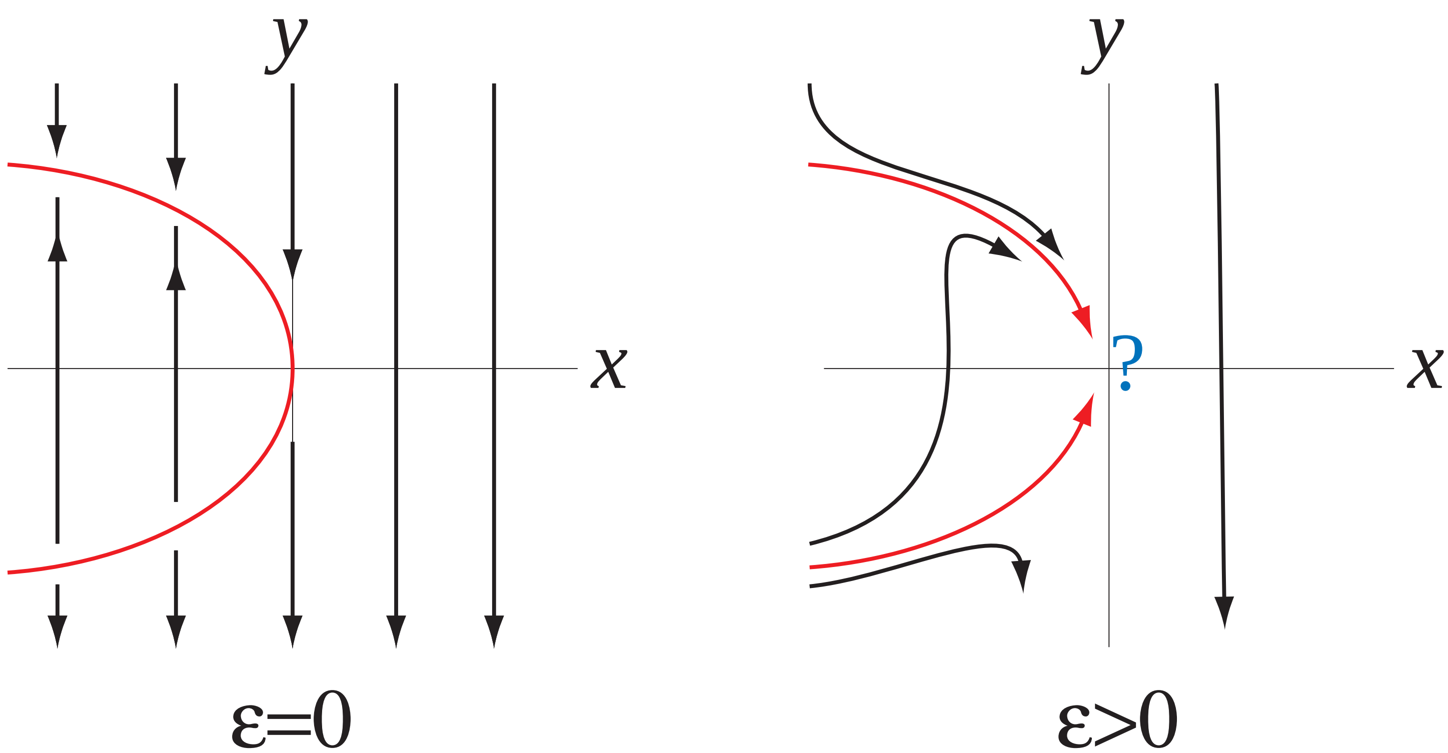
Loss of Normal Hyperbolicity



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“Happy families are all alike; every unhappy family is unhappy in its own way.”

Leo Tolstoy, *Anna Karenina*

Anna Karenina Principle: “By that sentence, Tolstoy meant that, in order to be happy, a marriage must succeed in many different respects: sexual attraction, agreement about money, child discipline, religion, in-laws, and other vital issues. Failure in any one of those essential respects can doom a marriage even if it has all the other ingredients needed for happiness.

“This principle can be extended to understanding much else about life besides marriage.”

Jared Diamond, *Guns, Germs, and Steel*

Principle of Fragility of Good Things: “Good things (e.g. stability) are more fragile than bad things. It seems that in good situations a number of requirements must hold simultaneously, while to call a situation bad even one failure suffices.”

Vladimir Arnold, *Catastrophe Theory*

Plan

(1) Saddle-node bifurcation in the fast equation

(2) Rarefactions in the Dafermos regularization of a system of conservation laws

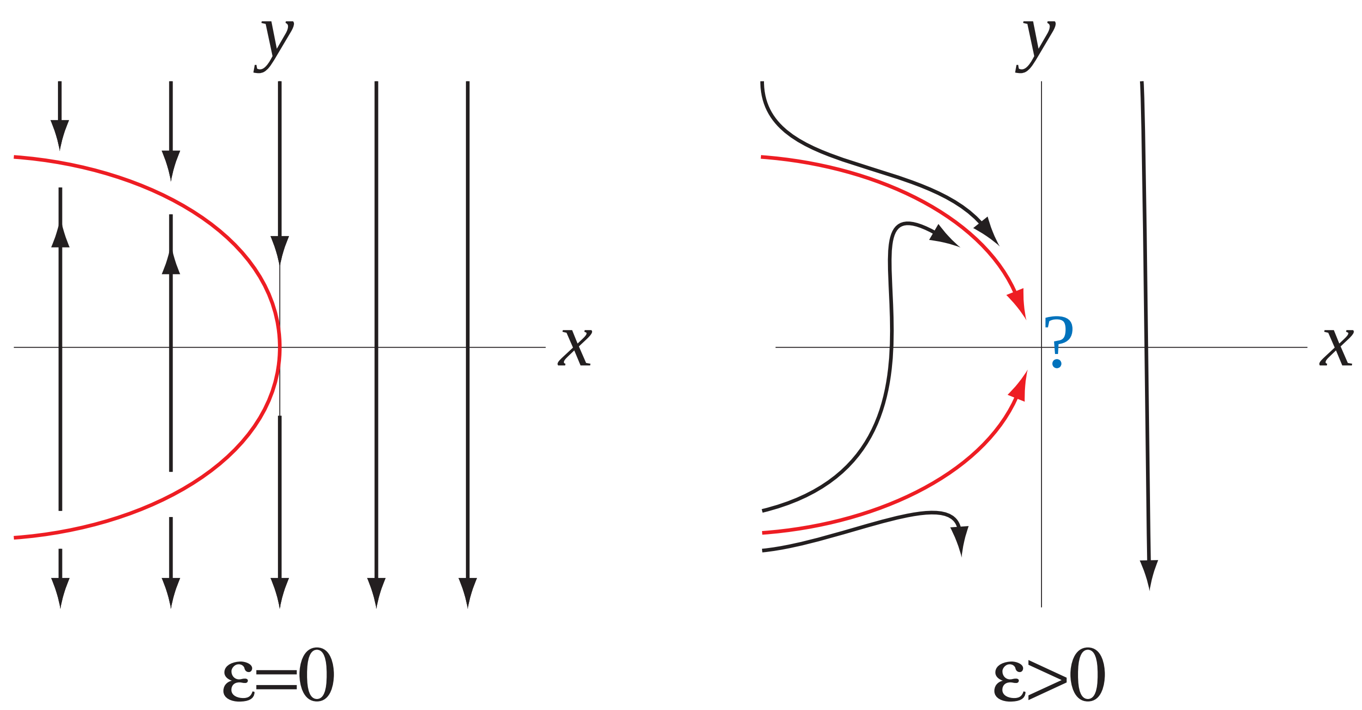
(3) Crystalline interphase boundaries

I. Saddle-Node Bifurcation in the Fast Equation

M. Krupa and P. Szmolyan, 2001, expanding on ideas of F. Dumortier and R. Roussarie.

System:

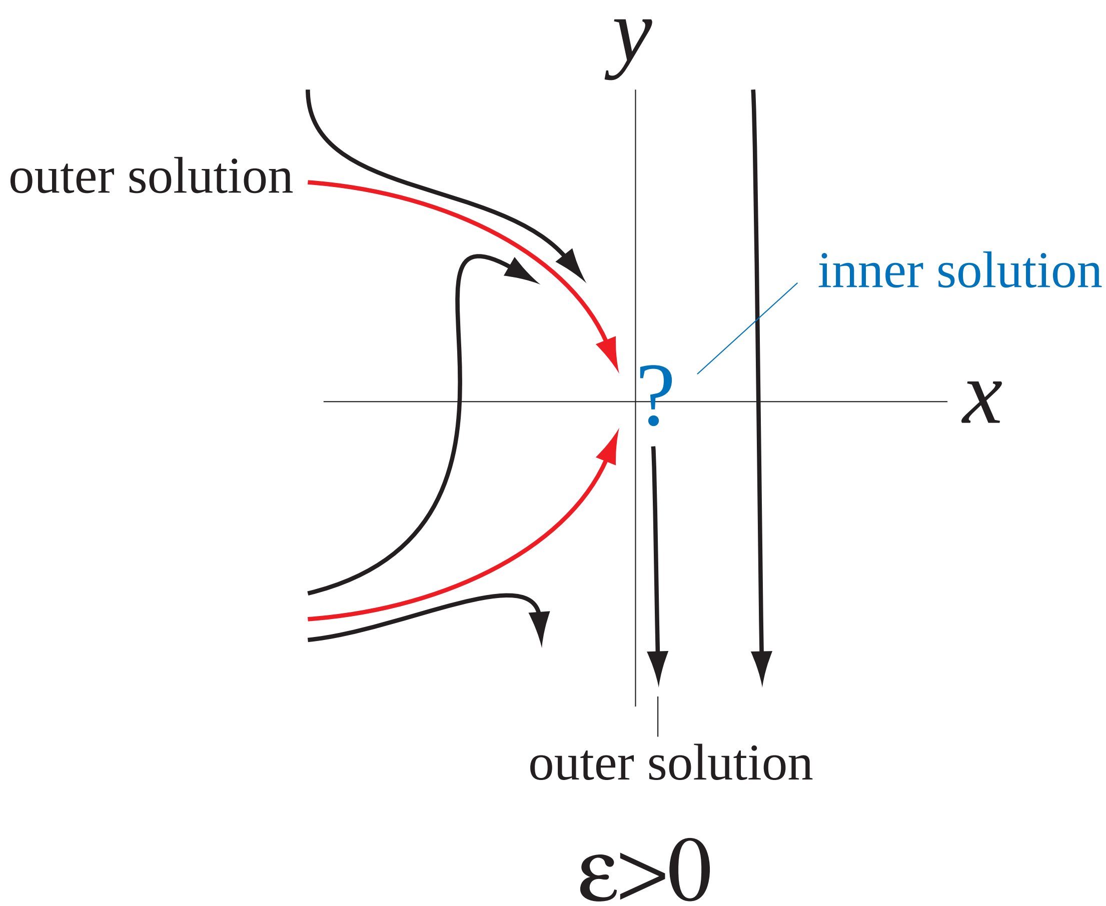
$$\dot{x} = \varepsilon(1 + \ldots)$$
$$\dot{y} = -x - y^2 + \ldots \qquad (x^2,\, xy \text{ terms allowed})$$



Question: For $\varepsilon > 0$, where does the normally attracting invariant curve go?

Classical analysis

$$\dot{x} = \varepsilon(1 + \ldots)$$
$$\dot{y} = -x - y^2 + \ldots \qquad (x^2,\ xy \text{ terms allowed})$$



To get inner solution, let $x = \varepsilon^{\frac{2}{3}} a$, $y = \varepsilon^{\frac{1}{3}} b$.

System:

$$\dot{x} = \varepsilon(1 + \dots)$$
$$\dot{y} = -x - y^2 + \dots \qquad (x^2, xy \text{ terms allowed})$$

Let $x = \varepsilon^{\frac{2}{3}}a$, $y = \varepsilon^{\frac{1}{3}}b$.

$$\varepsilon^{\frac{2}{3}}\dot{a} = \varepsilon\left(1 + O(\varepsilon^{\frac{1}{3}})\right)$$
$$\varepsilon^{\frac{1}{3}}\dot{y} = -\varepsilon^{\frac{2}{3}}a - \varepsilon^{\frac{2}{3}}b^2 + O(\varepsilon)$$

Simplify.

$$\dot{a} = \varepsilon^{\frac{1}{3}}\left(1 + O(\varepsilon^{\frac{1}{3}})\right)$$
$$\dot{b} = -\varepsilon^{\frac{1}{3}}a - \varepsilon^{\frac{1}{3}}b^2 + O(\varepsilon)$$

Rescale time (divide by $\varepsilon^{\frac{1}{3}}$).

$$a' = 1 + O(\varepsilon^{\frac{1}{3}})$$
$$b' = -a - b^2 + O(\varepsilon^{\frac{2}{3}})$$

Note three terms have order ε^0. Set $\varepsilon = 0$.

$$\frac{db}{da} = -a - b^2$$

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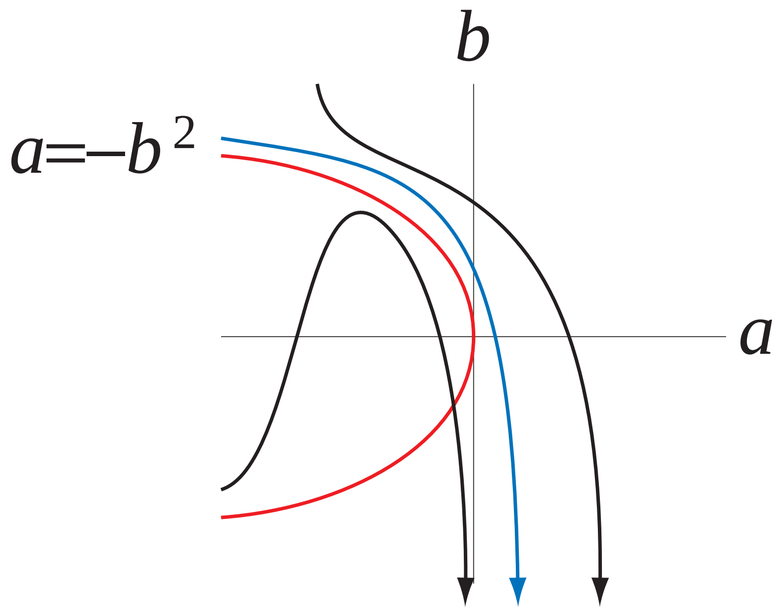
Convert this Riccati equation to a linear equation by the substitution $b = \frac{1}{c}\frac{dc}{da}$.

$$\frac{d^2c}{da^2} + ac = 0.$$

There is an explicit solution in terms of Airy functions. Convert back to get b in terms of a.



Use the blue one and try to match to the outer solutions using asymptotic expansions.

Solution is asymptotic to $a = k$ ($-k =$ first zero of Airy function), i.e., $x = k\varepsilon^{\frac{2}{3}}$.

Blow-Up: a Geometric Approach to Matching

Extend the original system:

$$
\begin{aligned}
\dot{x} &= \varepsilon(1+\ldots) \\
\dot{y} &= -x-y^2+\ldots \\
\dot{\varepsilon} &= 0
\end{aligned}
$$

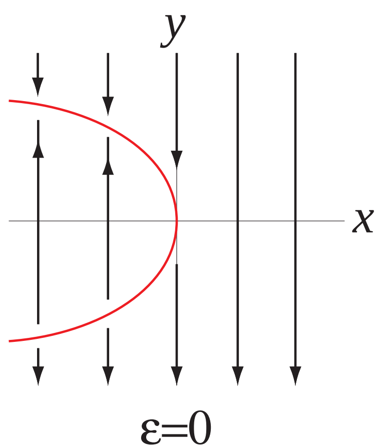
The blow-up transformation for this problem is a map from $S^2 \times [0,\infty)$ (blow-up space) to $xy\varepsilon$-space. Let $\left((\overline{x},\overline{y},\overline{\varepsilon}),\overline{r}\right)$ be a point of $S^2 \times [0,\infty)$, so $\overline{x}^2+\overline{y}^2+\overline{\varepsilon}^2=1$. Then

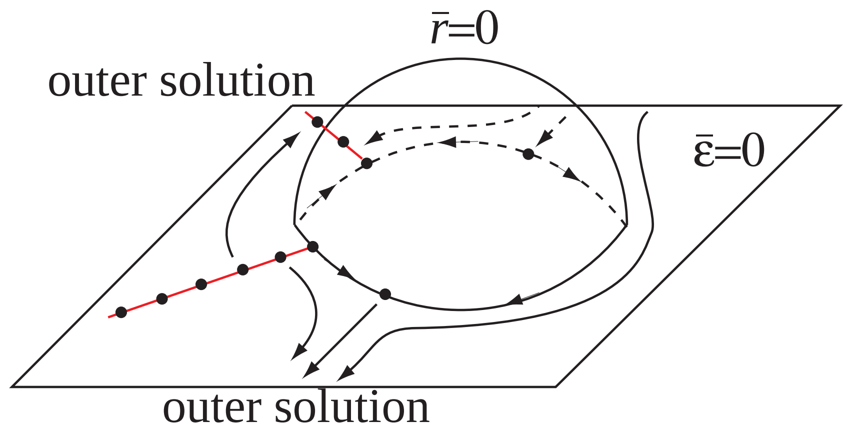
$$
x=\overline{r}^2\overline{x}, \quad y=\overline{ry}, \quad \varepsilon=\overline{r}^3\overline{\varepsilon}.
$$

The origin has been “blown up” to a sphere (“quasi-homogeneous” spherical coordinates).

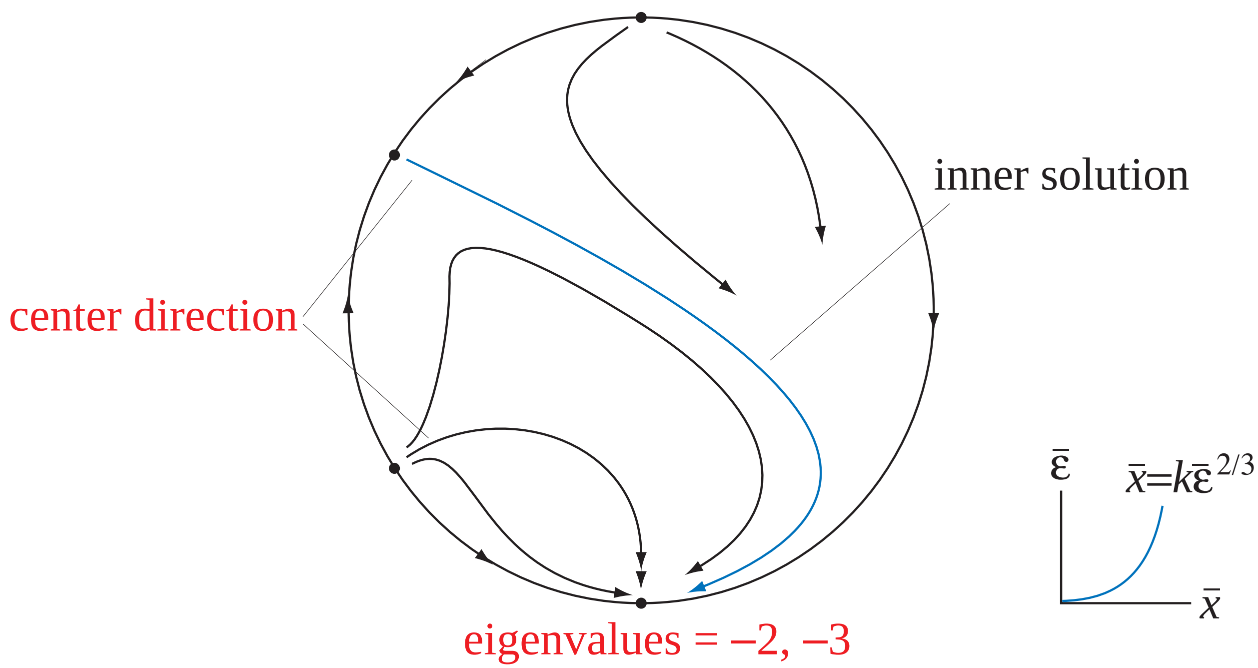
Under this transformation the system pulls back to a vector field X on $S^2 \times [0,\infty)$ for which the sphere $\overline{r}=0$ consists entirely of equilibria. The vector field we shall study is $\tilde{X}=\overline{r}^{-1}X$. Division by $\overline{r}$ desingularizes the vector field on the sphere $\overline{r}=0$ but leaves it invariant. It is equivalent to rescaling time.

In blow-up space there is no loss of normal hyperbolicity.





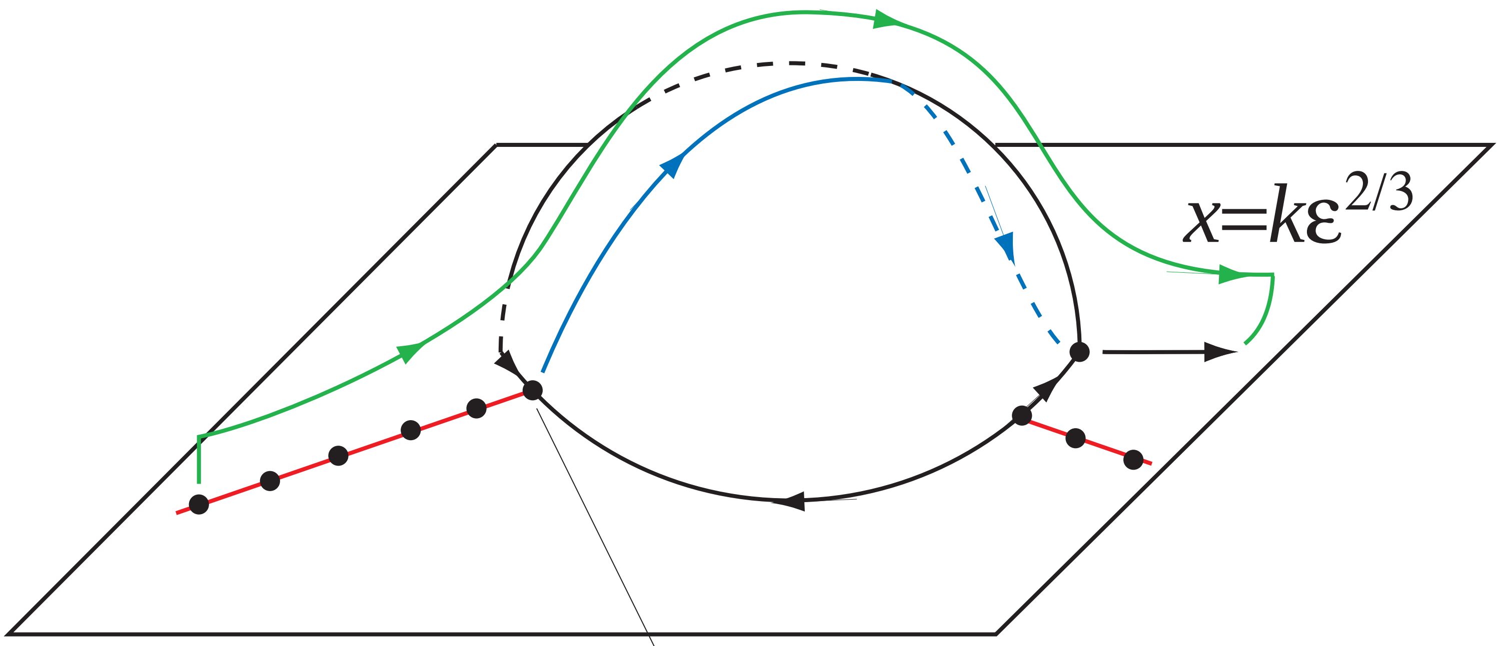
Top view of sphere:



Coordinate system on top of the sphere gives inner solution.

Coordinate systems on side and front of sphere allow geometric matching to outer solutions.

Flow past sphere:



Center manifold of this point is shown. It is the extension of the normally hyperbolic invariant manifold of outer solutions in $xy\varepsilon$-space.

Calculations

$$\dot{x} = \varepsilon(1 + \ldots)$$
$$\dot{y} = -x - y^2 + \ldots$$
$$\dot{\varepsilon} = 0$$

$$x = \bar{r}^2\bar{x}, \quad y = \bar{r}\bar{y}, \quad \varepsilon = \bar{r}^3\bar{\varepsilon}, \qquad \big((\bar{x},\bar{y},\bar{\varepsilon}),\bar{r}\big) \in S^2 \times [0,\infty)$$

Chart for $\bar{\varepsilon} > 0$

$$x = r^2 a, \quad y = rb, \quad \varepsilon = r^3, \qquad a \in \mathbb{R},\, b \in \mathbb{R},\, r \geq 0 \qquad (r = \bar{\varepsilon}^{\frac{1}{3}}\bar{r},\, a = \bar{\varepsilon}^{-\frac{2}{3}}\bar{x},\, b = \bar{\varepsilon}^{-\frac{1}{3}}\bar{y})$$

$$r^2\dot{a} = r^3(1 + O(r))$$
$$r\dot{b} = -r^2 a - r^2 b^2 + O(r^3)$$
$$\dot{r} = 0$$

Simplify and rescale time (divide by r).

$$a' = 1 + O(r)$$
$$b' = -a - b^2 + O(r^2)$$
$$r' = 0$$

We have seen this before: “rescaling chart.”

Side and front charts are used for geometric matching.

Chart for $\bar{x} < 0$

$$\begin{aligned} \dot{x} &= \varepsilon(1 + \dots) \\ \dot{y} &= -x - y^2 + \dots \\ \dot{\varepsilon} &= 0 \end{aligned}$$

$$x = -r^2, \quad y = rb, \quad \varepsilon = r^3 c, \qquad b \in \mathbb{R},\ c \in \mathbb{R},\ r \geq 0.$$

$$\begin{aligned} -2r\dot{r} &= r^3 c(1 + O(r)) \\ \dot{r}b + r\dot{b} &= r^2 - r^2 b^2 + O(r^3) \\ 3r^2\dot{r}c + r^3\dot{c} &= 0 \end{aligned}$$

Solve for $\dot{r}$, $\dot{b}$, $\dot{c}$.

$$\begin{aligned} \dot{r} &= -\frac{1}{2} r^2 c(1 + O(r)) \\ \dot{b} &= r - rb^2 + \frac{1}{2} rbc + O(r^2) \\ \dot{c} &= \frac{3}{2} rc^2 (1 + O(r)) \end{aligned}$$

Rescale time (divide by r).

$$r' = -\frac{1}{2}rc + O(r^2)$$
$$b' = 1 - b^2 + \frac{1}{2}bc + O(r)$$
$$c' = \frac{3}{2}c^2(1 + O(r))$$

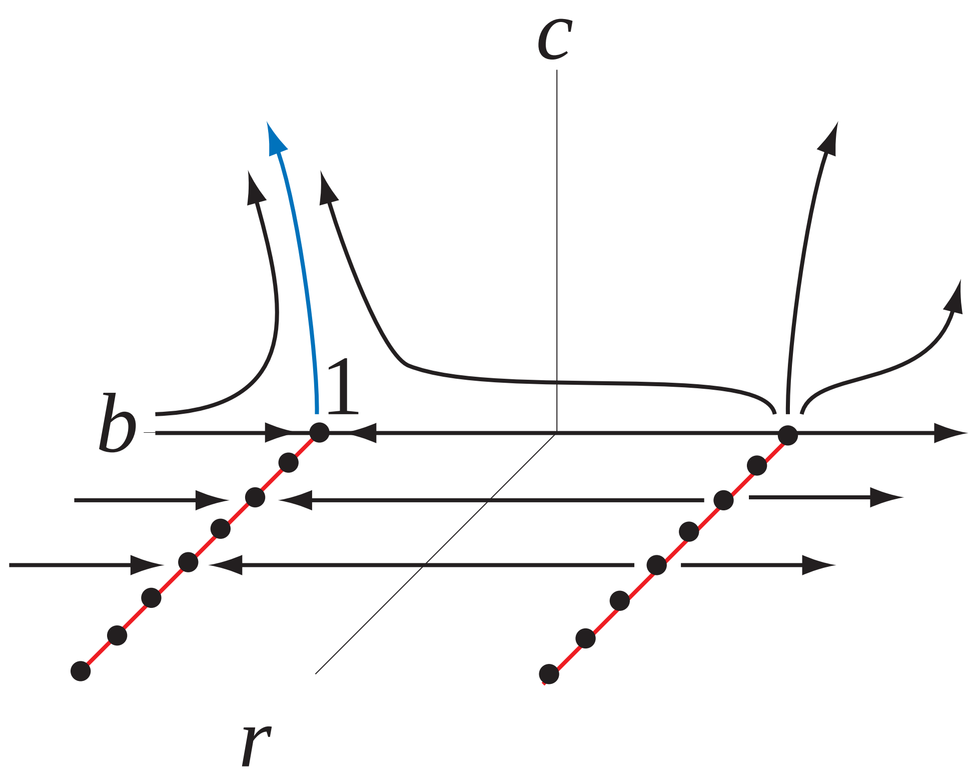


Chart for $\overline{y} < 0$

$$\begin{aligned}
\dot{x} &= \varepsilon(1 + \ldots) \\
\dot{y} &= -x - y^2 + \ldots \\
\dot{\varepsilon} &= 0
\end{aligned}$$

$$x = r^2 a, \quad y = -r, \quad \varepsilon = r^3 c, \qquad a \in \mathbb{R},\ c \in \mathbb{R},\ r \geq 0.$$

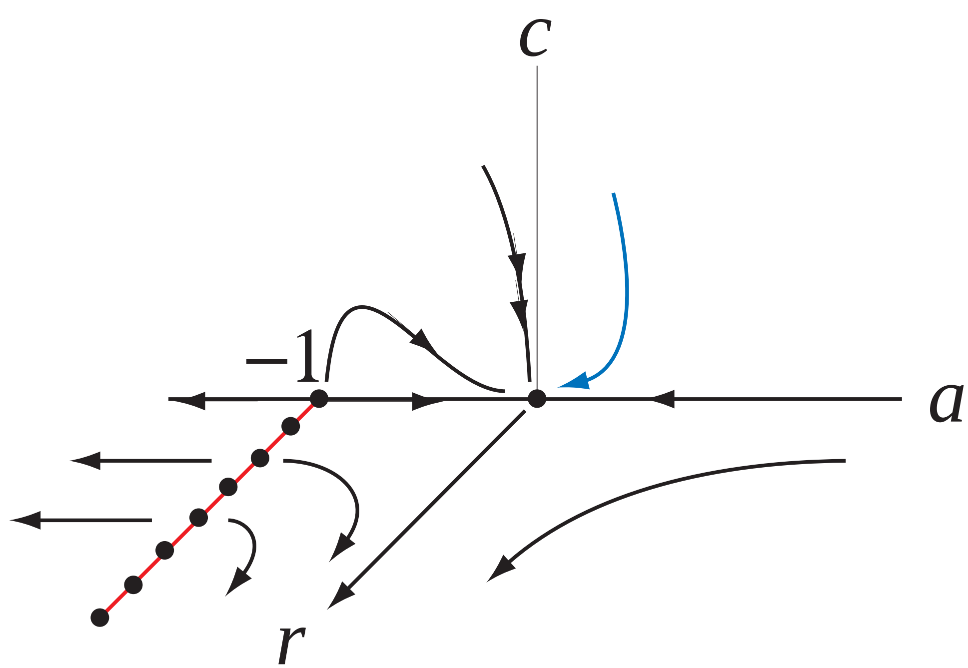
$$\begin{aligned}
2r\dot{r}a + r^2\dot{a} &= r^3 c(1 + O(r)) \\
-\dot{r} &= -r^2 a - r^2 + O(r^3) \\
3r^2\dot{r}c + r^3\dot{c} &= 0
\end{aligned}$$

Solve for $\dot{a}$, $\dot{r}$, $\dot{c}$.

$$\begin{aligned}
\dot{a} &= rc(1 + O(r)) - 2ra(a + 1 + O(r)) \\
\dot{r} &= r^2(a + 1 + O(r)) \\
\dot{c} &= -3cr(a + 1 + O(r))
\end{aligned}$$

Rescale time (divide by r).

$$a' = c(1+O(r)) - 2a(a+1+O(r))$$
$$r' = r(a+1+O(r)$$
$$c' = -3c(a+1+O(r))$$



$a = kc^{\frac{2}{3}}$ so $x = r^2 a = r^2 k c^{\frac{2}{3}} = k(r^3 c)^{\frac{2}{3}} = k\varepsilon^{\frac{2}{3}}$

To deal with loss of normal hyperbolicity in a manifold of equilibria for $\varepsilon = 0$:

(1) Identify manifolds of possible outer solutions.

(2) Extend the system by making ε into a variable.

(3) Decide on blow-up coordinates. Expect ε to be among the variables that are blown up.

(4) Use one chart to identify inner solution.

(5) Use other charts to match.

II. Gain-of-Stability Turning Points

(Rarefactions in the Dafermos Regularization)

Consider the system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= (A(u) - xI)v, \\ \dot{x} &= \varepsilon, \end{aligned}$$

with $(u, v, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $A(u)$ an $n \times n$ matrix.

Let $n = k + l + 1$. Assume that on an open set U in $\mathbb{R}^n$:

- There are numbers $\lambda_1 < \lambda_2$ such that $A(u)$ has
  - k eigenvalues with real part less than λ_1,
  - l eigenvalues with real part greater than λ_2,
  - a simple real eigenvalue $\lambda(u)$ with $\lambda_1 < \lambda(u) < \lambda_2$.
- $A(u)$ has an eigenvector $r(u)$ for the eigenvalue $\lambda(u)$, and $D\lambda(u)r(u) > 0$.

Notice ux-space is invariant for every ε. For $\varepsilon = 0$ it consists of equilibria, but loses normal hyperbolicity along the surface $x = \lambda(u)$. (Not in standard form for a slow-fast system.)

Goal: For $\varepsilon > 0$, find a solution that connects $u = u^-$ to $u = u^+$ as x passes $\lambda(u)$.

Simplify by restricting to a normally hyperbolic invariant manifold.

$$\dot{u} = v,$$
$$\dot{v} = (A(u) - xI)v,$$
$$\dot{x} = \varepsilon,$$

Near $x = \lambda(u)$, there is a normally hyperbolic invariant manifold with coordinates (u, z_1, x, ε) with z_1 a coordinate along $r(u)$ in v-space.

For $\varepsilon = 0$, within the normally hyperbolic invariant manifold, the equilibria $z_1 = 0$ still lose normal hyperbolicity when $x = \lambda(u)$.

We therefore make the change of variables $x = \lambda(u) + \sigma$ and blow up the set $z_1 = \sigma = \varepsilon = 0$:

$$u = u,$$
$$z_1 = \bar{r}^2 \bar{z}_1,$$
$$\sigma = \bar{r}\bar{\sigma},$$
$$\varepsilon = \bar{r}^2 \bar{\varepsilon},$$

with $\bar{z}_1{}^2 + \bar{\sigma}^2 + \bar{\varepsilon}^2 = 1$ (quasi-homogeneous “spherical cylindrical” coordinates).

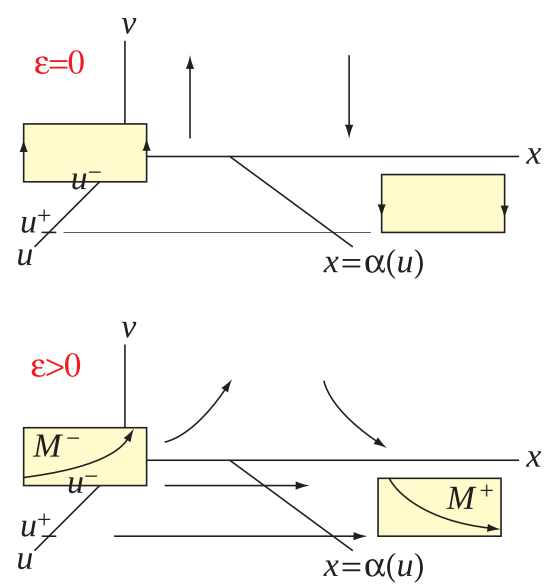
For the new system, the spherical cylinder $\bar{r} = 0$ consists entirely of equilibria. Divide by $\bar{r}$ to desingularize.

System with $n = 1$, so $u \in \mathbb{R}$ and $v \in \mathbb{R}$:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= (\alpha(u) - x)v, \quad \alpha' > 0 \\ \dot{x} &= \varepsilon. \end{aligned}$$

ux-space is invariant for every ε. For $\varepsilon = 0$ it consists of equilibria, but loses normal hyperbolicity along the curve $x = \alpha(u)$.

Look for a solution with $u(-\infty) = u^-$, $u(\infty) = u^+$, $u^- < u^+$.



Second picture shows possible outer solutions.

Preliminary change of coordinates: $x = \alpha(u) + \sigma$. Also, extend system.

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= -\sigma v,\\ \dot{\sigma} &= \varepsilon - \alpha'(u)v,\\ \dot{\varepsilon} &= 0.\end{aligned}$$

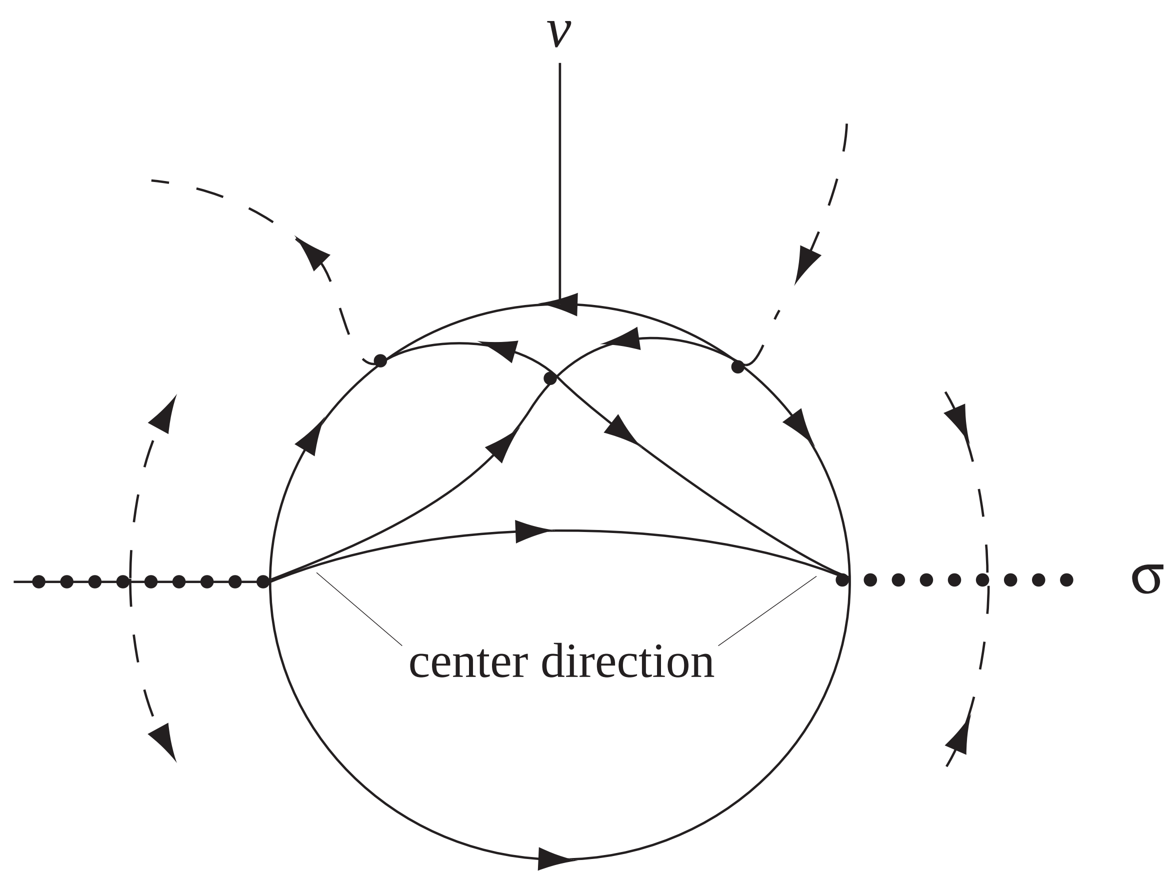
For $\varepsilon = 0$, normal hyperbolicity of $u\sigma$-space is lost along the u-axis ($\sigma = 0$).

Blow-up:

$$\begin{aligned}u &= u,\\ v &= \bar{r}^2\bar{v},\\ \sigma &= \bar{r}\bar{\sigma},\\ \varepsilon &= \bar{r}^2\bar{\varepsilon},\end{aligned}$$

with $u \in \mathbb{R}$, $(\bar{v}, \bar{\sigma}, \bar{\varepsilon}) \in S^2$, $\bar{r} \geq 0$.

Divide vector field on blow-up space by $\bar{r}$ to desingularize. The spherical cylinder $\bar{r} = 0$ remains invariant, and on it $\dot{u} = 0$.



Flow on blow-up space for fixed u. The ε-axis points toward you.

- No loss of normal hyperbolicity.
- Dashed curves do not have constant u.
- The “plane” $u = \text{constant}$, $\bar{v} = 0$ is invariant.

Use chart for $\bar{\varepsilon} > 0$ (rescaling chart) to find inner solution

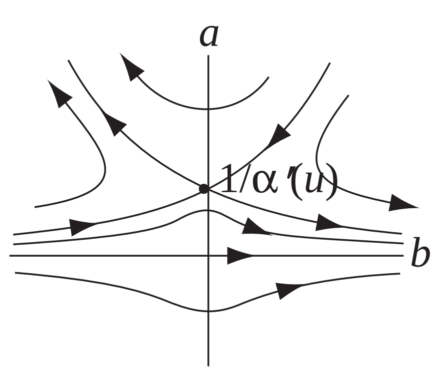
$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= -\sigma v,\\ \dot{\sigma} &= \varepsilon - \alpha'(u)v,\\ \dot{\varepsilon} &= 0.\end{aligned}$$

$$v = r^2 a, \quad \sigma = rb, \quad \varepsilon = r^2, \qquad a \in \mathbb{R},\ b \in \mathbb{R},\ r \geq 0.$$

Substitute, solve for $\dot{a}$ and $\dot{b}$, rescale time (divide by r):

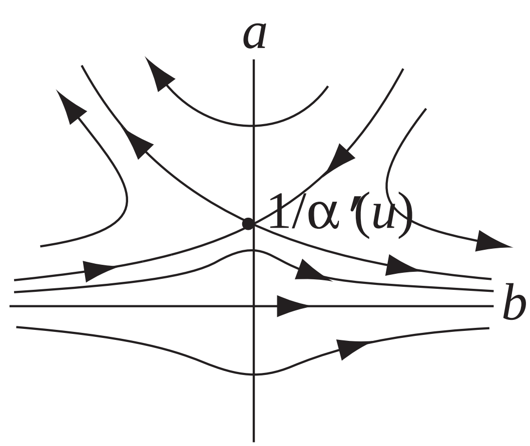
$$\begin{aligned}\dot{u} &= ra,\\ \dot{a} &= -ab,\\ \dot{b} &= 1 - \alpha'(u)a,\\ \dot{r} &= 0.\end{aligned}$$

Note three terms in $\dot{a}$ and $\dot{b}$ equations have order ε^0. First three equations are a slow-fast system with small parameter r. For $r = 0$, flow with $u =$ constant:



$$\dot{u} = ra,$$
$$\dot{a} = -ab,$$
$$\dot{b} = 1 - \alpha'(u)a,$$
$$\dot{r} = 0.$$

For $r = 0$, flow with $u =$ constant:



Slow manifold is $I_0 = \{(u, a, b) : a = \frac{1}{\alpha'(u)},\ b = 0\}$. For $r = 0$, I_0 is a normally hyperbolic curve of equilibria.

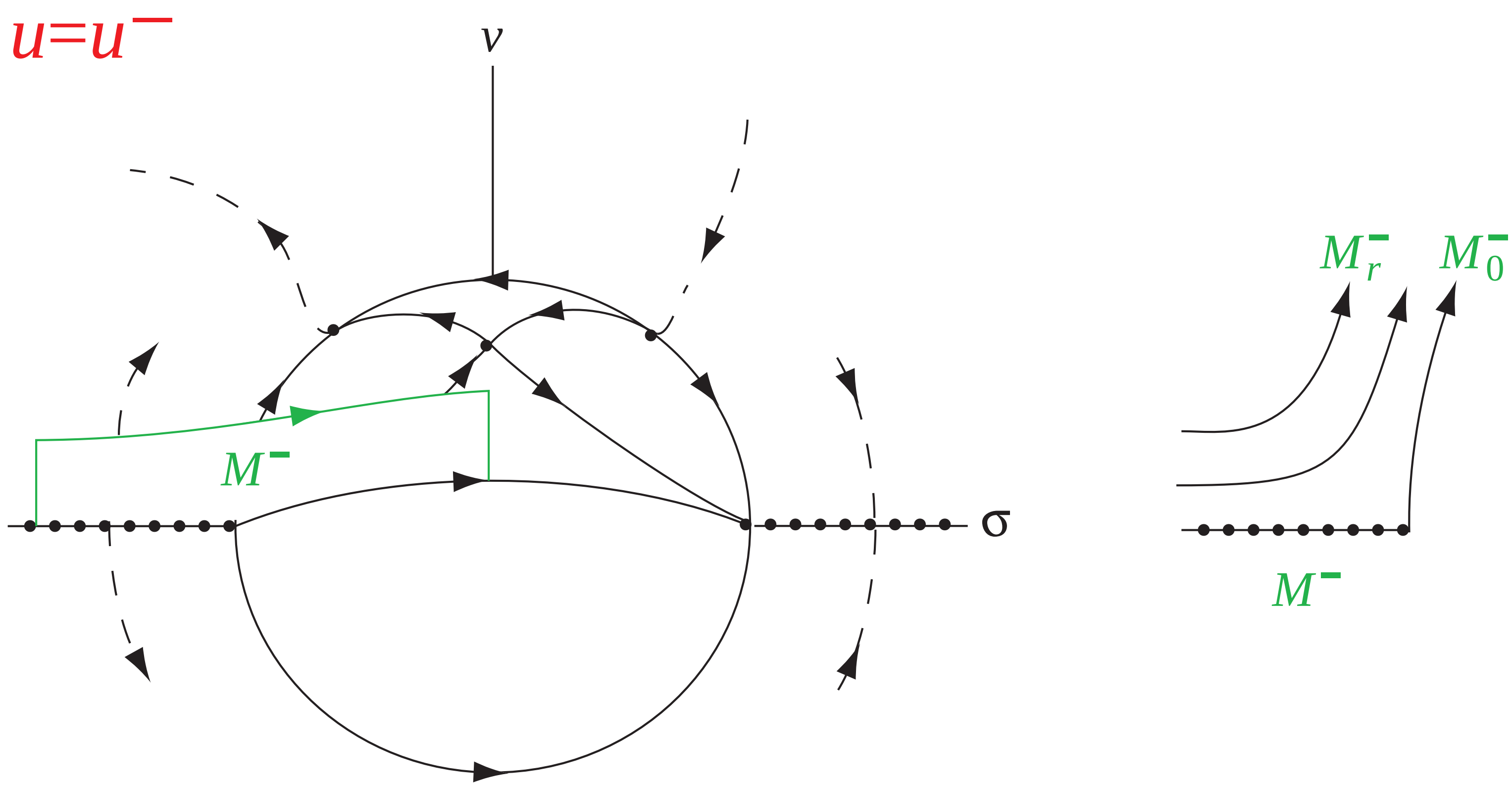
Therefore for $r > 0$ there is a nearby normally hyperbolic invariant curve I_r parameterized by u.

On I_r, to lowest order the differential equation (slow equation) is $\dot{u} = r\frac{1}{\alpha'(u)} > 0$: gives inner solution. The Exchange Lemma can help with matching.

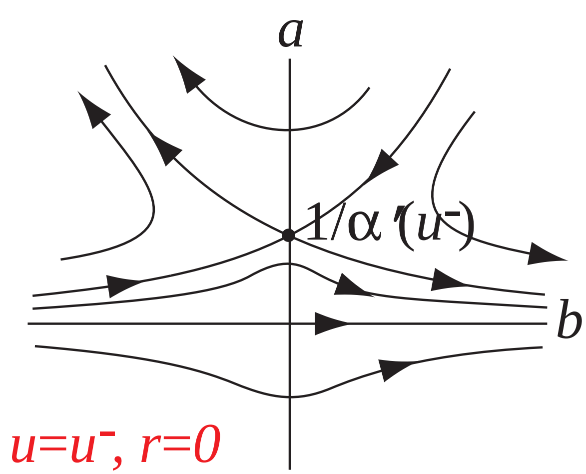
Matching: In $uv\sigma\varepsilon$-space, $\{(u,v,\sigma,\varepsilon) : v = 0,\ \sigma < -\delta < 0\}$ is 3-dimensional normally repelling invariant manifold.

Let $M^- = \{(u,v,\sigma,\varepsilon) : u = u^-,\ v = 0,\ \sigma < -\delta < 0\}$ a 2-dimensional invariant subset.

In blow-up space M^- extends as a normally hyperbolic invariant manifold to the sphere $\bar{r} = 0$: it's now $\{(u,(\bar{v},\bar{\sigma},\bar{\varepsilon}),\bar{r}) : u = u^-, \bar{v} = 0,\ \bar{\sigma} < 0\}$.



$W^u(M^-)$ (dimension = 3) includes an open subset of $\{u^-\} \times S^2 \times \{0\}$ that we call $W^u(M_0^-)$.



For $r = 0$, in uab-space, $W^u(M_0^-)$ (dimension = 2) is transverse to $W^s(I_0)$ (dimension = 2).

For small r, $W^u(M_r^-)$ is close to $W^u(M_0^-)$.

By the Exchange Lemma, for small $r > 0$, $W^u(M_r^-)$ is close to $W^u(I_r)$ when u reaches u^+.

(Transversality to $W^s(I_r)$ is exchanged for closeness to $W^u(I_r)$.)

For $r = 0$, in uab-space, $W^u(I_0)$ is transverse to $W^s(M_0^+)$.

Therefore for small $r > 0$, $W^u(M_r^-)$ is transverse to $W^s(M_r^+)$.

This gives, for small $r > 0$, an intersection of $W^u(M_r^-)$ and $W^s(M_r^+)$

Heteroclinic orbits in a Hamiltonian system

Motivation

Sourdis and Fife, *Existence of heteroclinic orbits for a corner layer problem in anisotropic interfaces*, Advances in Differential Equations **12** (2007), 623–668:

The physical motivation comes from a multi-order-parameter phase field model, developed by Braun et al. for the description of crystalline interphase boundaries. The smallness of ε is related to large anisotropy. [The heteroclinic orbit represents a moving interface between ordered and disordered states.] The mathematical interest stems from the fact that the smoothness and normal hyperbolicity of the critical manifold fails at certain points. Thus the well-developed geometric singular perturbation theory does not apply. The existence of such a heteroclinic, and its dependence on ε, is proved via a functional analytic approach.

Second-order system

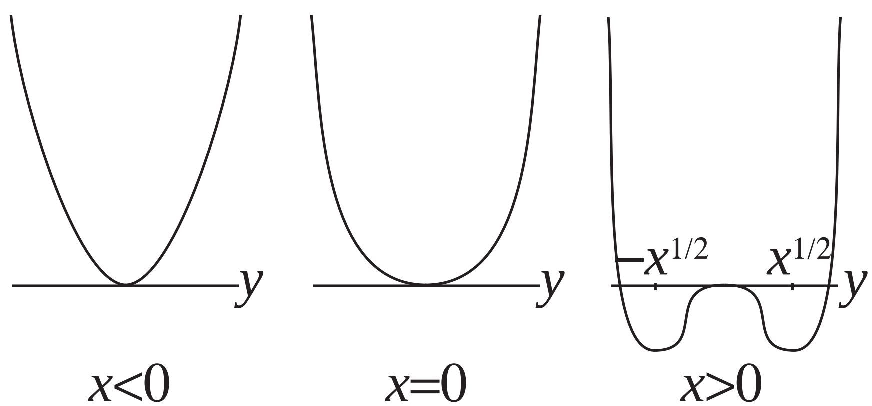
We consider

(1) $$x_{\tau\tau} = g_x(x,y),$$

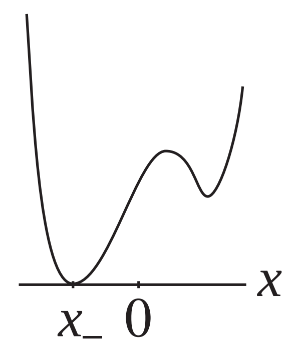
(2) $$\varepsilon^2 y_{\tau\tau} = g_y(x,y),$$

where

(3) $$g(x,y) = \frac{1}{4}y^4 - \frac{1}{2}xy^2 + h(x).$$



Graph of $(1/4)y^4-(1/2)xy^2$



Graph of $h(x)$

First-order system

Write (1)–(2) as a first-order system (the slow system) with $u_1 = x$, $u_3 = y$:

(4) $$u_{1\tau} = u_2,$$

(5) $$u_{2\tau} = g_x(u_1, u_3) = -\frac{1}{2}u_3^2 + h'(u_1),$$

(6) $$\varepsilon u_{3\tau} = u_4,$$

(7) $$\varepsilon u_{4\tau} = g_y(u_1, u_3) = u_3^3 - u_1 u_3.$$

In (4)–(7) let $\tau = \varepsilon\sigma$. We obtain the fast system:

(8) $$u_{1\sigma} = \varepsilon u_2,$$

(9) $$u_{2\sigma} = \varepsilon g_x(u_1, u_3) = \varepsilon\left(-\frac{1}{2}u_3^2 + h'(u_1)\right),$$

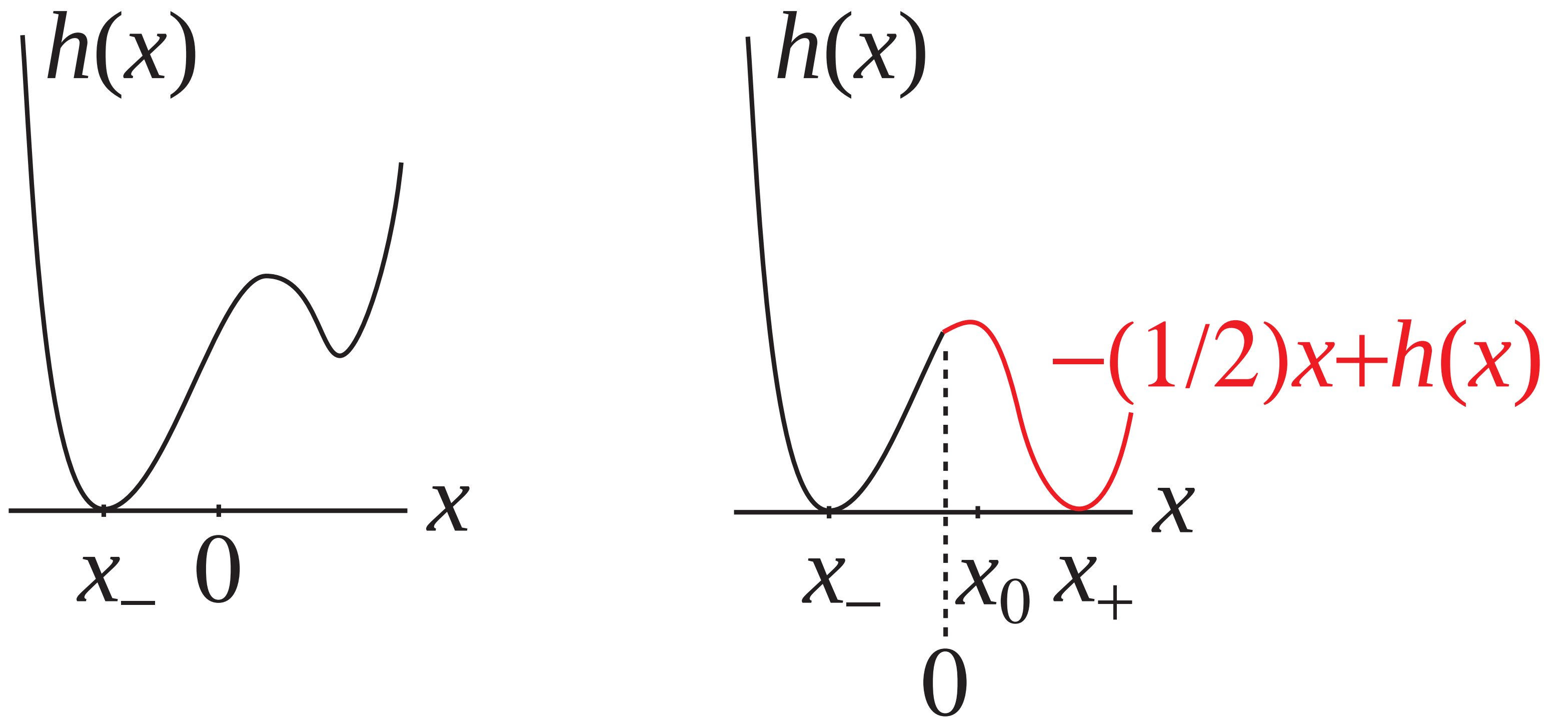
(10) $$u_{3\sigma} = u_4,$$

(11) $$u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3 = u_3(u_3^2 - u_1).$$

Equilibria of the fast system for $\varepsilon > 0$:

$$(u_1, 0, 0, 0) \text{ with } h'(u_1) = 0, \quad (u_1, 0, \pm u_1^{\frac{1}{2}}, 0) \text{ with } -\frac{1}{2}u_1 + h'(u_1) = 0.$$

Assumptions on h:



Fast system

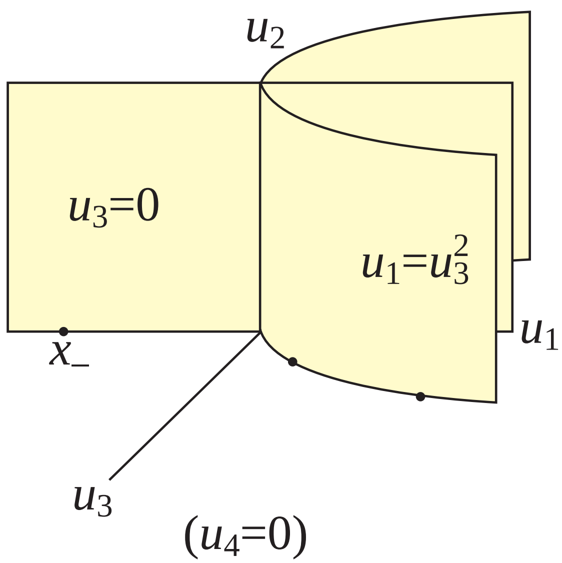
$$u_{1\sigma} = \varepsilon u_2,$$

$$u_{2\sigma} = \varepsilon g_x(u_1, u_3) = \varepsilon\left(-\frac{1}{2}u_3^2 + h'(u_1)\right),$$

$$u_{3\sigma} = u_4,$$

$$u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1u_3 = u_3(u_3^2 - u_1)$$

Equilibria of the fast system for $\varepsilon = 0$ (yellow) and for $\varepsilon > 0$ (black dots):



$$(x_-, 0, 0, 0), \quad (x_0, 0, \pm x_0^{\frac{1}{2}}, 0), \quad (x_+, 0, \pm x_+^{\frac{1}{2}}, 0).$$

For each ε, the fast system has the first integral

$$H(u_1,u_2,u_3,u_4) = \frac{1}{2}u_2^2 + \frac{1}{2}u_4^2 - g(u_1,u_3).$$

Note:

$$H(x_-,0,0,0) = H(x_+,0,x_+^{\frac{1}{2}},0) = 0.$$

Goal: show that for small $\varepsilon > 0$, there is a heteroclinic solution of the fast system from $(x_-,0,0,0)$ to $(x_+,0,x_+^{\frac{1}{2}},0)$.

For $\varepsilon > 0$, $(x_-,0,0,0)$ and $(x_+,0,x_+^{\frac{1}{2}},0)$ are hyperbolic equilibria of the fast system with two negative eigenvalues and two positive eigenvalues.

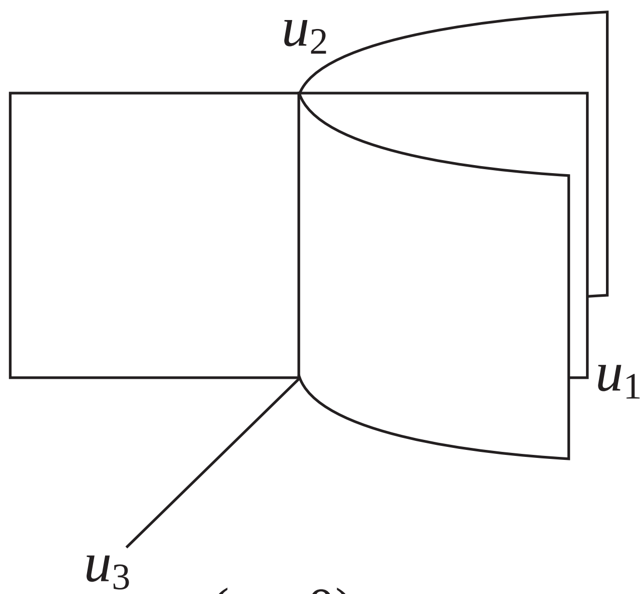
Manifolds of possible outer solutions: The heteroclinic solution should correspond to an intersection of the 2-dimensional manifolds $W_\varepsilon^u(x_-,0,0,0)$ and $W_\varepsilon^s(x_+,0,x_+^{\frac{1}{2}},0)$ that is transverse within the 3-dimensional manifold $H^{-1}(0)$ (which is indeed a manifold away from equilibria).

Fast limit and slow systems

Set $\varepsilon = 0$ in the fast system to obtain the fast limit system:

(12) $$u_{1\sigma} = 0,$$
(13) $$u_{2\sigma} = 0,$$
(14) $$u_{3\sigma} = u_4,$$
(15) $$u_{4\sigma} = g_y(u_1, u_3) = u_3(u_3^2 - u_1).$$

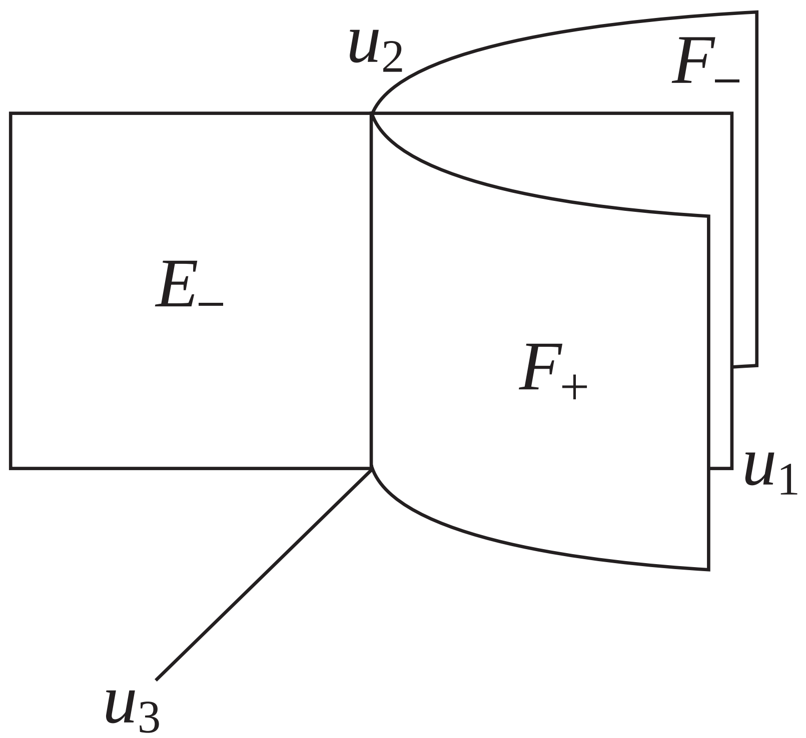
Equilibria:



$(u_4{=}0)$

Three manifolds of normally hyperbolic equilibria:

$$E_- = \{(u_1, u_2, 0, 0) : u_1 < 0 \text{ and } u_2 \text{ arbitrary}\},$$
$$F_- = \{(u_1, u_2, -u_1^{\frac{1}{2}}, 0) : u_1 > 0 \text{ and } u_2 \text{ arbitrary}\},$$
$$F_+ = \{(u_1, u_2, u_1^{\frac{1}{2}}, 0) : u_1 > 0 \text{ and } u_2 \text{ arbitrary}\}.$$



Each has one positive eigenvalue and one negative eigenvalue. (On E_+ there are two pure imaginary eigenvalues. On the u_2-axis all eigenvalues are 0.)

Set $\varepsilon = 0$ in the slow system to obtain the slow limit system:

$$u_{1\tau} = u_2, \tag{16}$$

$$u_{2\tau} = g_x(u_1, u_3) = -\frac{1}{2}u_3^2 + h'(u_1), \tag{17}$$

$$0 = u_4, \tag{18}$$

$$0 = g_y(u_1, u_3) = u_3(u_3^2 - u_1). \tag{19}$$

$E_\pm$, $F_\pm$ are manifolds of solutions of (18)–(19). Equations (16)–(17) give the slow system on these manifolds.

Slow system on E_- ($u_1 < 0$, u_2 arbitrary):

$$u_{1\tau} = u_2, \tag{20}$$

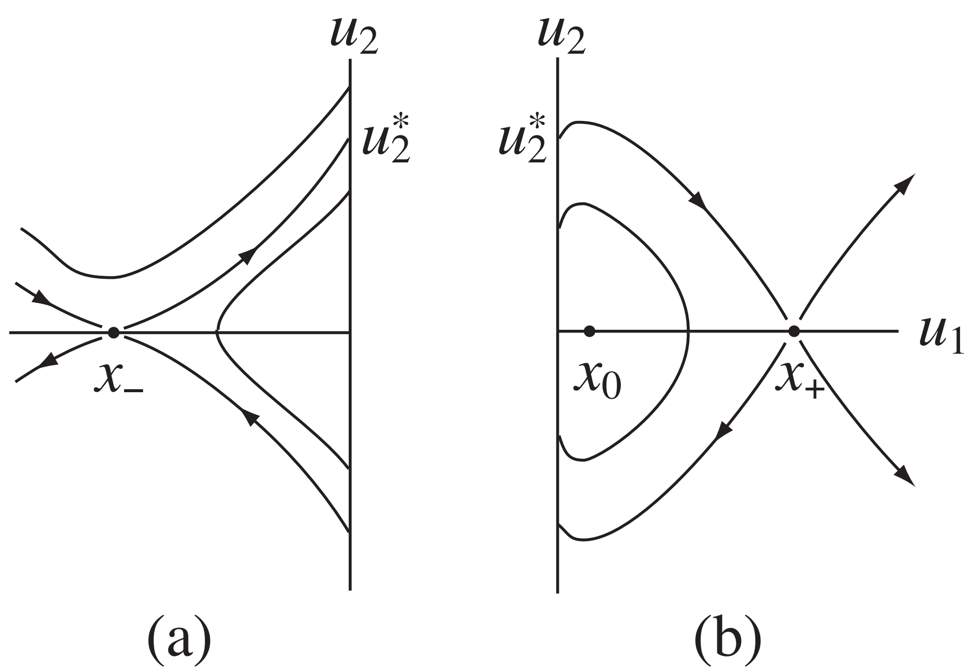
$$u_{2\tau} = g_x(u_1, 0) = h'(u_1). \tag{21}$$

Slow system on F_+ ($u_1 > 0$, u_2 arbitrary):

$$u_{1\tau} = u_2, \tag{22}$$

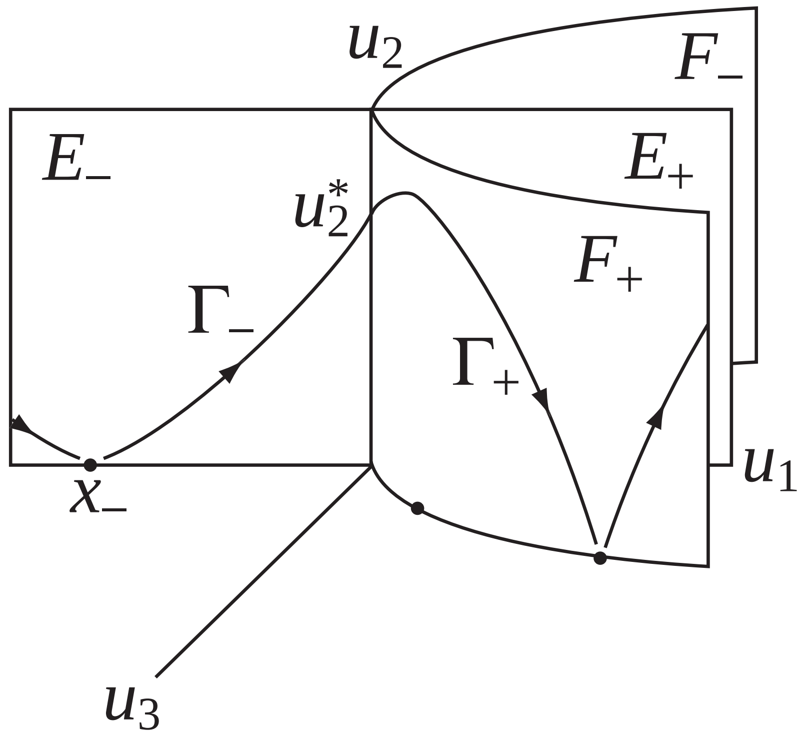
$$u_{2\tau} = g_x(u_1, u_1^{\frac{1}{2}}) = -\frac{1}{2}u_1 + h'(u_1). \tag{23}$$

Phase portraits of slow system on E_- and F_+ in u_1u_2-coordinates, both extended to $u_1 = 0$:



- In (a), $(x_-, 0)$ is a hyperbolic saddle, and a branch of its unstable manifold meets the u_2 axis at a point $(0, u_2^*)$.
- In (b), $(x_+, 0)$ is a hyperbolic saddle, and a branch of its stable manifold meets the u_2 axis at the same point $(0, u_2^*)$.

Slow limit system on E_- and F_+:



We want to show that for small $\varepsilon > 0$, there is a heteroclinic solution of the fast system from $(x_-, 0, 0, 0)$ to $(x_+, 0, x_+^{\frac{1}{2}}, 0)$ that is close to $\Gamma_- \cup \Gamma_+$.

Blow-up

To the fast system append the equation $\varepsilon_\sigma = 0$:

(24) $$u_{1\sigma} = \varepsilon u_2,$$

(25) $$u_{2\sigma} = \varepsilon g_x(u_1, u_3) = \varepsilon(-\frac{1}{2}u_3^2 + h'(u_1)),$$

(26) $$u_{3\sigma} = u_4,$$

(27) $$u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3,$$

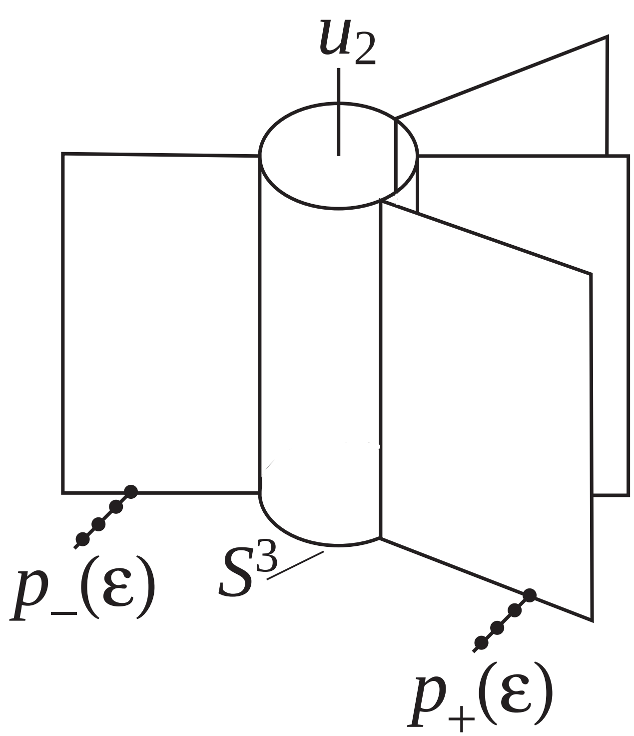
(28) $$\varepsilon_\sigma = 0.$$

For $\varepsilon = 0$, the u_2-axis consists of equilibria of (24)–(27) that are not normally hyperbolic within $u_1u_2u_3u_4$-space

In $u_1u_2u_3u_4\varepsilon$-space, we blow up the u_2-axis to the product of the u_2-axis with a 3-sphere. The 3-sphere is a blow-up of the origin in $u_1u_3u_4\varepsilon$-space.

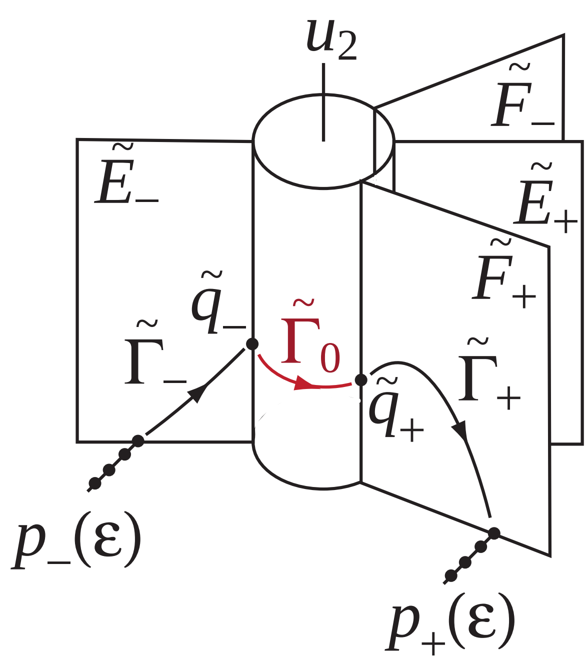
The blowup transformation is a map from $\mathbb{R} \times S^3 \times [0, \infty)$ to $u_1u_2u_3u_4\varepsilon$-space. Let $(u_2, (\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\varepsilon}), \bar{r})$ be a point of $\mathbb{R} \times S^3 \times [0, \infty)$; we have $\bar{u}_1{}^2 + \bar{u}_3{}^2 + \bar{u}_4{}^2 + \bar{\varepsilon}^2 = 1$. Then

(29) $$u_1 = \bar{r}^2\bar{u}_1, \quad u_2 = u_2, \quad u_3 = \overline{ru}_3, \quad u_4 = \bar{r}^2\bar{u}_4, \quad \varepsilon = \bar{r}^3\bar{\varepsilon}.$$



Under this transformation (24)–(28) pulls back to a vector field X on $\mathbb{R} \times S^3 \times [0, \infty)$ for which the cylinder $\overline{r} = 0$ consists entirely of equilibria. The vector field we shall study is $\tilde{X} = \overline{r}^{-1} X$. Division by $\overline{r}$ desingularizes the vector field on the cylinder $\overline{r} = 0$ but leaves it invariant.

Let $p_{-}(\varepsilon)$ (respectively $p_{+}(\varepsilon)$) be the unique point in $\mathbb{R} \times S^3 \times [0, \infty)$ that corresponds to $(x_{-}, 0, 0, 0, \varepsilon)$ (respectively $(x_{+}, 0, x_{+}^{\frac{1}{2}}, 0, \varepsilon)$). We wish to show that for small $\varepsilon > 0$ there is an integral curve of $\tilde{X}$ from $p_{-}(\varepsilon)$ to $p_{+}(\varepsilon)$.



In blow-up space:

- $\tilde{\Gamma}_-$ corresponds to Γ_- and approaches a point $\tilde{q}_- = (u_2^*, \hat{q}_-, 0)$ on the blow-up cylinder.
- $\tilde{\Gamma}_+$ corresponds to Γ_+ and approaches a point $\tilde{q}_+ = (u_2^*, \hat{q}_+, 0)$ on the blow-up cylinder.
- On the blow-up cylinder, each 3-sphere $u_2 =$ constant is invariant.

Proposition (inner solution). There is an integral curve $\tilde{\Gamma}_0$ of $\tilde{X}$ from $\tilde{q}_-$ to $\tilde{q}_+$ that lies in the 3-dimensional hemisphere given by $u_2 = u_2^*$, $\bar{r} = 0$, $\bar{\varepsilon} > 0$.

Theorem. For small $\varepsilon > 0$ there is an integral curve $\tilde{\Gamma}(\varepsilon)$ of $\tilde{X}$ from $p_-(\varepsilon)$ to $p_+(\varepsilon)$. As $\varepsilon \to 0$, $\tilde{\Gamma}(\varepsilon) \to \tilde{\Gamma}_- \cup \tilde{\Gamma}_0 \cup \tilde{\Gamma}_+$.

We shall need three charts on blow-up space:

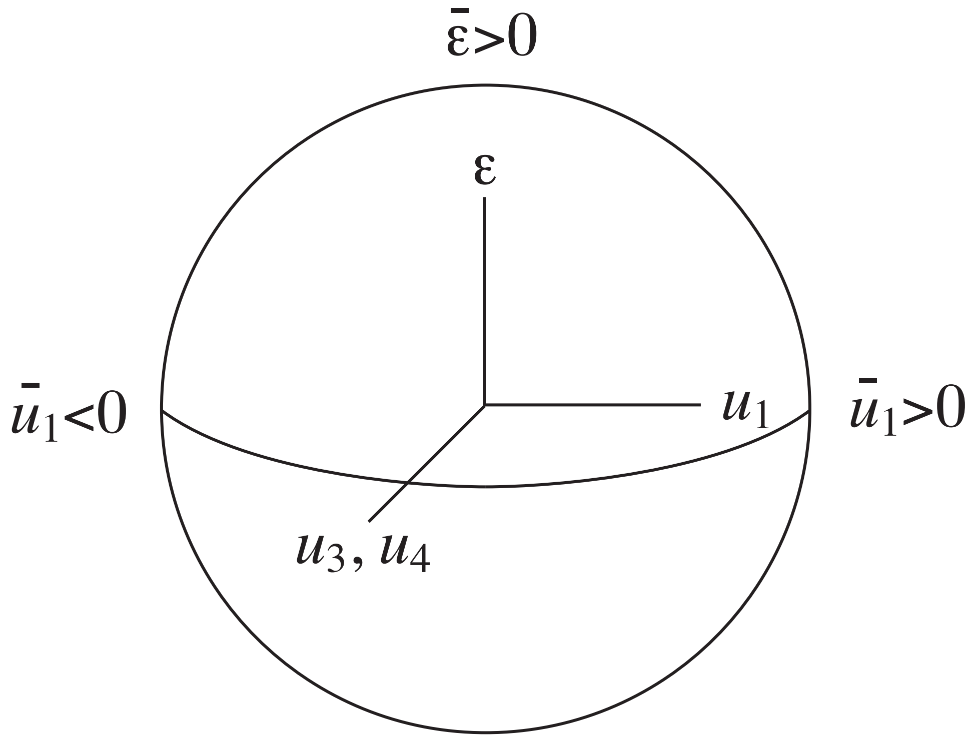


Chart for $\overline{\varepsilon} > 0$

$$u_1 = r^2 b_1, \quad u_2 = u_2, \quad u_3 = r b_3, \quad u_4 = r^2 b_4, \quad \varepsilon = r^3, \tag{30}$$

with $r \geq 0$. After division by r, (24)–(28) becomes

$$b_{1s} = u_2, \tag{31}$$

$$u_{2s} = r^2\left(-\frac{1}{2} r^2 b_3^2 + h'(r^2 b_1)\right), \tag{32}$$

$$b_{3s} = b_4, \tag{33}$$

$$b_{4s} = b_3^3 - b_1 b_3, \tag{34}$$

$$r_s = 0. \tag{35}$$

Note: $r = 0$ implies $u_{2s} = 0$.

Chart for $\overline{u_1} < 0$

(36) $$u_1 = -v^2, \quad u_2 = u_2, \quad u_3 = va_3, \quad u_4 = v^2 a_4, \quad \varepsilon = v^3\delta,$$

with $v \geq 0$. After division by v, (24)–(28) becomes

(37) $$v_t = -\frac{1}{2} v\delta u_2,$$

(38) $$u_{2t} = v^2\delta(-\frac{1}{2} v^2 a_3^2 + h'(-v^2)),$$

(39) $$a_{3t} = a_4 + \frac{1}{2}\delta u_2 a_3,$$

(40) $$a_{4t} = a_3^3 + a_3 + \delta u_2 a_4,$$

(41) $$\delta_t = \frac{3}{2}\delta^2 u_2.$$

Note: $v = 0$ implies $u_{2t} = 0$.

Chart for $\bar{u}_1 > 0$

(42) $$u_1 = w^2, \quad u_2 = u_2, \quad u_3 = wc_3, \quad u_4 = w^2 c_4, \quad \varepsilon = w^3\gamma.$$

with $w \geq 0$. After division by w, (24)–(28) becomes

(43) $$w_t = \frac{1}{2} w\gamma u_2,$$

(44) $$u_{2t} = w^2\gamma\left(-\frac{1}{2}w^2c_3^2 + h'(w^2)\right),$$

(45) $$c_{3t} = c_4 - \frac{1}{2}\gamma u_2 c_3,$$

(46) $$c_{4t} = c_3^3 - c_3 - \gamma u_2 c_4,$$

(47) $$\gamma_t = -\frac{3}{2}\gamma^2 u_2.$$

Note: $w = 0$ implies $u_{2t} = 0$.

Construction of the inner solution $\tilde{\Gamma}_0$

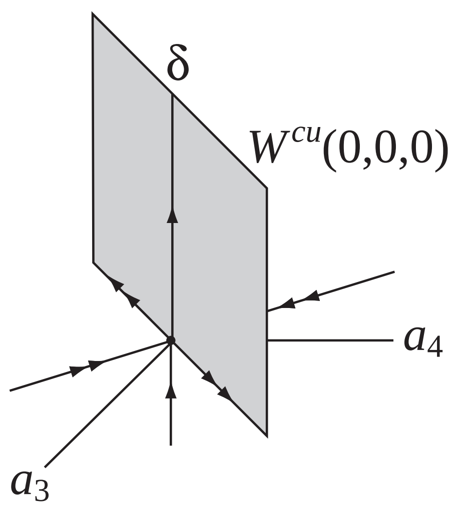
Consider the restriction of the vector field $\tilde{X}$ to the invariant 3-sphere $S = \{u_2^*\} \times S^3 \times \{0\}$, $S^3 = \{(\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\varepsilon}) : \bar{u}_1^2 + \bar{u}_3^2 + \bar{u}_4^2 + \bar{\varepsilon}^2 = 1\}$.

Chart on the open subset of S with $\bar{u}_1 < 0$: use (a_3, a_4, δ). In this chart:

(48) $$a_{3t} = a_4 + \frac{1}{2}\delta u_2^* a_3,$$

(49) $$a_{4t} = a_3^3 + a_3 + \delta u_2^* a_4,$$

(50) $$\delta_t = \frac{3}{2}\delta^2 u_2^*.$$



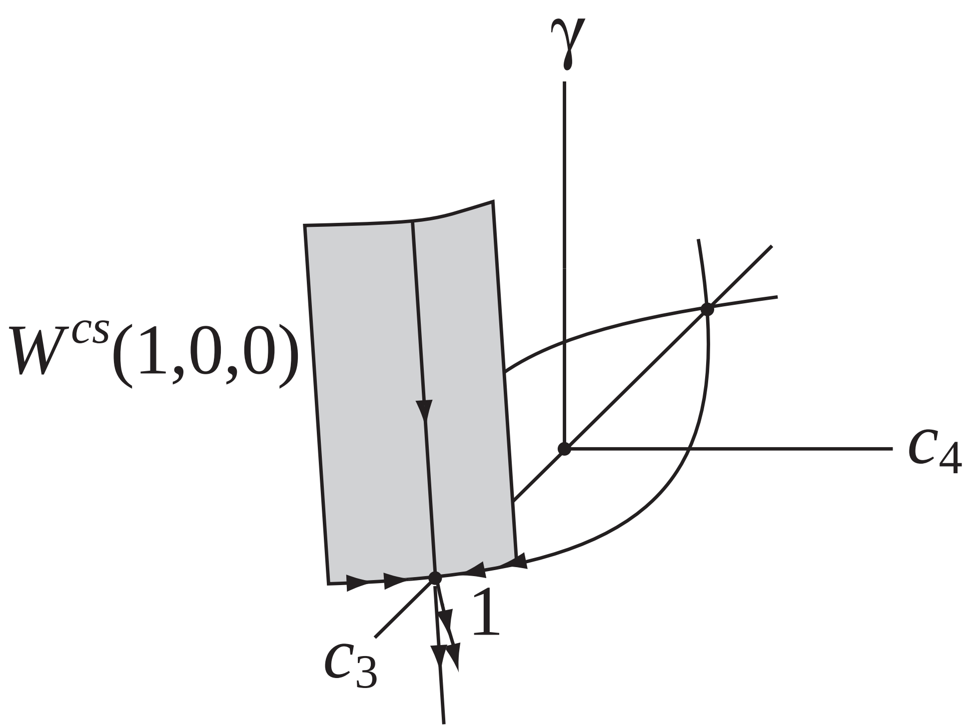
Only equilibrium is $\hat{q}^- = (0,0,0)$. Hyperbolicity is recovered in the a_3- and a_4-directions.

Chart on the open subset of S with $\bar{u}_1 > 0$: Use (c_3, c_4, γ). In this chart:

(51) $$c_{3t} = c_4 - \frac{1}{2}\gamma u_2^* c_3,$$

(52) $$c_{4t} = c_3^3 - c_3 - \gamma u_2^* c_4,$$

(53) $$\gamma_t = -\frac{3}{2}\gamma^2 u_2^*.$$



Three equilibria, $\hat{q}^+ = (1,0,0)$. Hyperbolicity is recovered in the c_3- and c_4-directions.

Chart on the open subset of S with $\bar{\varepsilon} > 0$: use (b_1, b_3, b_4). In this chart, we have:

(54) $$b_{1s} = u_2^*,$$

(55) $$b_{3s} = b_4,$$

(56) $$b_{4s} = b_3^3 - b_1 b_3 = b_3(b_3^2 - b_1).$$

The solution of (54) with $b_1(0) = 0$ is $b_1 = u_2^* s$. Substitute into (56) and combining (55) and (56) into a second-order equation:

(57) $$b_{3ss} = b_3(b_3^2 - u_2^* s)$$

By Sourdis and Fife, (57) has a solution $b_3(s)$ with $b_{3s} > 0$ such that

(S1) $b_3(s) = O\left(|s|^{-\frac{1}{4}} e^{-\frac{2}{3}(u_2^*)^{\frac{1}{2}}|s|^{\frac{3}{2}}}\right)$ as $s \to -\infty$,
(S2) $b_3(s) = (u_2^* s)^{\frac{1}{2}} + O\left(s^{-\frac{5}{2}}\right)$ as $s \to \infty$,
(S3) $b_{3s}(s) \leq C|s|^{-\frac{1}{2}}$, $s \neq 0$.

$(u_2^* s, b_3(s), b_{3s}(s))$ is a solution of (54)–(56). It represents an intersection of $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ in the 3-sphere S.

Transversality

$W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ are 2-dimensional submanifolds of the 3-sphere S.

Let $\tilde{\Gamma}_0 = (u_2^*, \hat{\Gamma}_0, 0)$. They intersect along $\hat{\Gamma}_0$.

Proposition. $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ intersect transversally within S along $\hat{\Gamma}_0$.

Proof. The linearization of

$$\begin{aligned} b_{1s} &= u_2^*, \\ b_{3s} &= b_4, \\ b_{4s} &= b_3^3 - b_1 b_3 \end{aligned}$$

along $(u_2^* s, b_3(s), b_{3s}(s))$ is

(58) $$\begin{pmatrix} B_{1s} \\ B_{3s} \\ B_{4s} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b_3(s) & 3b_3(s)^2 - u_2^* s & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_3 \\ B_4 \end{pmatrix}.$$

We must show there are no solutions with appropriate behavior at $s = \pm\infty$ other than multiples of (u_2^*, b_{3s}, b_{3ss}).

There is a complementary 2-dimensional space of solutions of (58) with $B_1(s) = 0$ and $(B_3(s), B_4(s))$ a solution of

(59)
$$\begin{pmatrix} B_{3s} \\ B_{4s} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3b_3(s)^2 - u_2^* s & 0 \end{pmatrix} \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}$$

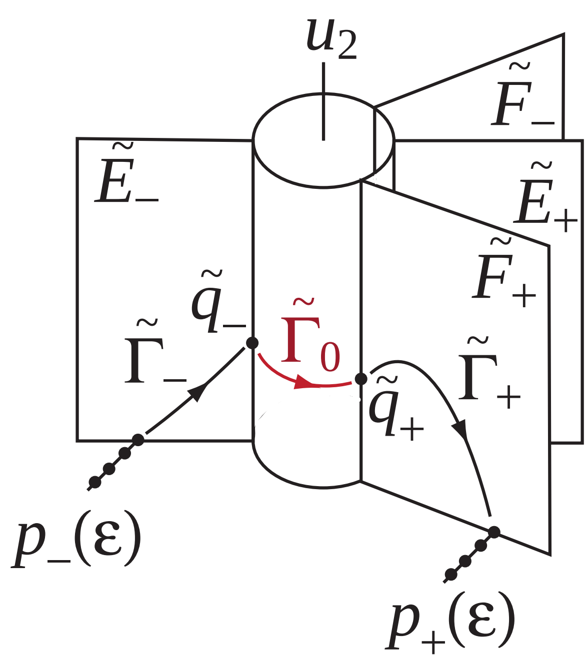
We must show that no nontrivial solution has appropriate behavior at $s = \pm\infty$.

(59) is equivalent to the second order linear system

(60)
$$B_{3ss} = (3b_3(s)^2 - u_2^* s)B_3.$$

By Alikakos, Bates, Cahn, Fife, Fusco, and Tanoglu, *Analysis of the corner layer problem in anisotropy*, Discrete Contin. Dyn. Syst. **6** (2006), 237–255, (60) has no nontrivial solutions in L^2, hence no solution with the correct asymptotic behavior.

Matching

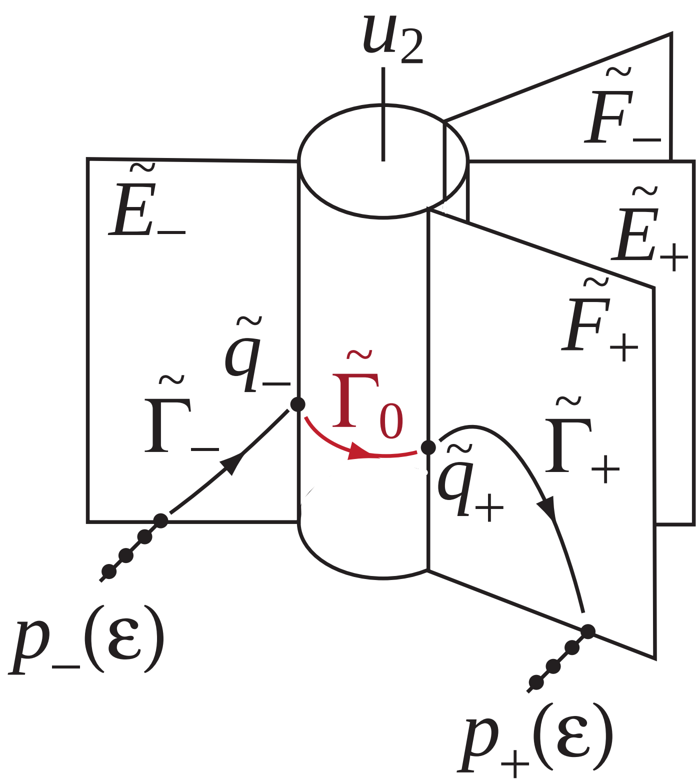


Recall: for each ε, the fast system has the first integral

$$H(u_1,u_2,u_3,u_4) = \frac{1}{2}u_2^2 + \frac{1}{2}u_4^2 - \left(\frac{1}{4}u_3^4 - \frac{1}{2}u_1u_3^2 + h(u_1)\right).$$

H gives rise to a first integral $\tilde{H}$ on blow-up space:

$$\tilde{H}(u_2,(\bar{u}_1,\bar{u}_3,\bar{u}_4,\bar{\varepsilon}),\bar{r}) = \frac{1}{2}u_2^2 + \bar{r}^4\left(\frac{1}{2}\bar{u}_4^2 - \frac{1}{4}\bar{u}_3^4 + \frac{1}{2}\bar{u}_1\bar{u}_3^2\right) - h(\bar{r}^2\bar{u}_1).$$



Let N_ε denote the set of points in blow-up space at which $\tilde{H} = 0$ and $\overline{r}^3\overline{\varepsilon} = \varepsilon$.

Away from equilibria of $\tilde{X}$, each N_ε is a manifold of dimension 3.

For the vector field $\tilde{X}$ and $\varepsilon > 0$, the equilibria $p_-(\varepsilon)$ and $p_+(\varepsilon)$ have 2-dimensional unstable and stable manifolds.

We will prove the theorem by showing that for small $\varepsilon > 0$, $W^u(p_-(\varepsilon))$ and $W^s(p_+(\varepsilon))$ have a nonempty intersection that is transverse within N_ε.

Chart for $\bar{u}_1 < 0$:

$$\begin{aligned}
v_t &= -\frac{1}{2} v \delta u_2, \\
u_{2t} &= v^2 \delta(-\frac{1}{2} v^2 a_3^2 + h'(-v^2)), \\
a_{3t} &= a_4 + \frac{1}{2} \delta u_2 a_3, \\
a_{4t} &= a_3^3 + a_3 + \delta u_2 a_4, \\
\delta_t &= \frac{3}{2} \delta^2 u_2.
\end{aligned}$$

The 3-dimensional space $a_3 = a_4 = 0$ is invariant, and is normally hyperbolic near the plane of equilibria $a_3 = a_4 = \delta = 0$. One eigenvalue is positive, one is negative.

The plane of equilibria corresponds to E_-. Normal hyperbolicity within $\delta = 0$ is *not* lost at $v = 0$, which corresponds to $u_1 = 0$.

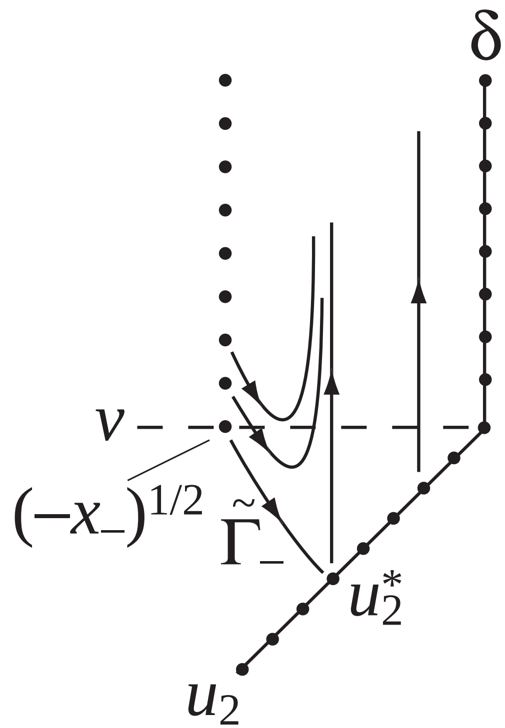
Restrict to $a_3 = a_4 = 0$ and divide by δ:

$$\dot{v} = -\frac{1}{2} v u_2, \tag{61}$$

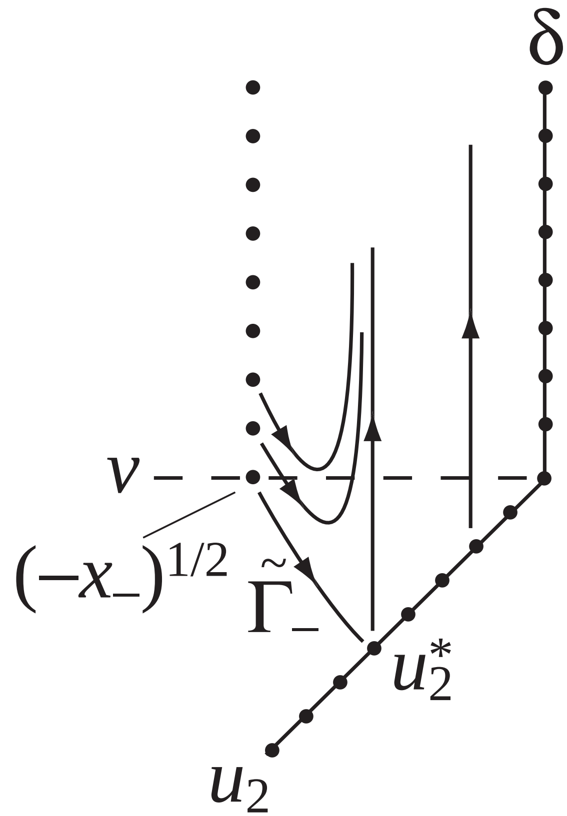
$$\dot{u}_2 = v^2 h'(-v^2), \tag{62}$$

$$\dot{\delta} = \frac{3}{2} \delta u_2. \tag{63}$$

$$\dot{v} = -\frac{1}{2} v u_2,$$
$$\dot{u}_2 = v^2 h'(-v^2),$$
$$\dot{\delta} = \frac{3}{2} \delta u_2.$$



Equilibria on the lines $\{(v, u_2, \delta) : v = (-x_{-})^{\frac{1}{2}}, u_2 = 0\}$ and $\{(v, u_2, \delta) : v = \delta = 0, u_2 \neq 0\}$ are normally hyperbolic within $vu_2\delta$-space, with one positive eigenvalue and one negative eigenvalue.



Lemma. As $\delta_0 \to 0+$, $W^u((-x_-)^{\frac{1}{2}}, 0, \delta_0)$ approaches $W^u(0, u_2^*, 0)$ in the C^1 topology. (Both have dimension 1.) (*Corner Lemma.*)

Lemma. In the chart for $\bar{u}_1 < 0$, as $\delta_0 \to 0+$, $W^u((-x_-)^{\frac{1}{2}}, 0, 0, 0, \delta_0)$ approaches the manifold of unstable fibers over $W^u(0, u_2^*, 0)$ in the C^1 topology. (Both have dimension 2.)

The latter corresponds to $W^{cu}(\hat{q}_1)$ in $S = \{u_2^*\} \times S^3 \times \{0\}$.

Chart for $\bar{u}_1 > 0$:

$$
\begin{aligned}
w_t &= \frac{1}{2} w \gamma u_2, \\
u_{2t} &= w^2 \gamma\left(-\frac{1}{2} w^2 c_3^2 + h'(w^2)\right), \\
c_{3t} &= c_4 - \frac{1}{2} \gamma u_2 c_3, \\
c_{4t} &= c_3^3 - c_3 - \gamma u_2 c_4, \\
\gamma_t &= -\frac{3}{2} \gamma^2 u_2.
\end{aligned}
$$

The equilibria of the plane $c_3 = 1$, $c_4 = \gamma = 0$ have, transverse to the plane, one positive eigenvalue, one negative eigenvalue, one zero eigenvalue.

Therefore this plane is part of a 3-dimensional normally hyperbolic invariant manifold S_2, with equations

$$
c_3 = 1 + \gamma^2 \tilde{c}_3(w, u_2, \gamma), \quad c_4 = \gamma \tilde{c}_4(w, u_2, \gamma).
$$

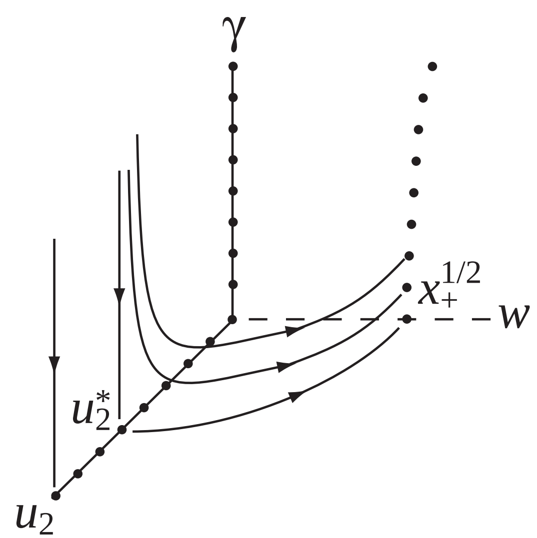
The plane of equilibria corresponds to F_+. Normal hyperbolicity within $\gamma = 0$ is *not* lost at $w = 0$, which corresponds to $u_1 = 0$.

Restrict to S_2 and divide by γ:

$$w_t = \frac{1}{2} w u_2, \tag{64}$$

$$u_{2t} = w^2\left(-\frac{1}{2} w^2 (1+\gamma^2 \tilde{c}_3)^2 + h'(w^2)\right), \tag{65}$$

$$\gamma_t = -\frac{3}{2} \gamma u_2. \tag{66}$$



Lemma. As $\gamma_0 \to 0+$, $W^s(x_+^{\frac{1}{2}}, 0, \gamma_0)$ approaches $W^s(0, u_2^*, 0)$ in the C^1 topology. (Both have dimension 1.)

Lemma. In the chart for $\bar{u}_1 > 0$, as $\gamma_0 \to 0+$, $W^s(x_+^{\frac{1}{2}}, 0, 1, 0, \gamma_0)$ approaches the manifold of stable fibers over $W^s(0, u_2^*, 0)$ in the C^1 topology. (Both have dim 2.)

The latter corresponds to $W^{cs}(\hat{q}_+)$ in $S = \{u_2^*\} \times S^3 \times \{0\}$.

Conclusion: in blow-up space,

As $\varepsilon \to 0+$, $W^u(p_-(\varepsilon))$ approaches $W^{cu}(\hat{q}_-)$ in the C^1 topology.

As $\varepsilon \to 0+$, $W^s(p_+(\varepsilon))$ approaches $W^{cs}(\hat{q}_+)$ in the C^1 topology.

We showed $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ meet transversally within the 3-sphere $\bar{r} = 0$, $u_2 = u_2^*$, **which is** N_0.

In the chart for $\bar{\varepsilon} > 0$, H corresponds to

$$H_b(b_1, u_2, b_3, b_4, r) = \frac{1}{2}u_2^2 + r^4(\frac{1}{2}b_4^2 - \frac{1}{4}b_3^4 + \frac{1}{2}b_1 b_3^2) + h(r^2 b_1).$$

N_0 corresponds to the set of (b_1, u_2, b_3, b_4, r) such that $H_b = 0$ and $r = 0$. The functions H_b and r have linearly independent gradients provided $u_2 \neq 0$. Therefore, where $u_2 \neq 0$, the sets $N_{\varepsilon^{\frac{1}{3}}} = N_r$ depend smoothly on r. Since $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ meet transversally within N_0, it follows that $W^u(p_-(\varepsilon))$ and $W^s(p_+(\varepsilon))$ meet transversally within N_ε for ε small.