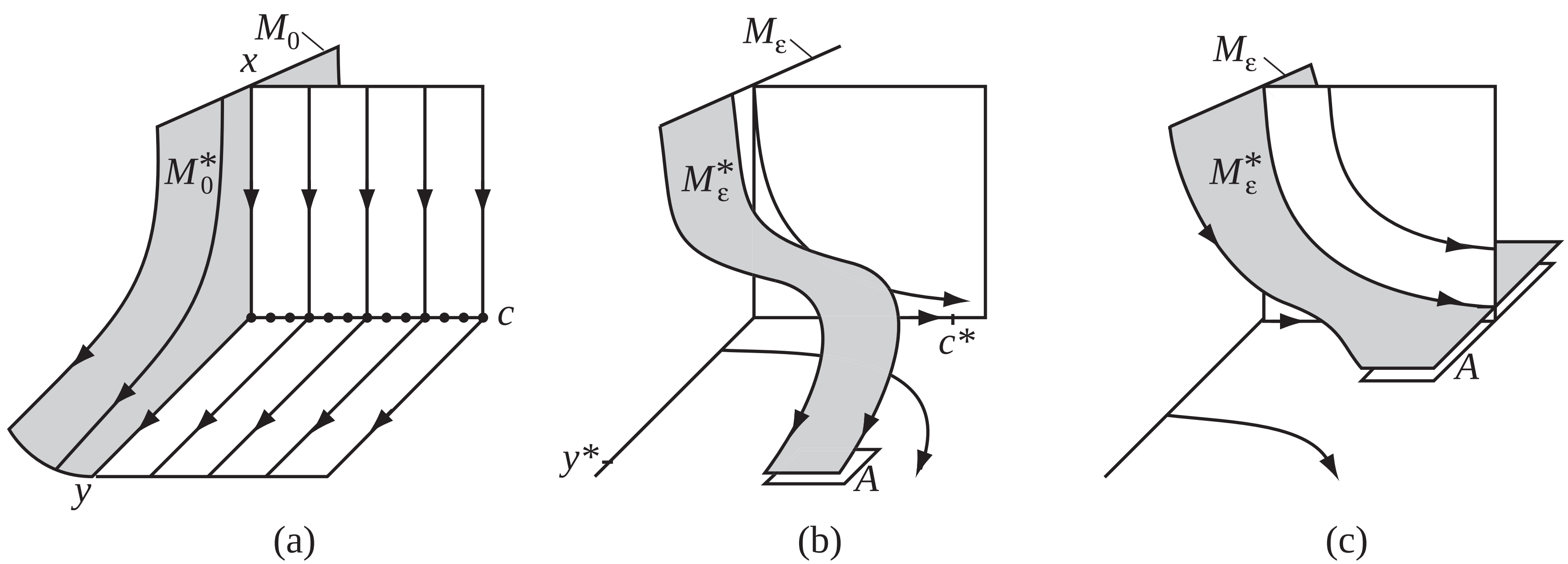
Exchange Lemmas



Steve Schecter

North Carolina State University

Plan

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(3) Exchange Lemma of Jones and Kopell

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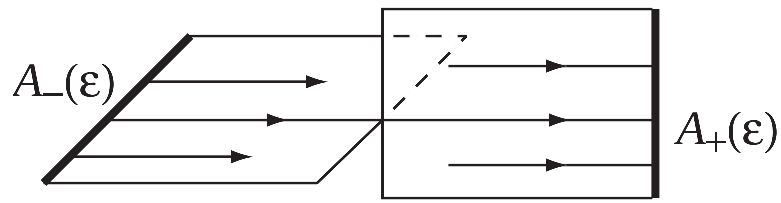
Boundary Value Problems

$$\dot{\xi} = F(\xi,\varepsilon), \quad \xi(t_-) \in A_-(\varepsilon), \quad \xi(t_+) \in A_+(\varepsilon),$$



To show existence of a solution: show that the manifold of solutions that start on $A_-(\varepsilon)$ and the manifold of solutions that end on $A_+(\varepsilon)$ meet transversally.

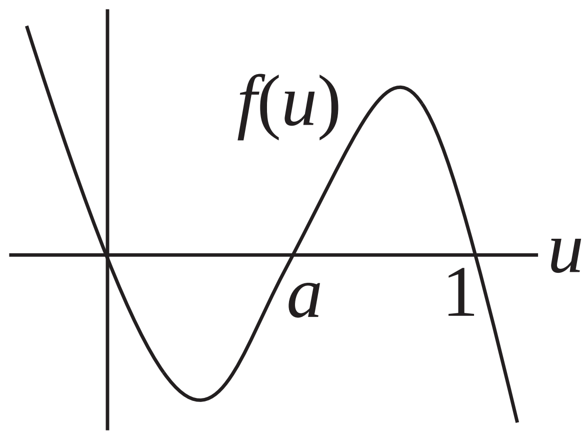
Remarks

- The problem with $\varepsilon = 0$ may be degenerate in some major way .
- Such problems are called *singularly perturbed*.
- The geometric approach to these problems, which focuses on *tracking manifolds of potential solutions* rather than on asymptotic expansions of solutions, is called *geometric singular perturbation theory* (Fenichel, Kopell, Jones, . . .).

Fitzhugh-Nagumo Equation

$$
\begin{aligned}
u_t &= u_{xx} + f(u) - w \\
w_t &= \varepsilon(u - \gamma w)
\end{aligned}
$$

$$
f(u) = u(u-a)(1-u), \quad 0 < a < \frac{1}{2}, \qquad 0 < \gamma, \qquad 0 < \varepsilon << 1.
$$



u = voltage potential across nerve axon membrane.
w = negative feedback effects.
Is there a traveling wave $(u,w)(\xi)$, $\xi = x - ct$ for some velocity c, such that $u(\xi) \to 0$ as $\xi \to \pm\infty$?

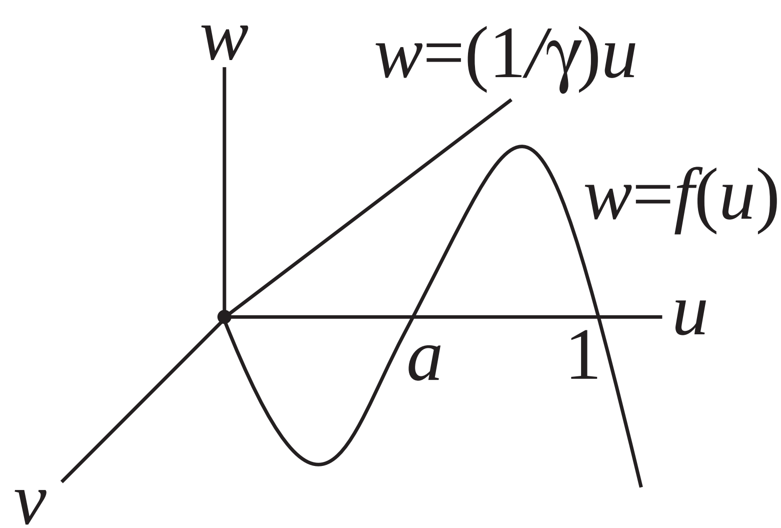
$$
\begin{aligned}
-cu_\xi &= u_{\xi\xi} + f(u) - w \\
-cw_\xi &= \varepsilon(u - \gamma w)
\end{aligned}
$$

$$-cu_\xi = u_{\xi\xi} + f(u) - w$$
$$-cw_\xi = \varepsilon(u - \gamma w)$$

Write as a first-order system, make c a variable:

$$u_\xi = v$$
$$v_\xi = -cv - f(u) + w$$
$$w_\xi = \frac{\varepsilon}{c}(\gamma w - u)$$
$$c_\xi = 0$$

Slow-fast system, 2 slow variables, 2 fast variables. Equilibria for $\varepsilon > 0$: $v = 0$, $w = f(u)$, $w = \frac{1}{\gamma}u$. For γ small there is just the origin for each c:

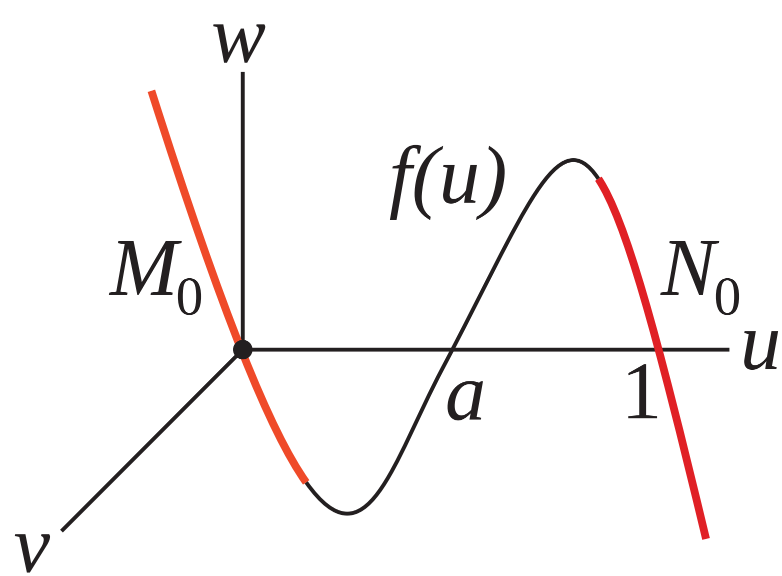


Origin has 1 negative eigenvalue and 2 eigenvalues with positive real part.

Set $\varepsilon = 0$:

$$\begin{aligned} u_\xi &= v \\ v_\xi &= -cv - f(u) + w \\ w_\xi &= 0 \\ c_\xi &= 0 \end{aligned}$$

Normally hyperbolic manifolds of equilibria, 1 positive eigenvalue, 1 negative eigenvalue:



M_0 and N_0 are actually 2-dimensional. Both are given by

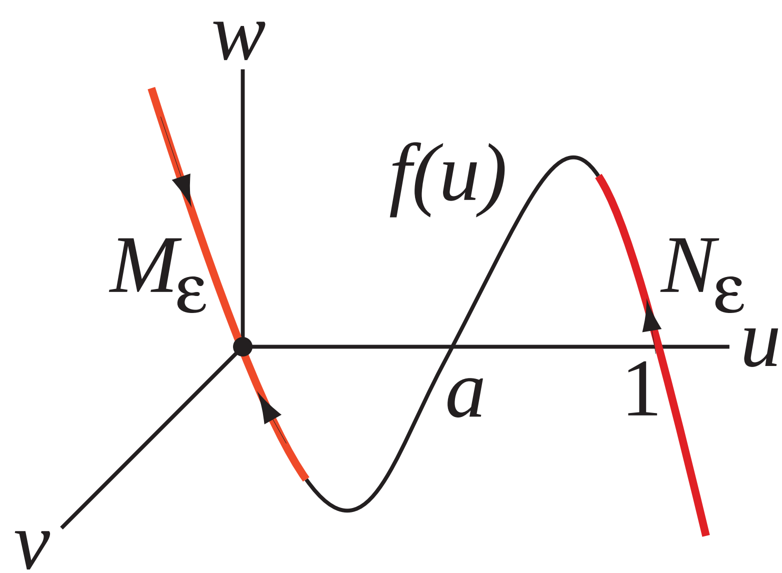
$$u = \text{“}f^{-1}(w)\text{”}, \quad v = 0, \quad c = \text{ arbitrary.}$$

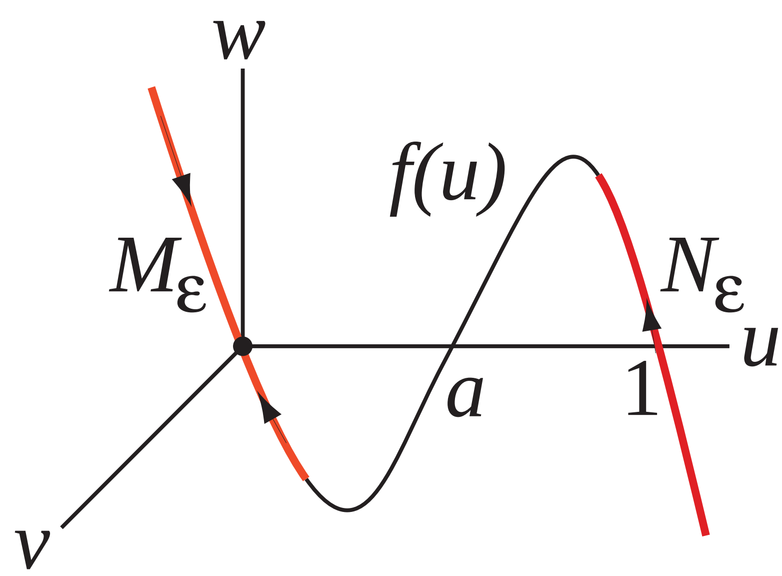
$$
\begin{aligned}
u_\xi &= v \\
v_\xi &= -cv - f(u) + w \\
w_\xi &= \frac{\varepsilon}{c}(\gamma w - u) \\
c_\xi &= 0
\end{aligned}
$$

For $\varepsilon > 0$, M_0 and N_0 persist as normally hyperbolic invariant manifolds. Differential equation on them:

$$
\begin{aligned}
w_\xi &= \frac{\varepsilon}{c}(\gamma w - f^{-1}(w)) + O(\varepsilon^2) \\
c_\xi &= 0
\end{aligned}
$$

For $c < 0$ and γ small:





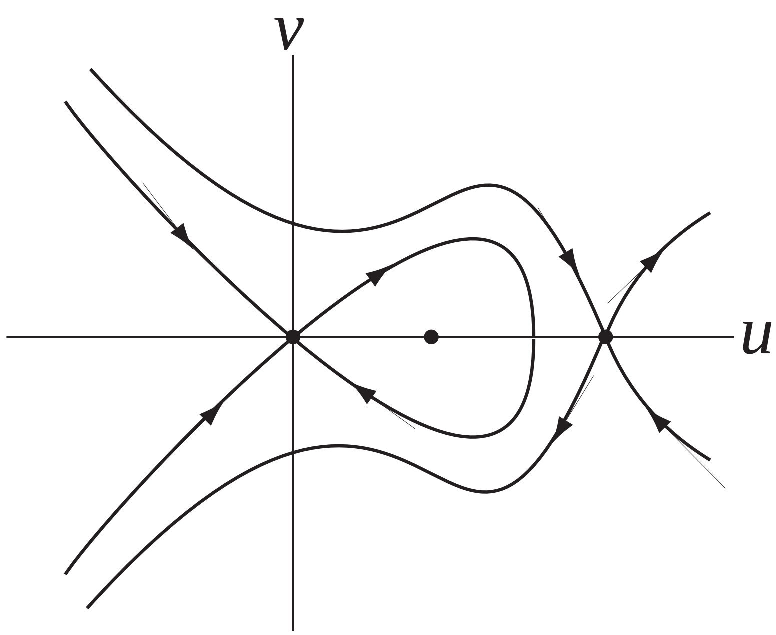
Let $J = \{(0,0,0,c) : c \text{ arbitrary}\}$. $J \subset M_\varepsilon$ for all ε. Look for a solution that at time t_- is in $W^u_\varepsilon(J)$ (dimension = 2) and at time t_+ is in $W^s(M_\varepsilon)$ (dimension=3).

$W^u_\varepsilon(J)$ exists and depends smoothly on ε from the theory of normally hyperbolic invariant manifolds.

Back to $\varepsilon = 0$:

$$\begin{aligned} u_\xi &= v \\ v_\xi &= -cv - f(u) + w \\ w_\xi &= 0 \\ c_\xi &= 0 \end{aligned}$$

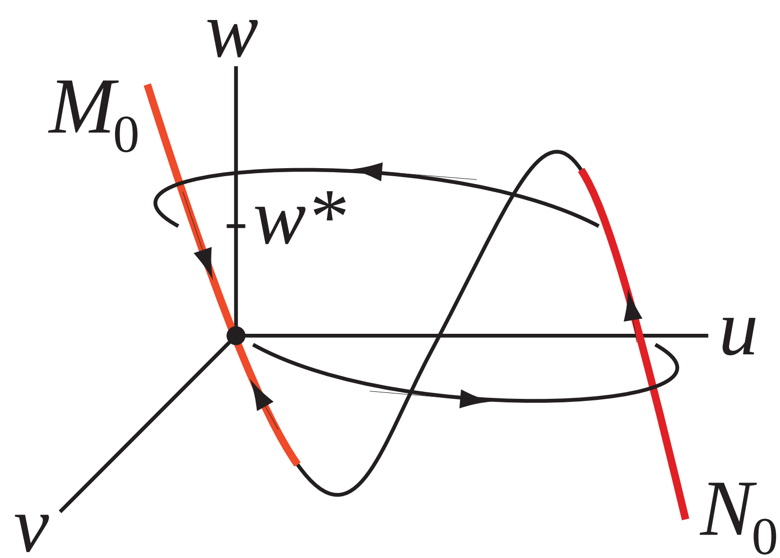
Phase portrait for $(w,c) = (0,0)$



For $(w,c) = (0,c^*)$, $c^* < 0$, $(0,0)$ connects to $(1,0)$

For $(w,c) = (w^*,c^*)$, $w^* > 0$, right equilibrium connects to left equilibrium.

Slow and fast orbits, $c = c^*$:



Is there a true homoclinic orbit nearby for $\varepsilon > 0$?

1. For $\varepsilon = 0$: $W^u_0(J)$ (dimension = 2) intersects $W^s(N_0)$ (dimension = 3) transversally in an orbit with $c = c^*$.

2. Therefore for ε small, $W^u_\varepsilon(J)$ intersects $W^s(N_\varepsilon)$ transversally in an orbit with $c = c(\varepsilon)$, $c(0) = c^*$.

3. For $\varepsilon = 0$: $W^u(N_0 \cap \{c = c^*\}$ (dimension = 2) intersects $W^s(M_0)$ (dimension = 3) transversally in an orbit with $w = w^*$.

4. Therefore for ε small, $W^u(N_\varepsilon \cap \{c = c(\varepsilon)\}$ intersects $W^s(M_\varepsilon)$ transversally in an orbit with “w near w^*.”

5. For ε small, does $W^u_\varepsilon(J)$ become close to $W^u(N_\varepsilon \cap \{c = c(\varepsilon)\}$? If so we are done.

Exchange Lemma of Jones and Kopell

Slow-Fast Systems

$$\dot{a} = f(a,b,\varepsilon), \quad \dot{b} = \varepsilon g(a,b,\varepsilon), \quad (a,b) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Set $\varepsilon = 0$:

$$\dot{a} = f(a,b,0), \quad \dot{b} = 0.$$

Assume:

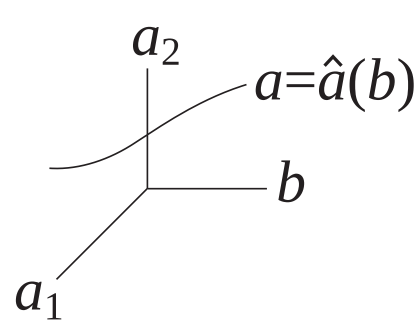
(1) $f(\hat{a}(b),b,0) = 0$.

(2) $D_a f(\hat{a}(b),b,0)$ has

- k eigenvalues with negative real part.
- l eigenvalues with positive real part.
- $k + l = n$.

(3) $g(\hat{a}(b),b,0) \neq 0$.

(1) and (2) say the m-dimensional manifold $a = \hat{a}(b)$ is a normally hyperbolic manifold of equlibria for $\varepsilon = 0$.

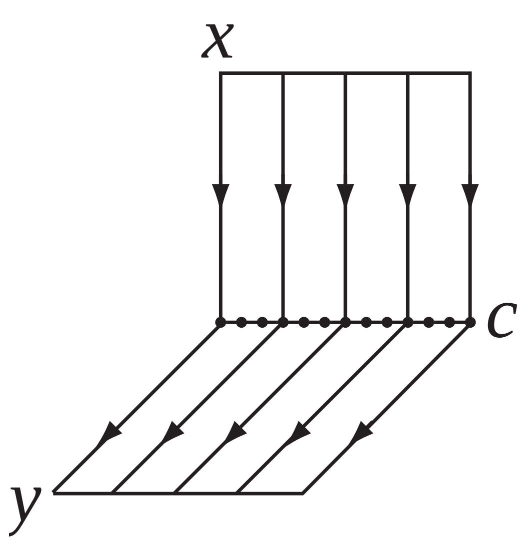


After a change of coordinates:

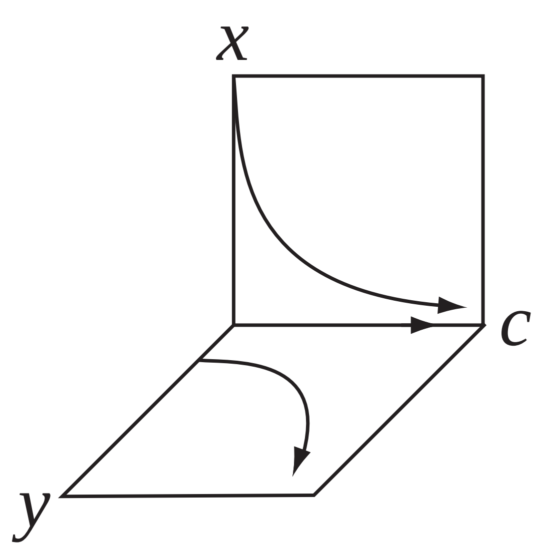
$$\begin{aligned}\dot{x} &= A(x,y,c,\varepsilon)x,\\ \dot{y} &= B(x,y,c,\varepsilon)y,\\ \dot{c} &= \varepsilon((1,0,\dots,0)+L(x,y,c,\varepsilon)xy),\end{aligned}$$

$(x,y,c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$ (contracting $\times$ expanding $\times$ center),
$A(0,0,c,0)$ has eigenvalues with negative real part,
$B(0,0,c,0)$ has eigenvalues with positive real part.

Note that on xc-space and yc-space, $\dot{c}$ depends on only on c and ε. Gives “fast foliation.”



Flow with $\varepsilon = 0$.



Flow with $\varepsilon > 0$.

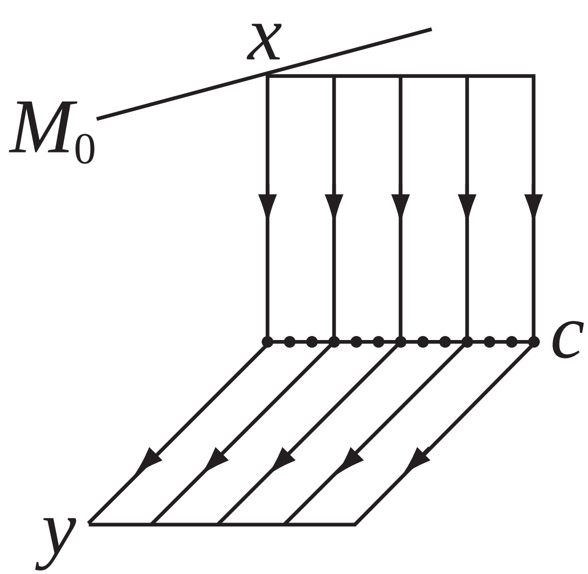
Exchange Lemma of Jones and Kopell

Assume:

(1) For each ε, M_ε is a submanifold of xyc-space of dimension l.

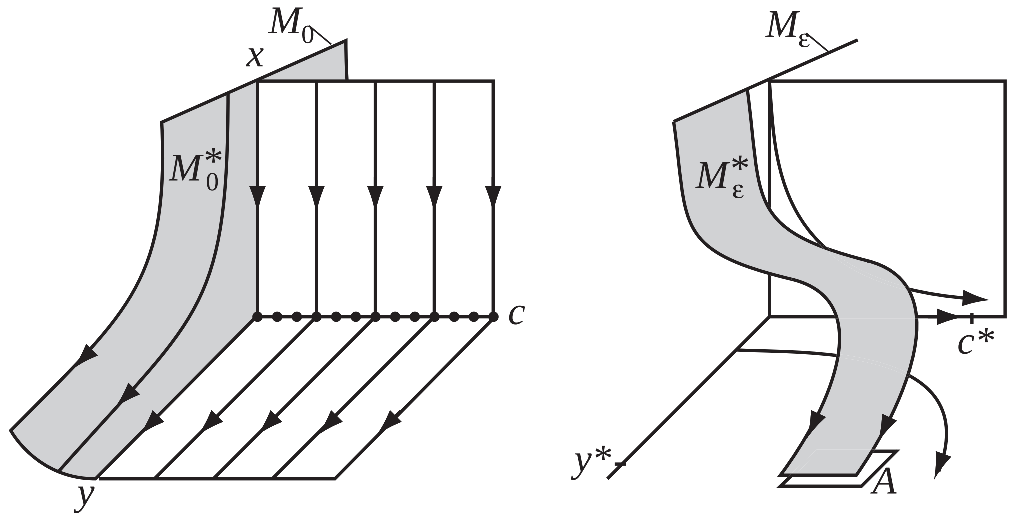
(2) $M = \{(x,y,c,\varepsilon) : (x,y,c) \in M_\varepsilon\}$ is itself a manifold.

(3) For each ε, M_ε meets xc-space transversally in a point $(x(\varepsilon),0,0)$.



$$\varepsilon = 0$$

Under the forward flow, each M_ε becomes a manifold M_ε^* of dimension $l+1$.



Theorem 1 (Exchange Lemma of Jones and Kopell, 1994). Let

$$c^* = (c_1^*, 0, \ldots, 0) \in \mathbb{R}^m$$

with $0 < c_1^*$. Let $y^* \neq 0$. Let A be a small neighborhood of (y^*, c_1^*) in yc_1-space. Then for small $\varepsilon_0 > 0$ there are smooth functions $\tilde{x} : A \times [0, \varepsilon_0) \to \mathbb{R}^k$ and $\tilde{c} : A \times [0, \varepsilon_0) \to \mathbb{R}^{m-1}$such that:

(1) $\tilde{x}(y, c_1, 0) = 0$.

(2) $\tilde{c}(y, c_1, 0) = \tilde{c}(0, c_1, \varepsilon) = 0$.

(3) As $\varepsilon \to 0$, $(\tilde{x}, \tilde{c}) \to 0$ exponentially, along with its derivatives with respect to all variables.

(4) For $0 < \varepsilon < \varepsilon_0$, $\{(x, y, c_1, \ldots, c_m) : (y, c_1) \in A,\, x = \tilde{x}(y, c_1, \varepsilon),$ and $(c_2, \ldots, c_m) = \tilde{c}(y, c_1, \varepsilon)\}$ is contained in M_ε^*.

Transversality to xc-space is “exchanged” for closeness to yc-space.

Brunovsky's Reformulation of Jones and Kopell's Exchange Lemma as an Inclination Lemma

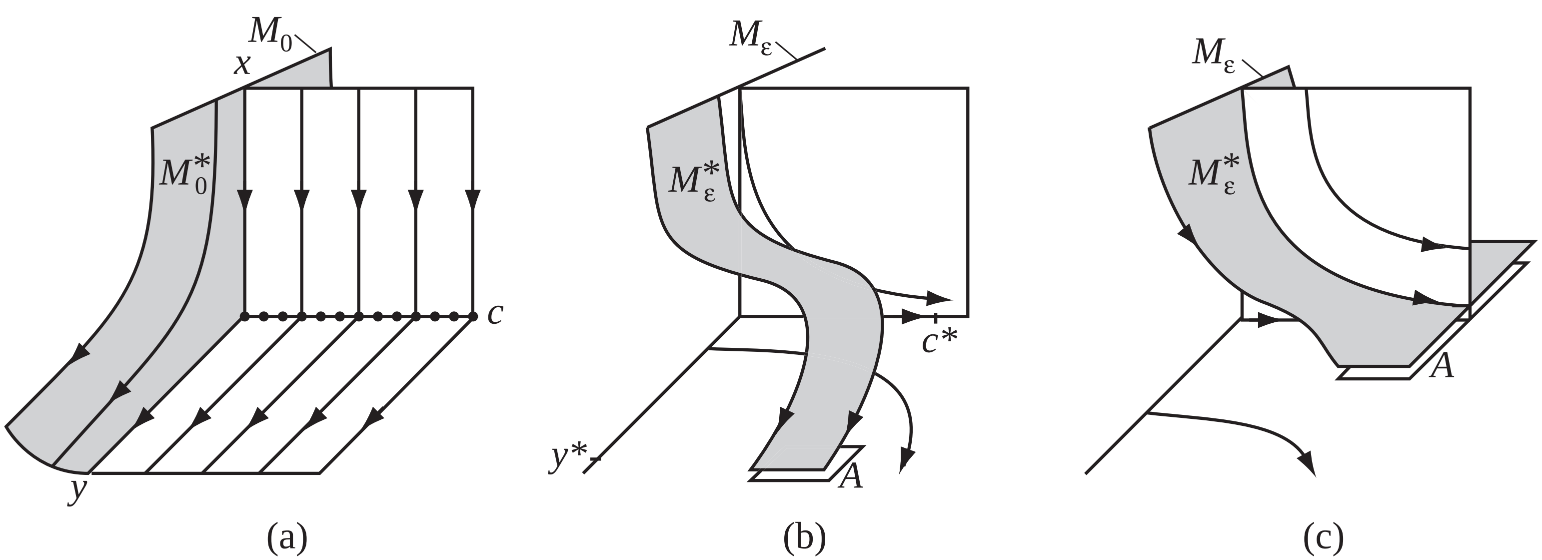
Theorem 2 (1999). Let $c^* = (c_1^*, 0, \ldots, 0) \in \mathbb{R}^m$ with $0 < c_1^*$. Let A be a small neighborhood of $(0, c_1^*)$ in yc_1-space. Then for small $\varepsilon_0 > 0$ there are smooth functions $\tilde{x} : A \times [0, \varepsilon_0) \to \mathbb{R}^k$ and $\tilde{c} : A \times [0, \varepsilon_0) \to \mathbb{R}^{m-1}$such that:

(1) $\tilde{x}(y, c_1, 0) = 0$.

(2) $\tilde{c}(y, c_1, 0) = \tilde{c}(0, c_1, \varepsilon) = 0$.

(3) As $\varepsilon \to 0$, $(\tilde{x}, \tilde{c}) \to 0$ exponentially, along with its derivatives with respect to all variables.

(4) For $0 < \varepsilon < \varepsilon_0$, $\{(x, y, c_1, \ldots, c_m) : (y, c_1) \in A, x = \tilde{x}(y, c_1, \varepsilon),$ and $(c_2, \ldots, c_m) = \tilde{c}(y, c_1, \varepsilon)\}$ is contained in M_ε^*.



Exchange Lemma of Jones and Tin

Consider again:

$$
\begin{aligned}
\dot{x} &= A(x,y,c,\varepsilon)x, \\
\dot{y} &= B(x,y,c,\varepsilon)y, \\
\dot{c} &= \varepsilon((1,0,\ldots,0)+L(x,y,c,\varepsilon)xy),
\end{aligned}
$$

$(x,y,c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$ (contracting $\times$ expanding $\times$ center),
$A(0,0,c,0)$ has eigenvalues with negative real part,
$B(0,0,c,0)$ has eigenvalues with positive real part.

Assume:

(1) For each ε, M_ε is a submanifold of xyc-space of dimension $l+p$, $0 \le p \le m-1$.

(2) $M = \{(x,y,c,\varepsilon) : (x,y,c) \in M_\varepsilon\}$ is itself a manifold.

(3) M_0 meets xc-space transversally in a manifold N_0 of dimension p.

(4) N_0 projects smoothly to a submanifold P_0 of c-space of dimension p.

(5) The vector $(1,0,\ldots,0)$ is not tangent to P_0.

Then:

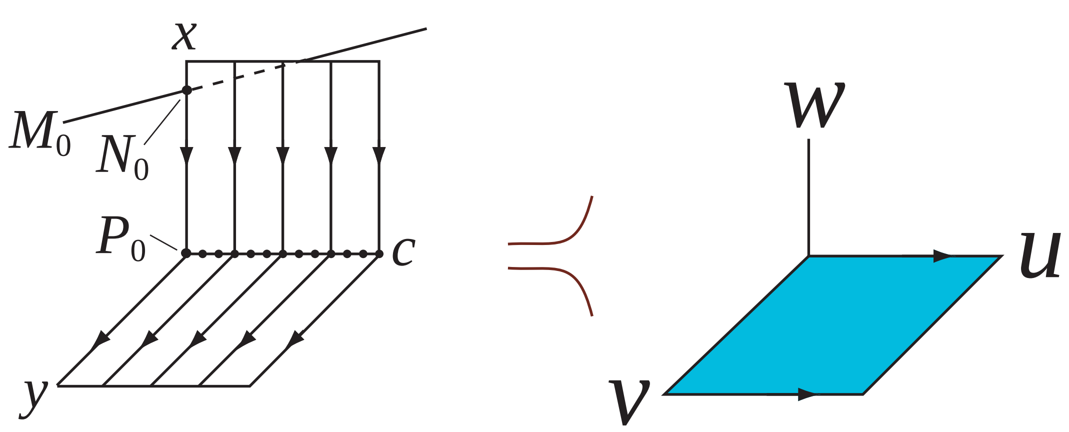
(1) Each M_ε meets xc-space transversally in a manifold N_ε of dimension p.

(2) N_ε projects smoothly to a submanifold P_ε of c-space of dimension p.

(3) The vector $(1, 0, \ldots, 0)$ is not tangent to P_ε.

After a change of coordinates $c = (u, v, w) \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^{m-1-p}$ that takes each P_ε to v-space, the system can be put in the form

$$\begin{aligned}
\dot{x} &= A(x, y, u, v, w, \varepsilon)x, \\
\dot{y} &= B(x, y, u, v, w, \varepsilon)y, \\
\dot{u} &= \varepsilon(1 + e(x, y, u, v, w, \varepsilon)xy), \\
\dot{v} &= \varepsilon F(x, y, u, v, w, \varepsilon)xy, \\
\dot{w} &= \varepsilon G(x, y, u, v, w, \varepsilon)xy.
\end{aligned}$$



Under the forward flow, each M_ε becomes a manifold M_ε^* of dimension $l+p+1$. Each P_ε becomes a manifold P_ε^* of dimension $p+1$, which in our coordinates is just uv-space.

Theorem 3 (Exchange Lemma of Jones and Tin, 2009). Let $0 < u^*$. Let A be a small neighborhood of $(0,u^*,0)$ in yuv-space. Then for small $\varepsilon_0 > 0$ there are smooth function $\tilde{x} : A \times [0,\varepsilon_0) \to \mathbb{R}^k$ and $\tilde{w} : A \times [0,\varepsilon_0) \to \mathbb{R}^{m-p-1}$such that:

(1) $\tilde{x}(y,u,v,0) = 0$.

(2) $\tilde{w}(y,u,v,0) = \tilde{w}(0,u,v,\varepsilon) = 0$.

(3) As $\varepsilon \to 0$, $(\tilde{x},\tilde{w}) \to 0$ exponentially, along with its derivatives with respect to all variables.

(4) For $0 < \varepsilon < \varepsilon_0$, $\{(x,y,u,v,w) : (y,u,v) \in A$ and $(x,w) = (\tilde{x},\tilde{w})(y,u,v,\varepsilon)\}$ is contained in M_ε^*.

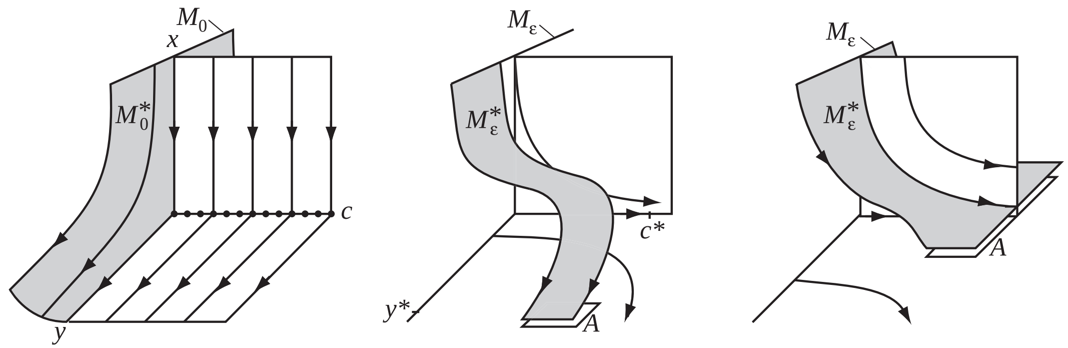
Remark

The theorem also applies to

$$
\begin{aligned}
\dot{x} &= A(x,y,c,\varepsilon)x,\\
\dot{y} &= B(x,y,c,\varepsilon)y,\\
\dot{c} &= \varepsilon(1,0,\ldots,0) + L(x,y,c,\varepsilon)xy,
\end{aligned}
$$

It is really about perturbations of systems with a family of normally hyperbolic equilibria, not about slow-fast systems.

General Exchange Lemma



(a) (b) (c)

Important Features of the Exchange Lemma

(1) There is a normally hyperbolic invariant manifold (c-space) and a small parameter ε.

(2) There is a collection of submanifolds M_ε of xyc-space such that $M = \{(x,y,c,\varepsilon) : (x,y,c) \in M_\varepsilon\}$ is itself a manifold. M_ε meets xc-space transversally in a manifold N_ε (in picture, a point).

(3) N_ε projects along the stable fibration of xc space to a submanifold P_ε of c-space of the same dimension (in picture, a point).

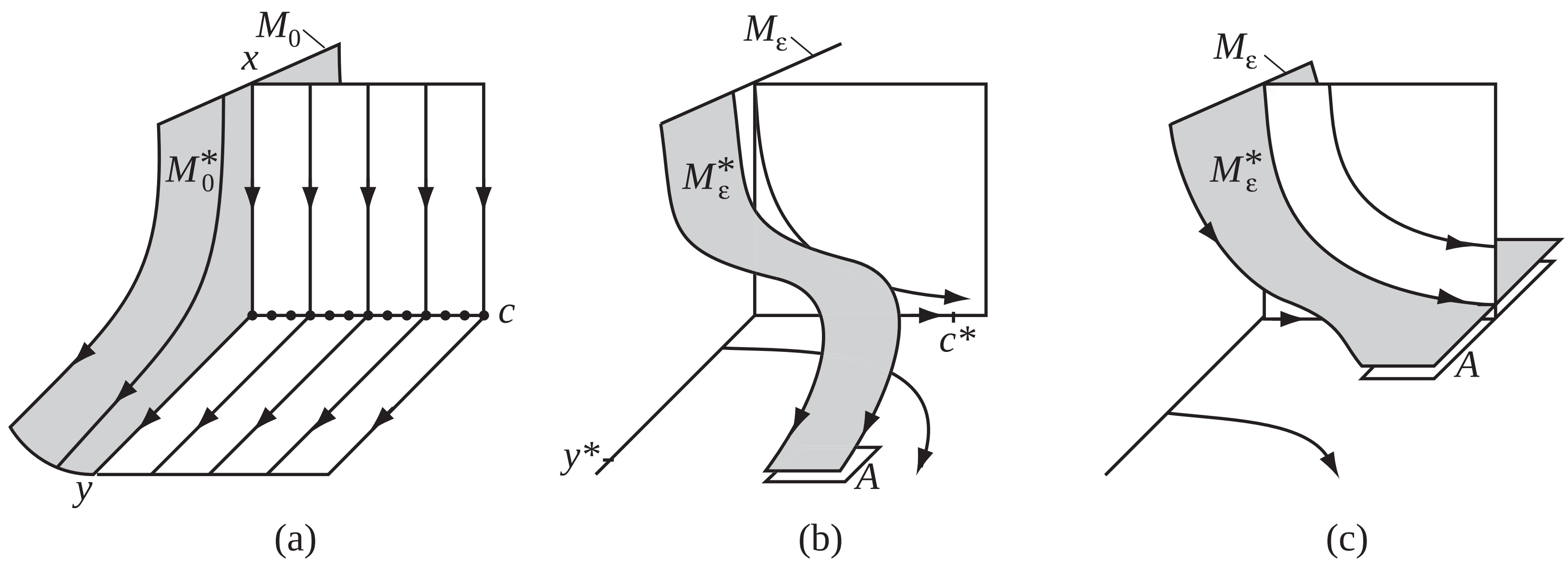
(4) For $\varepsilon > 0$, the vector field is not tangent to P_ε.

(5) For small $\varepsilon > 0$, the flow on c-space is followed for a long time.

(6) It takes P_ε to a set P_ε^* of dimension one greater. As $\varepsilon \to 0$, the limit of $P_\varepsilon^* \neq$ where the limiting DE takes P_0. Nevertheless, the limit of P_ε^* exists and has the same dimension. Call it P_0^*.

(7) As $\varepsilon \to 0$, $M_\varepsilon^* \to W^u(P_0^*)$.

General Exchange Lemma (S., 2007). (1)–(6) plus technical assumptions imply (7).



What's the point?

- To understand the flow on the normally hyperbolic invariant manifold for $\varepsilon > 0$ may require rectification, blowing-up, etc.

- Once you've done this work, the General Exchange Lemma deals with the remaining dimensions.

Loss-of-Stability Turning Points: Liu's Exchange Lemma

Liu considers a slow-fast system

$$\dot{a} = f(a,b,\varepsilon),$$
$$\dot{b} = \varepsilon g(a,b,\varepsilon),$$

with $a \in \mathbb{R}^{k+l+1}$ and $b \in \mathbb{R}^{m-1}$, $m \geq 2$. Assume:

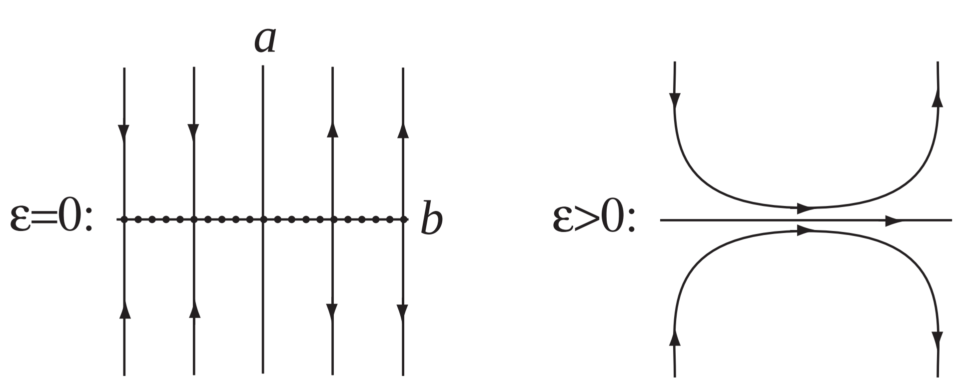
(1) $f(0,b,\varepsilon) = 0$. (Hence for each ε, b-space is invariant, and for $\varepsilon = 0$ it consists of equilibria.)

(2) $D_a f(0,b,0)$ has

- k eigenvalues with negative real part;
- l eigenvalues with positive real part;
- a last eigenvalue $\nu(b)$ such that $\nu(0) = 0$.

(3) $D\nu(0)g(0,0,0) > 0$.

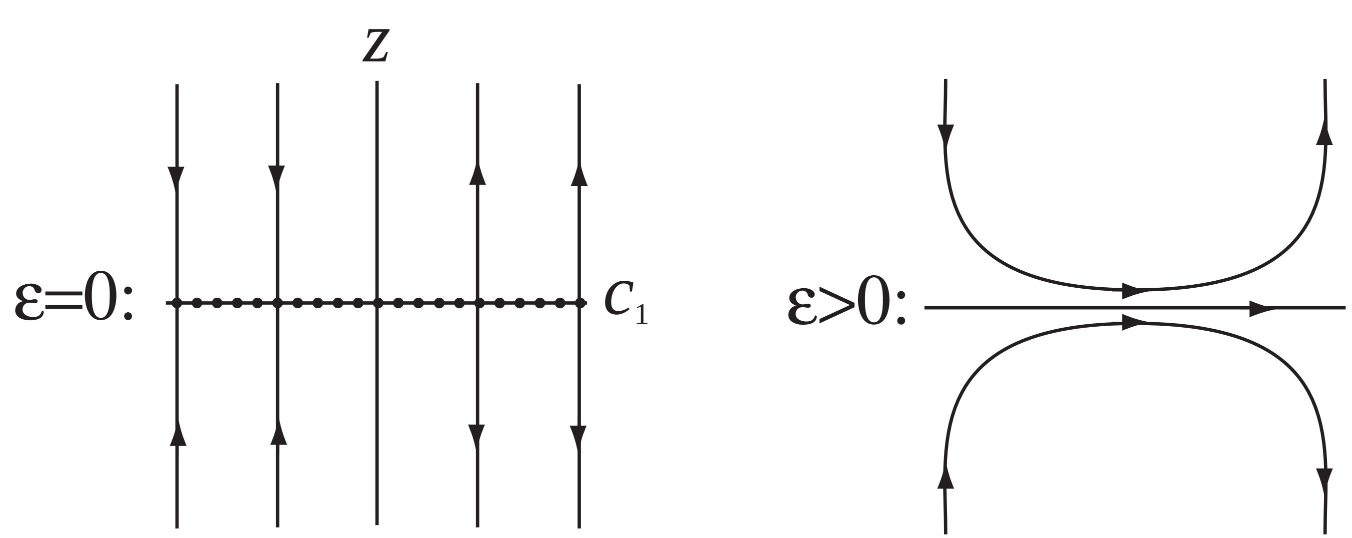
$k+l=0$, $m=2$



After a change of coordinates:

$$
\begin{aligned}
\dot{x} &= A(x,y,z,c,\varepsilon)x,\\
\dot{y} &= B(x,y,z,c,\varepsilon)y,\\
\dot{z} &= h(z,c,\varepsilon)z + k(x,y,z,c,\varepsilon)xy,\\
\dot{c} &= \varepsilon((1,0,\dots,0) + l(z,c,\varepsilon)z + L(x,y,z,c,\varepsilon)xy),
\end{aligned}
$$

$$
\begin{aligned}
(x,y,z,c) &\in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}^{m-1},\\
A(0,0,0,c,0) &\text{ has eigenvalues with negative real part,}\\
B(0,0,0,c,0) &\text{ has eigenvalues with positive real part,}\\
h(0,(0,c_2,\dots,c_{m-1}),0) &= 0,\\
\frac{\partial h}{\partial c_1} &> 0.
\end{aligned}
$$



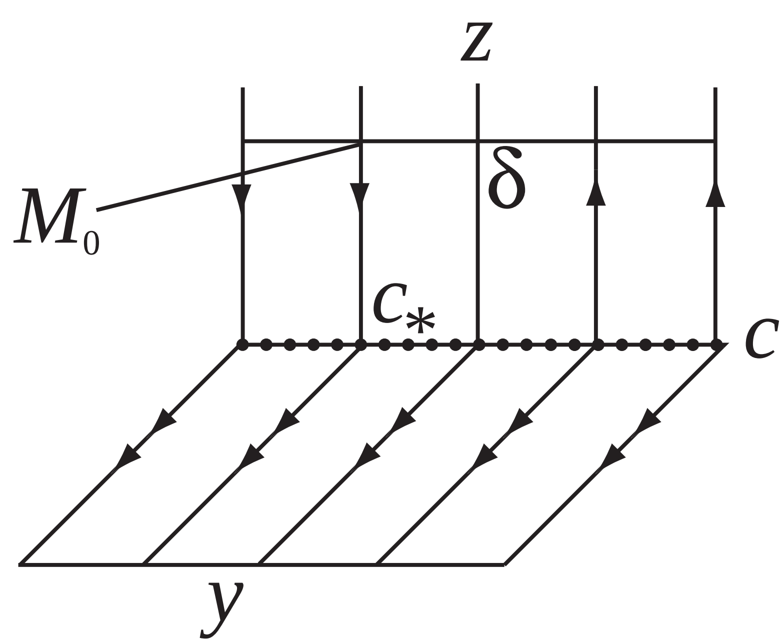
Assume:

(1) $m = 2$ (for simplicity, so c-space is one-dimensional).

(2) For each ε, M_ε is a submanifold of $xyzc$-space of dimension l.

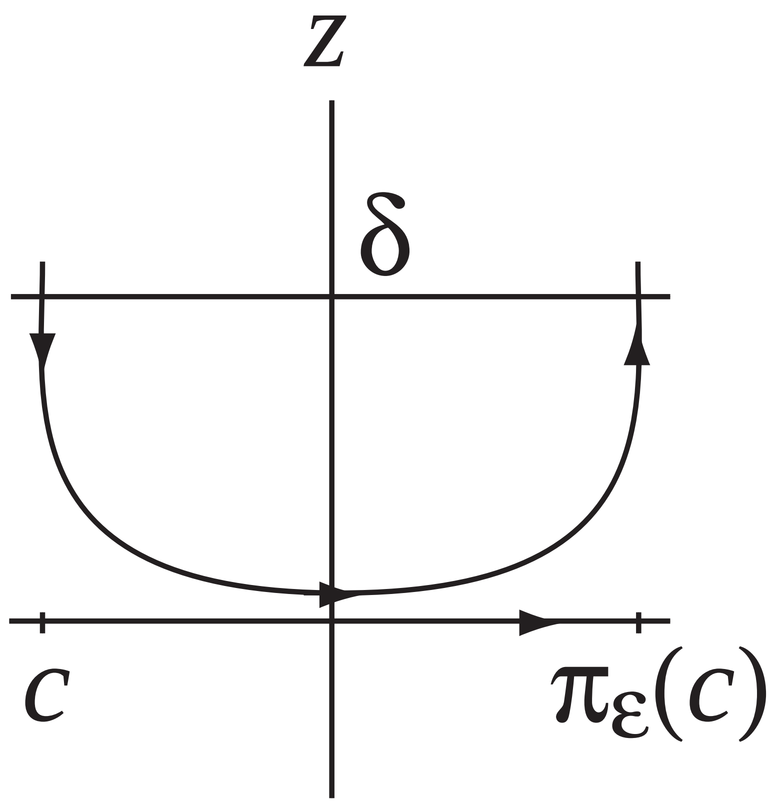
(3) $M = \{(x,y,z,c,\varepsilon) : (x,y,z,c) \in M_\varepsilon\}$ is itself a manifold.

(4) M_0 meets xzc-space transversally at a point $(x_*, 0, \delta, c_*)$ with $\delta \neq 0$ and $c_* < 0$. We may assume that $M \subset \{(x,y,z,c,\varepsilon) : z = \delta\}$.



Each M_ε meets xzc-space transversally at $(x,y,z,c) = (x(\varepsilon), 0, c(\varepsilon), \delta)$ with $(x(0), c(0)) = (x_*, c_*)$.

For $\varepsilon > 0$ define Poincaré maps on $z = \delta$ by $c \to \pi_\varepsilon(c)$.



Define π_0 implicitly by

$$\int_c^{\pi_0(c)} h(0,u,0)\,du = 0.$$

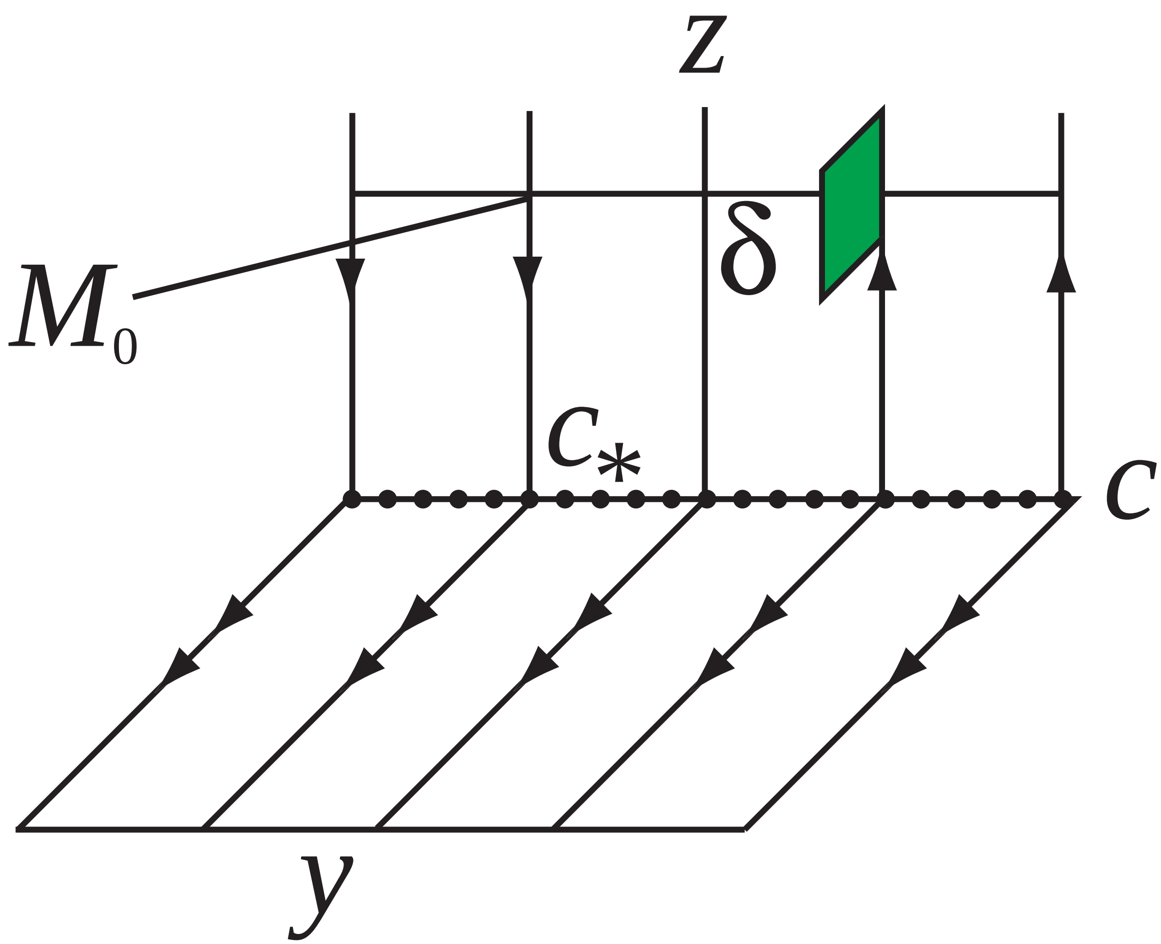
$\pi_\varepsilon \to \pi_0$, along with its derivatives, as $\varepsilon \to 0$ (De Maesschalck, 2008).

Under the forward flow, each M_ε becomes a manifold M_ε^* of dimension $l+1$.

Theorem 4 (Liu's Exchange Lemma, 2000). In zc-space, consider a short integral curve C_ε through $(z,c) = (\delta, \pi_\varepsilon(c(\varepsilon)))$. Let

$$A_\varepsilon = \{(x,y,z,c) : x = 0, \|y\| \text{ is small, } (z,c) \in C_\varepsilon\}.$$

Then M_ε^* is close to A_ε. As $\varepsilon \to 0$ the distance goes to 0 exponentially.



Gain-of-Stability Turning Points

(Rarefactions in the Dafermos Regularization)

Consider the system

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= (A(u) - xI)v,\\ \dot{x} &= \varepsilon,\end{aligned}$$

with $(u,v,x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $A(u)$ an $n \times n$ matrix.

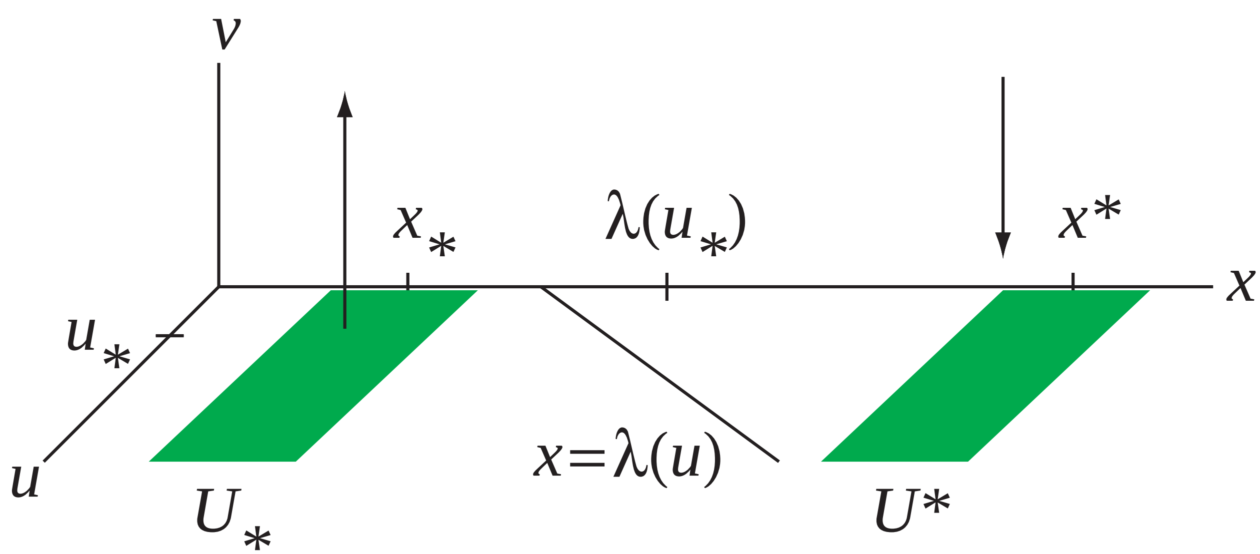
Let $n = k + l + 1$. Assume that on an open set U in $\mathbb{R}^n$:

- There are numbers $\lambda_1 < \lambda_2$ such that $A(u)$ has
  - k eigenvalues with real part less than λ_1,
  - l eigenvalues with real part greater than λ_2,
  - a simple real eigenvalue $\lambda(u)$ with $\lambda_1 < \lambda(u) < \lambda_2$.
- $A(u)$ has an eigenvector $r(u)$ for the eigenvalue $\lambda(u)$ such that $D\lambda(u)r(u) = 1$.

Notice ux-space is invariant for every ε. For $\varepsilon = 0$ it consists of equilibria, but loses normal hyperbolicity along the surface $x = \lambda(u)$.

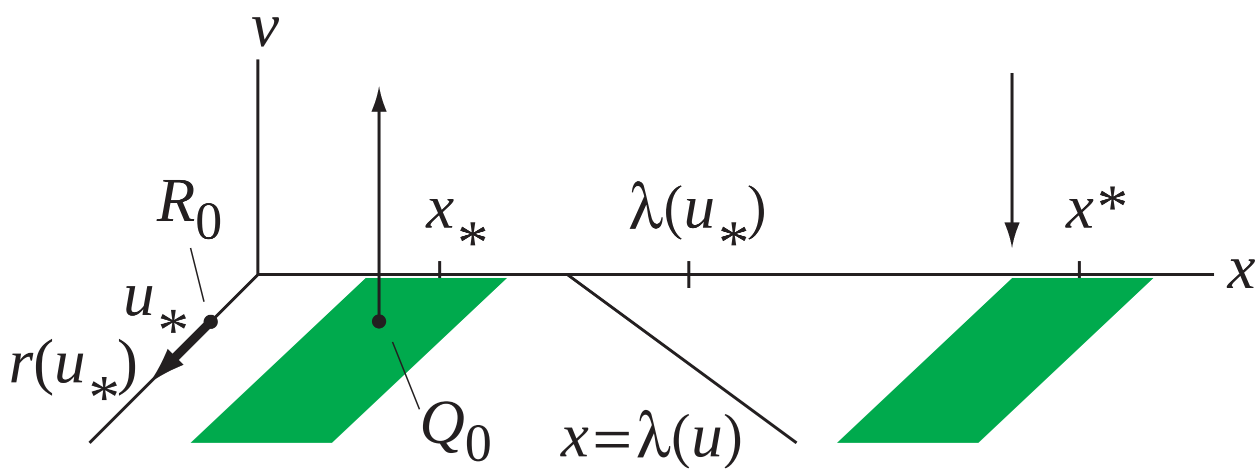
Choose $u_* \in U$, x_*, x^* such that $\lambda_1 < x_* < \lambda(u_*) < x^* < \lambda_2$. Let

$$U_* = \{(u,v,x) : u \in U, v = 0, |x - x_*| < \delta\},$$
$$U^* = \{(u,v,x) : u \in U, v = 0, |x - x^*| < \delta\}.$$



For $\varepsilon = 0$, U_* and U^* are normally hyperbolic manifolds of equilibria of dimension $n+1$. For U_*, the stable and unstable manifolds of each point have dimensions k and $l+1$ respectively; for U^*, the stable and unstable manifolds of each point have dimensions $k+1$ and l respectively.

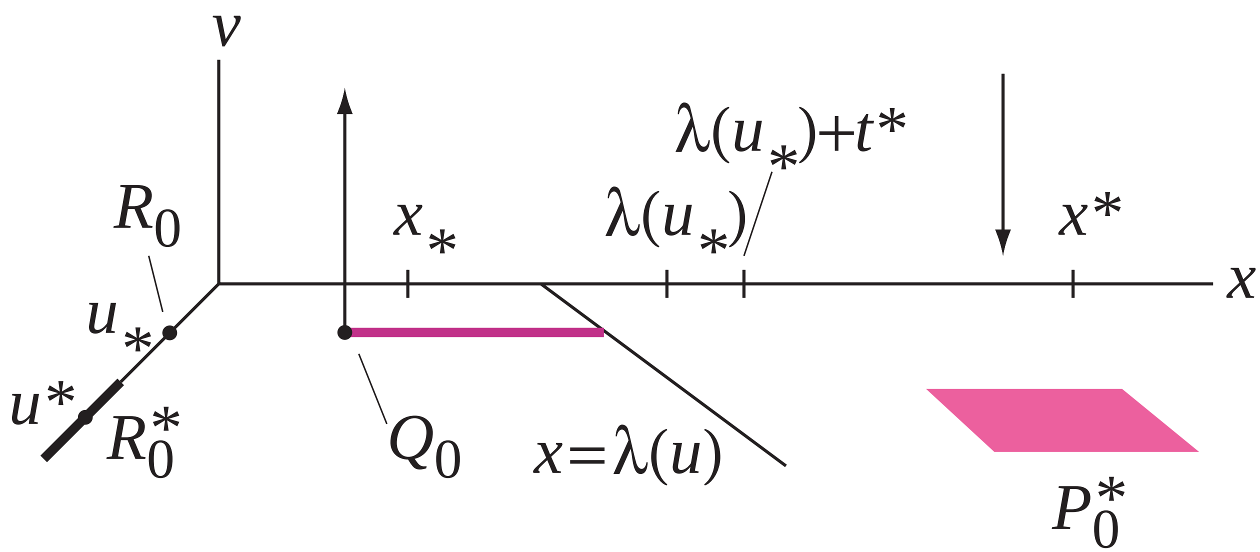
For $\varepsilon > 0$, U_* and U^* are normally hyperbolic invariant manifolds on which the system reduces to $\dot{u} = 0$, $\dot{x} = \varepsilon$.



For each $\varepsilon \geq 0$, let M_ε be a submanifold of uvx-space of dimension $l+1+p$, $0 \leq p \leq n-1$. Assume:

- $M = \{(u,v,x,\varepsilon) : (u,v,x) \in M_\varepsilon\}$ is itself a manifold.
- M_0 is transverse to $W_0^s(U_*)$ at a point in the stable fiber of $(u_*,0,x_*)$. The intersection of M_0 and $W_0^s(U_*)$ is a smooth manifold S_0 of dimension p.
- S_0 projects smoothly to a submanifold Q_0 of ux-space of dimension p.
- The vector $(\dot{u},\dot{x}) = (0,1)$ is not tangent to Q_0. Therefore Q_0 projects smoothly to a submanifold R_0 of u-space of dimension p.
- $r(u_*)$ is not tangent to R_0.

Under the flow, each M_ε becomes a manifold M_ε^* of dimension $l+2+p$.



Let $\phi(t,u)$ be the flow of $\dot{u}=r(u)$. Choose $t^*>0$ such that $\lambda(u_*)+t^*<x^*$. Let

$$R_0^*=\cup_{|t-t^*|<\delta}\phi(t,R_0),\quad P_0^*=\{(u,v,x):u\in R_0^*,v=0,|x-x^*|<\delta\}.$$

R_0^* and P_0^* have dimensions $p+1$ and $p+2$ respectively.

Let $u^*=\phi(t^*,u_*)$.

Theorem 7. Near $(u^*,0,x^*)$, M_ε^* is close to $W_0^u(P_0^*)$.

Generalized Deng's Lemma

In the literature, there are three ways to prove exchange lemmas:

- Jones and Kopell's approach, which is to follow the tangent space to M_ε forward using the extension of the linearized differential equation to differential forms.
- Brunovsky's approach, which is to locate M_ε^* by solving a boundary value problem in Silnikov variables.
- Krupa–Sandstede–Szmolyan approach (1997), using Lin's method.

We follow Brunovsky's approach, which is based on work of Bo Deng (1990). Brunovsky generalized a lemma of Deng that gives estimates on solutions of boundary value problems in Silnikov variables. Our proof of the Generalized Exchange Lemma is based on a further generalization of Deng's Lemma.

Let $(x,y,c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$. Let V be an open subset of $\mathbb{R}^m$. On a neighborhood of $\{0\} \times \{0\} \times V$, consider the C^{r+1} differential equation

$$\begin{aligned} \dot{x} &= A(x,y,c)x, \\ \dot{y} &= B(x,y,c)y, \\ \dot{c} &= C(c) + E(x,y,c)xy. \end{aligned}$$

Let $\phi(t,c)$ be the flow of $\dot{c} = C(c)$. For each $c \in V$ there is a maximal interval I_c containing 0 such that $\phi(t,c) \in V$ for all $t \in I_c$. Let the linearized solution operator of the system, with $\varepsilon = 0$, along the solution $(0,0,\phi(t,c^0))$ be

$$\begin{pmatrix} \overline{x}(t) \\ \overline{y}(t) \\ \overline{c}(t) \end{pmatrix} = \begin{pmatrix} \Phi^s(t,s,c^0) & 0 & 0 \\ 0 & \Phi^u(t,s,c^0) & 0 \\ 0 & 0 & \Phi^c(t,s,c^0) \end{pmatrix} \begin{pmatrix} \overline{x}(s) \\ \overline{y}(s) \\ \overline{c}(s) \end{pmatrix}$$

Assume:

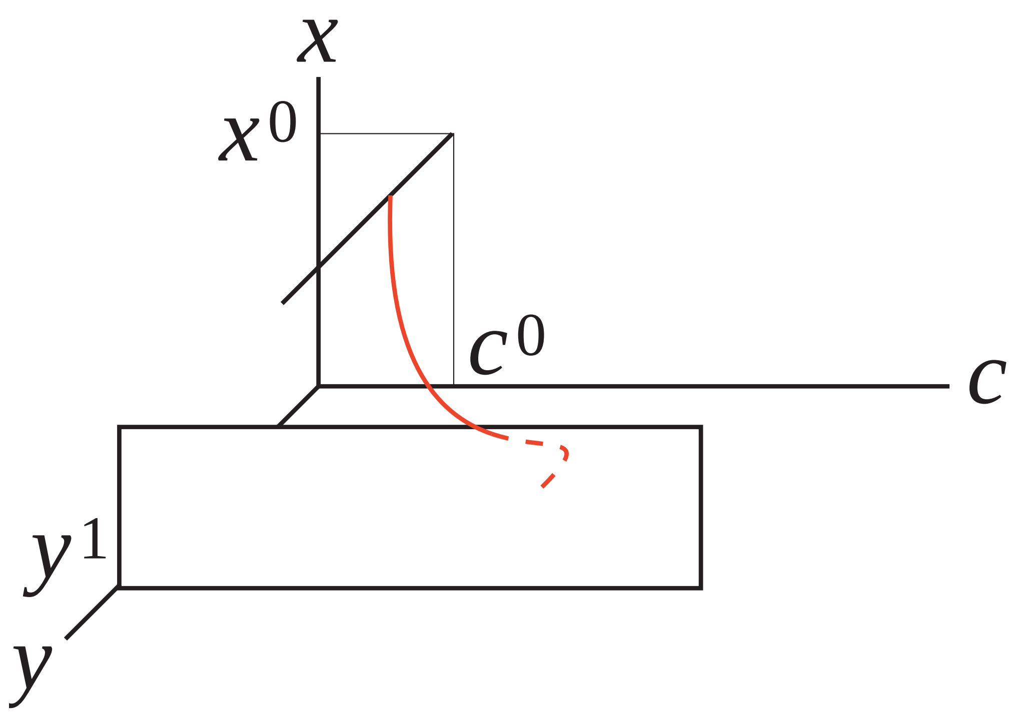
(E1) There are numbers $\lambda_0 < 0 < \mu_0$, $\beta > 0$, and $M > 0$ such that for all $c^0 \in N$ and $s,\, t \in I_{c^0}$,

$$\begin{aligned} \|\Phi^s(t,s,c^0)\| &\le Me^{\lambda_0(t-s)} && \text{if } t \ge s, \\ \|\Phi^u(t,s,c^0)\| &\le Me^{\mu_0(t-s)} && \text{if } t \le s, \\ \|\Phi^c(t,s,c^0)\| &\le Me^{\beta|t-s|} && \text{for all } t,\, s. \end{aligned}$$

(E2) $\lambda_0 + r\beta < 0 < \lambda_0 + \mu_0 - r\beta$.

We wish to study solutions of Silnikov's boundary value problem on an interval $0 \le t \le \tau$:

$$x(0) = x^0, \quad y(\tau) = y^1, \quad c(0) = c^0.$$



We denote the solution of Silnikov's boundary value problem

$$x(0) = x^0, \quad y(\tau) = y^1, \quad c(0) = c^0.$$

by $(x, y, c)(t, \tau, x^0, y^1, c^0)$.

Theorem 9 (Generalized Deng's Lemma, S. 2008). Let V_0 and V_1 be compact subsets of V such that $V_0 \subset \text{Int}\,(V_1)$. For each $c^0 \in V_0$ let J_{c^0} be the maximal interval such that $\phi(t, c^0) \in \text{Int}\,(V_1)$ for all $t \in J_{c^0}$. Then for λ and μ a little closer to 0 than λ_0 and μ_0, there is a number $\delta_0 > 0$ such that if $\|x^0\| \leq \delta_0$, $\|y^1\| \leq \delta_0$, $c^0 \in V_0$, and $\tau > 0$ is in J_{c^0}, then Silnikov's boundary value problem has a solution $(x, y, c)(t, \tau, x^0, y^1, c^0)$ on the interval $0 \leq t \leq \tau$. Moreover, there is a number $K > 0$ such that for all (t, τ, x^0, y^1, c^0) as above,

$$\|x(t, \tau, x^0, y^1, c^0)\| \leq Ke^{\lambda t},$$
$$\|y(t, \tau, x^0, y^1, c^0)\| \leq Ke^{\mu(t-\tau)},$$
$$\|c(t, \tau, x^0, y^1, c^0) - \phi(t, c^0)\| \leq Ke^{\lambda t + \mu(t-\tau)}.$$

In addition, if $\mathbf{i}$ is any $|\mathbf{i}|$-tuple of integers between 1 and $2+k+l+m$, with $1 \leq |\mathbf{i}| \leq r$, then

$$\|D_{\mathbf{i}}x(t, \tau, x^0, y^1, c^0)\| \leq Ke^{(\lambda+|\mathbf{i}|\beta)t},$$
$$\|D_{\mathbf{i}}y(t, \tau, x^0, y^1, c^0)\| \leq Ke^{(\mu-|\mathbf{i}|\beta)(t-\tau)},$$
$$\|D_{\mathbf{i}}c(t, \tau, x^0, y^1, c^0) - D_{\mathbf{i}}\phi(t, c^0)\| \leq Ke^{(\lambda+|\mathbf{i}|\beta)t+(\mu-|\mathbf{i}|\beta)(t-\tau)}.$$

Proof of Deng's Lemma

$$
\begin{aligned}
\dot{x} &= A(x,y,c)x, \\
\dot{y} &= B(x,y,c)y, \\
\dot{c} &= C(c) + E(x,y,c)xy.
\end{aligned}
$$

Let $c = \phi(t, c^0) + z$. Rewrite as

$$
\begin{aligned}
\dot{x} &= A(t,c^0)x + f(t,c^0,x,y,z), \\
\dot{y} &= B(t,c^0)y + g(t,c^0,x,y,z), \\
\dot{z} &= C(t,c^0)z + \theta(t,c^0,z) + h(t,c^0,x,y,z),
\end{aligned}
$$

with $A(t,c^0)$, $B(t,c^0)$,$C(t,c^0)$ linear. Silnikov's problem:

$$
x(0) = x^0, \quad y(\tau) = y^1, \quad c(0) = c^0.
$$

$(x(t),y(t),c(t))$ is a solution of Silnikov's problem if and only if $c(t) = \phi(t,c^0) + z(t)$ and $\eta(t) = (x(t),y(t),z(t))$ satisfies

$$
\begin{aligned}
x(t) &= \Phi^s(t,0,c^0)x^0 + \int_0^t \Phi^s(t,s,c^0) f(s,c^0,\eta(s))\, ds, \\
y(t) &= \Phi^u(t,\tau,c^0)y^1 + \int_\tau^t \Phi^u(t,s,c^0) g(s,c^0,\eta(s))\, ds, \\
z(t) &= \int_0^t \Phi^c(t,s,c^0)(\theta(s,c^0,z(s)) + h(s,c^0,\eta(s)))\, ds,
\end{aligned}
$$

$$x(t) = \Phi^s(t,0,c^0)x^0 + \int_0^t \Phi^s(t,s,c^0)\, f(s,c^0,\eta(s))\, ds,$$
$$y(t) = \Phi^u(t,\tau,c^0)y^1 + \int_\tau^t \Phi^u(t,s,c^0) g(s,c^0,\eta(s))\, ds,$$
$$z(t) = \int_0^t \Phi^c(t,s,c^0)(\theta(s,c^0,z(s)) + h(s,c^0,\eta(s)))\, ds,$$

Regard the right-hand side as a map from a weighted space of functions on $[0,\tau]$ into itself. Show there is a fixed point and estimate derivatives (which are fixed points of inhomogeneous linear maps).

How Exchange Lemmas are proved

Consider the Jones-Kopell Exchange Lemma. Situation: $(x,y,u,v) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}^{m-1}$,

$$\begin{aligned} \dot{x} &= Ax, \\ \dot{y} &= By, \\ \dot{u} &= \varepsilon + Cxy, \\ \dot{v} &= Exy, \end{aligned}$$

with A, B, C, E functions of (x,y,u,v,ε). Eigenvalues of A have negative real part, eigenvalues of B have positive real part.

M is given by

$$\begin{aligned} x &= x(\varepsilon) + L(y,\varepsilon)y, \\ u &= M(y,\varepsilon)y, \\ v &= N(y,\varepsilon)y. \end{aligned}$$

Given $\varepsilon > 0$ and (y^1, u^1) near (y, u^*), let $\tau = \frac{u^1}{\varepsilon}$. Find y^0 such that if we set

(1) $$x^0 = x(\varepsilon) + L(y^0,\varepsilon)y^0,$$

(2) $$u^0 = M(y^0,\varepsilon)y^0,$$

(3) $$v^0 = N(y^0,\varepsilon)y^0,$$

then the solution of Silnikov’s boundary value problem with

$$x(0)=x^0,\quad y(\tau)=y^1,\quad u(0)=u^0,\quad v(0)=v^0$$

has $z(0)=z_0$. Then M^* includes the graph of

$$(x(\tau,\tau,x^0,y^1,u^0,v^0),v(\tau,\tau,x^0,y^1,u^0,v^0)).$$

Note that the arguments depend on (y^1,u^1,ε). Now estimate (x,v) and their derivatives using Deng’s Lemma.

To find y^0 as a function of (y^1,u^1,ε), consider the mapping $((x^0,u^0,v^0),(y^1,u^1,\varepsilon))\to$ right hand side of (1)–(3), with

$$y^0=y(0,\tau,x^0,y^1,u^0,v^0) \text{ and } \tau=\frac{u^1}{\varepsilon}.$$

Show that for fixed (y^1,u^1,ε), this mapping is a contraction of a closed ball in (x^0,u^0,v^0)-space. Find the fixed point, then define y^0 by the above formula.