Stability of Traveling Waves for a Class of Reaction-diffusion Systems that Arise in Chemical Reaction Models

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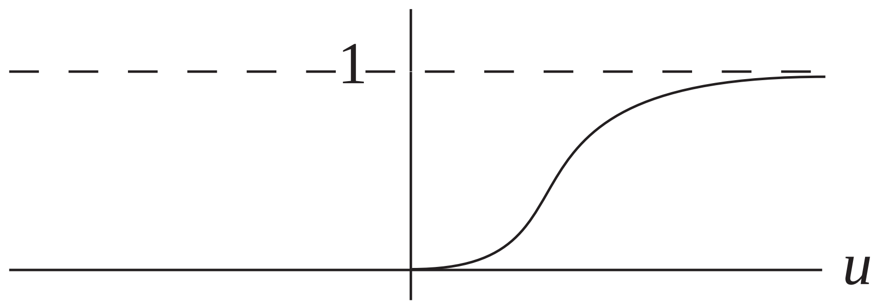
I. Combustion of a solid fuel in one space dimension

Model:

$$\partial_t u = \partial_{xx} u + v\rho(u),$$
$$\partial_t v = -\beta v\rho(u),$$

where

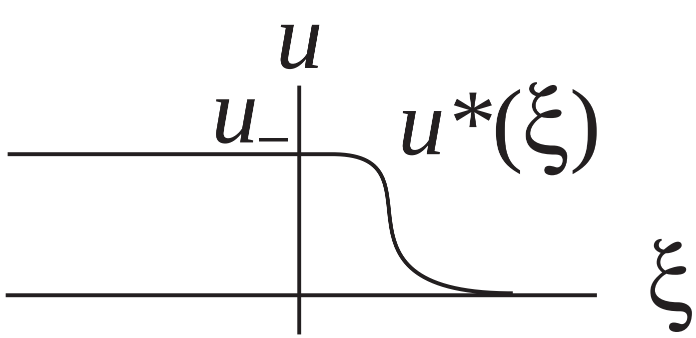
$$\rho(u) = \begin{cases} 0 & \text{if } u \le 0, \\ e^{-\frac{1}{u}} & \text{if } u > 0. \end{cases}.$$

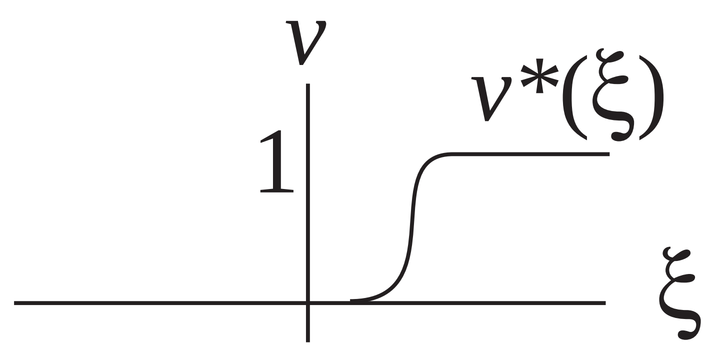


Graph of $\rho(u)$

- $u =$ temperature.
- $v =$ concentration of unburned fuel.
- $\rho =$ normalized reaction rate.
- $\beta > 0$ is the “exothermicity” parameter.
- $u = 0$ is a background temperature at which the reaction does not take place.

We are interested in combustion fronts $= (u^*, v^*)(\xi)$, $\xi = x - ct$.





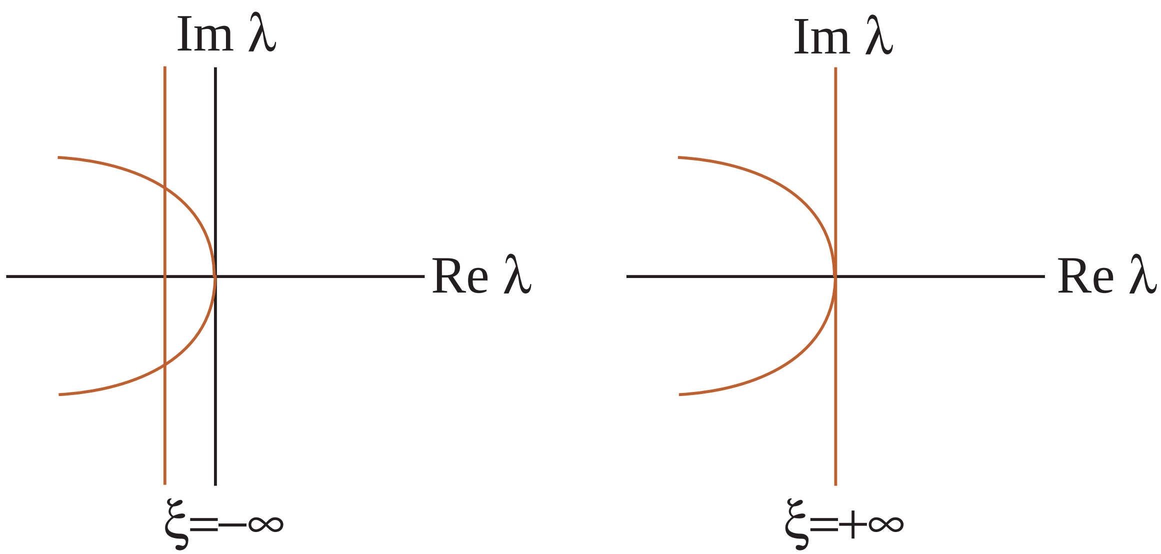
- $c > 0$ is the speed of the front.
- Behind the front: $(u^*, v^*) = (u_-, 0)$.
- $u_- > 0$ is the temperature of combustion, which is to be determined.
- Ahead of the front: $(u^*, v^*) = (0, u_+)$.
- $u_+ = 1$ is the concentration of fuel in the medium.

We assume $(u^*, v^*)(\xi)$ approaches its end states exponentially.

To study stability of the traveling wave:

- Write system in ξt-coordinates.
- Linearize at the equilibrium $(u^*, v^*)(\xi)$.
- Get a linear operator $\mathcal{L}_0$ on $\mathcal{E}_0 = BUC(\mathbb{R})$ or $H^1(\mathbb{R})$ (spaces suited to study of nonlinear stability because closed under multiplication), with norm $\| \|_0$.

Linearize at the end states, get constant-coefficient operators with the following spectra:



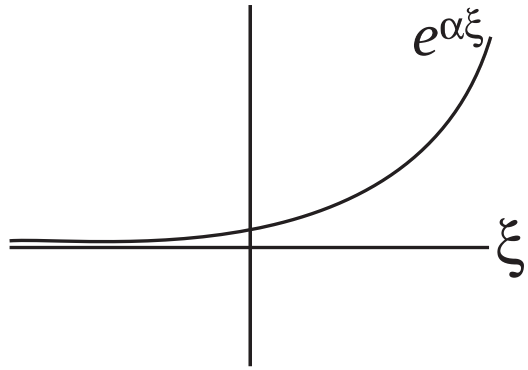
Facts:

(1) $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_0) \subset \mathrm{Re}\,\lambda \le 0$.
(2) Evans function has a simple zero at 0.
(3) Numerical study of the Evans function indicates no other zeros in $\mathrm{Re}\,\lambda \ge 0$ for small β.

We would like to conclude stability of the combustion front for small β, but:

(1) $\mathcal{L}_0$ is not sectorial.
(2) $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_0)$ touches the imaginary axis.

The essential spectrum can be moved to the left by using a norm with weight function $e^{\alpha\xi}$, $\alpha > 0$.



$$\mathcal{E}_\alpha = \{u(\xi) : e^{\alpha\xi}u(\xi) \in \mathcal{E}_0\}, \quad \|u\|_\alpha = \|e^{\alpha\xi}u(\xi)\|_0, \quad \alpha > 0 \text{ but not too big.}$$

A perturbation of the combustion front that is small in this norm is exponentially close to the front at the right but may be unbounded at the left.

The requirement that the perturbation be exponentially close at the right is natural: there are other traveling waves that approach the right state more slowly!

Unfortunately, the nonlinear terms in the PDE do not yield a map from $\mathcal{E}_\alpha$ to itself. Reason: consider

$$e^{\alpha\xi}\tilde{v}(\xi)\rho'(u^*(\xi))\tilde{u}(\xi), \quad \tilde{v},\tilde{u} \in \mathcal{E}_\alpha.$$

- $\rho'(u^*(\xi))$ is bounded.
- $e^{\alpha\xi}\tilde{v}(\xi)$ is bounded.
- $\tilde{u}(\xi)$ is not necessarily bounded.

Theorem on Combustion Fronts. Let $(\tilde{u},\tilde{v})(\xi)$ be small in *both* $\mathcal{E}_0$ and $\mathcal{E}_\alpha$. Consider the solution with value $(u^*+\tilde{u}, v^*+\tilde{v})(\xi)$ at $t=0$, which we denote $(u,v)(\xi,t)$. Then there is a number q^* such that

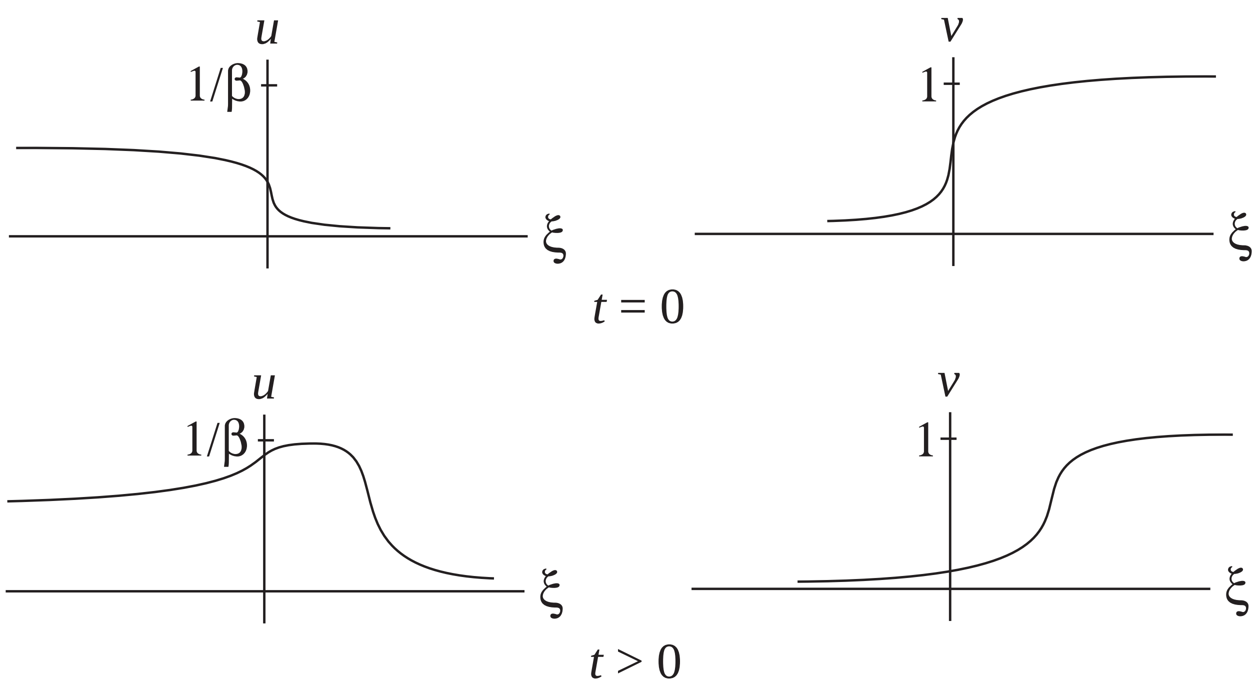
(1) $\|(u,v)(\xi,t)-(u^*,v^*)(\xi-q^*)\|_\alpha$ decays exponentially.
(2) $\|u(\xi,t)-u^*(\xi-q^*)\|_0$ stays small.
(3) $\|v(\xi,t)-v^*(\xi-q^*)\|_0$ decays exponentially.

In addition, suppose $\|(\tilde{u},\tilde{v})(\xi)\|_{L^1}$ is small. Then:

(4) $\|u(\xi,t)-u^*(\xi-q^*)\|_{L^1}$ stays small.
(5) $\|u(\xi,t)-u^*(\xi-q^*)\|_{L^\infty}$ decays like $t^{-\frac{1}{2}}$.

A. de Souza and GLS, "Stability of gasless combustion fronts in one-dimensional solids," Arch. Ration. Mech. Anal., to appear.

Explanation: (1) says that *at the right* the solution soon looks like the traveling wave. Consider the following perturbed initial condition:



(3) says, for example, that if we add some fuel behind the front, it will rapidly burn. Reason: behind the front the temperature is high!

(2), (4), and (5) say, for example, that if we add some heat behind the front, it will diffuse.

Hence our result yields, from the spectral information commonly found about the traveling wave, rather detailed information about the wave's stability.

II. General results (linear and nonlinear)

Reaction-diffusion system:

(1)
$$Y_t = DY_{xx} + R(Y).$$

$Y \in \mathbb{R}^n$, $D = \text{diag}(d_1, \ldots, d_n)$ with all $d_i \geq 0$, $R(Y)$ is smooth.

$Y_*(\xi)$, $\xi = x - ct$, is a traveling wave, $\lim_{\xi \to \pm\infty} Y_*(\xi) = Y_\pm$.

There are numbers $K > 0$ and $\omega > 0$ such that

$$\text{for } \xi \leq 0,\ \|Y_*(\xi) - Y_-\| \leq Ke^{\omega\xi},$$
$$\text{for } \xi \geq 0,\ \|Y_*(\xi) - Y_+\| \leq Ke^{-\omega\xi}.$$

Replace x by $\xi = x - ct$ in (1):

(2)
$$Y_t = DY_{\xi\xi} + cY_\xi + R(Y),$$

The traveling wave $Y_*(\xi)$ is a stationary solution of (2). Y_* is *stable* (more precisely: exponentially stable with asymptotic phase) in the space $\mathcal{X}$ if a small perturbation of Y_* of the form $Y = Y_* + \tilde{Y}$ with $\tilde{Y} \in \mathcal{X}$ decays exponentially in $\mathcal{X}$ to some shift of Y_*.

Linearize (2) about Y_*:

(3) $$\tilde{Y}_t = D\tilde{Y}_{\xi\xi} + c\tilde{Y}_\xi + DR(Y_*)\tilde{Y} = L\tilde{Y}.$$

$\mathcal{L} : \mathcal{X} \to \mathcal{X}$ is the operator on $\mathcal{X}$ given by $\tilde{Y} \to L\tilde{Y}$, with its natural domain.

Y_* is *spectrally stable* in the space $\mathcal{X}$ if the spectrum of $\mathcal{L}$ is contained in the half-plane $\mathrm{Re}\,\lambda \le -\mu < 0$, with the exception of a simple eigenvalue 0. (A traveling wave has an eigenvalue 0, with eigenvector $Y_*'(\xi)$, in any space that contains Y_*'.) Y_* is *linearly exponentially stable* in $\mathcal{X}$ if every solution of (3) decays exponentially to a multiple of Y_*'.

Linear Theorem. Consider the linear PDE

$$\tilde{Y}_t = D\tilde{Y}_{\xi\xi} + c\tilde{Y}_\xi + A(\xi)\tilde{Y} = L\tilde{Y};$$

$D = \mathrm{diag}(d_1, \dots, d_n)$ with all $d_i \ge 0$, $A(\xi)$ is smooth, and there are matrices $A_\pm$ such that $A(\xi) \to A_\pm$ exponentially as $\xi \to \pm\infty$. Let $\mathcal{X}$ denote one of the standard Banach spaces $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$, and let $\mathcal{L}$ denote the operator on $\mathcal{X}$ associated with L. Assume (1) $\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}) < 0\}$ and (2) $\{\lambda : \mathrm{Re}\,\lambda \ge 0\}$ is contained in the resolvent set of $\mathcal{L}$ except possibly for an eigenvalue 0 with generalized null space $\mathcal{N}$. Let $\mathcal{P}$ be the Riesz spectral projection for $\mathcal{L}$ whose kernel is equal to $\mathcal{N}$. Then there are positive numbers K and ν such that $\|e^{t\mathcal{L}\mathcal{P}}\| \le Ke^{-\nu t}$.

Proofs:

- All d_i positive: $\mathcal{L}$ is sectorial, the result is in Henry.
- Some $d_i = 0$, $A_+ = A_-$: proved by Bates and Jones (Dynamics Reported **2**, 1989).
- General case: proved in GLS, "Stability of traveling waves for degenerate systems of reaction diffusion equations," Indiana Math. J., to appear., and by Jens Rottmann-Matthes in his Bielefeld thesis.

Consequences:

- The Linear Theorem implies that if the traveling wave Y_* is spectrally stable in any of the spaces $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$, or in one of these spaces with weight function bounded away from 0, then it is linearly exponentially stable in that space.
- For $\mathcal{X} = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$, or in one of these spaces with weight function bounded away from 0, linearized exponential stability of the traveling wave Y_* implies (nonlinear) stability (Bates and Jones).

Example: traveling *fronts* in Fitzhugh-Nagumo.

Nonlinear results: Let $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$.

Assume: $Y_*(\xi)$ is spectrally stable in $\mathcal{E}_\alpha$ with $0 < \alpha < \omega$.

Intuitively, this assumption is enough to prove stability at the right but not at the left.

We take Y_- to be 0.

In appropriate variables $Y = (U,V)$, we assume $R(U,0) = 0$, i.e.,

$$R(U,V) = \begin{pmatrix} \tilde{R}_1(U,V)V \\ \tilde{R}_2(U,V)V \end{pmatrix}. \tag{4}$$

Example: In a combustion problem with $n-1$ reactants, suppose the left state of a combustion front with positive velocity has temperature $y_1 = y_{1-} > 0$ and reactant concentrations $(y_2, \ldots, y_n) = (0, \ldots, 0)$. Let

$$u = y_1 - y_{1-}, \quad (v_1, \ldots, v_{n-1}) = (y_2, \ldots, y_n).$$

Since the reaction rate is 0 when the reactant concentrations are all 0, the reaction term takes the form (4).

Linearize (2) at $(0,0)$:

$$\tilde{U}_t = D_1\tilde{U}_{\xi\xi} + c\tilde{V}_\xi + \tilde{R}_1(0,0)\tilde{V} = L^{(1)}\tilde{U} + \tilde{R}_1(0,0)\tilde{V},$$
$$\tilde{V}_t = D_2\tilde{V}_{\xi\xi} + c\tilde{V}_\xi + \tilde{R}_2(0,0)\tilde{V} = L^{(2)}\tilde{V}.$$

Assume: in $\mathcal{E}_0$, the operator associated with $L^{(2)}$ has its spectrum in $\mathrm{Re}\,\lambda < -\rho < 0$ for some ρ.

Note that in $\mathcal{E}_0$ the operator associated with $L^{(1)}$ generates a bounded semigroup.

Nonlinear Theorem 1. Perturbations of the traveling wave that are initially small in $\mathcal{E}_0 \cap \mathcal{E}_\alpha$ decay exponentially in $\mathcal{E}_\alpha$ to some shift of the wave. In addition, the U-component of the perturbation stays small in $\mathcal{E}_0$, and the V-component of the perturbation decays exponentially $\mathcal{E}_0$.

Notice:

- In $\mathcal{E}_0$ the U-component of the perturbation may travel with velocity less than c without decay (*convective instability*).
- For $\mathcal{E}_0 = BUC(\mathbb{R})$, as $\xi \to -\infty$, the perturbation of the traveling wave need only be bounded.

Nonlinear Theorem 2. Suppose the linear equation $\tilde{U}_t = L^{(1)}\tilde{U}$ is parabolic, i .e., the corresponding d_i's are all positive. If the perturbation of the traveling wave is also small in L^1, then the U-component of the perturbation stays small in the L^1-norm and decays decays like $t^{-\frac{1}{2}}$ in the L^∞-norm.

III. Proof of Linear Theorem

We consider a more general class of linear PDEs:

(5) $$\begin{pmatrix} U \\ V \end{pmatrix}_t = \mathcal{L} \begin{pmatrix} U \\ V \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} d\partial_{\xi\xi} + a\partial_\xi + B_{11} & B_{12} \\ B_{21} & b\partial_\xi + B_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{A} & B_{12} \\ B_{21} & \mathcal{G} \end{pmatrix},$$

$$U(t,\xi) \in \mathbb{R}^{N_1}, \quad V(t,\xi) \in \mathbb{R}^{N_2}, \quad d = \mathrm{diag}(d_1, \dots, d_{N_1}) \text{ with all } d_k > 0,$$
$$a = (a_{kl}) \text{ of size } N_1 \times N_1, \quad b = \mathrm{diag}(b_1, \dots, b_{N_2}),$$

The matrices d, a, and b are constant.

Simplifying assumption:

- All b_k are nonzero.

Assume:

- $B = (B_{ij})$ is C^1, there are matrices $B^\pm$ such that $B(\xi) \to B^\pm$ as $\xi \to \pm\infty$, and $B'(\xi) \to 0$ exponentially as $\xi \to \pm\infty$.

Define $\mathcal{L}^\pm$, $\mathcal{A}^\pm$, $\mathcal{G}^\pm$ (constant coefficient operators).

Let $\mathcal{X}$ be a Banach space, and let $\mathcal{C} : \mathcal{X} \to \mathcal{X}$ be a closed, densely defined linear operator. Define

- $\rho(\mathcal{C})$, the set of $\lambda \in \mathbb{C}$ such that $\mathcal{C} - \lambda I$ has a bounded inverse.
- $\mathrm{Sp}(\mathcal{C}) = \mathbb{C} \setminus \rho(\mathcal{C})$. $\mathrm{Sp}(\mathcal{C})$ is the union of $\mathrm{Sp}_{\mathrm{d}}(\mathcal{C})$, which is the set of isolated eigenvalues of $\mathcal{C}$ of finite algebraic multiplicity, and $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{C})$, which is the rest.
- $\mathrm{s}(\mathcal{C}) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{C})\}$.
- $\mathrm{s}_{\mathrm{ess}}(\mathcal{C})$, the infimum of all real ω such that $\mathrm{Sp}(\mathcal{C}) \cap \{\lambda : \mathrm{Re}\,\lambda > \omega\}$ is a subset of $\mathrm{Sp}_{\mathrm{d}}(\mathcal{C})$ and has only finitely many points.
- $\rho_{\mathrm{F}}(\mathcal{C})$, the set of $\lambda \in \mathbb{C}$ such that $\mathcal{C} - \lambda I$ is Fredholm of index zero. ($\mathcal{C}$ is *Fredholm* if its range is closed, its kernel has finite dimension n, and its range has finite codimension m. The *index* of a Fredholm operator $\mathcal{C}$ is $n - m$.)
- $\mathrm{Sp}_{\mathrm{F}}(\mathcal{C}) = \mathbb{C} \setminus \rho_{\mathrm{F}}(\mathcal{C})$.
- $\mathrm{s}_{\mathrm{F}}(\mathcal{C}) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}_{\mathrm{F}}(\mathcal{C})\}$.

For a *bounded* linear operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$, we define:

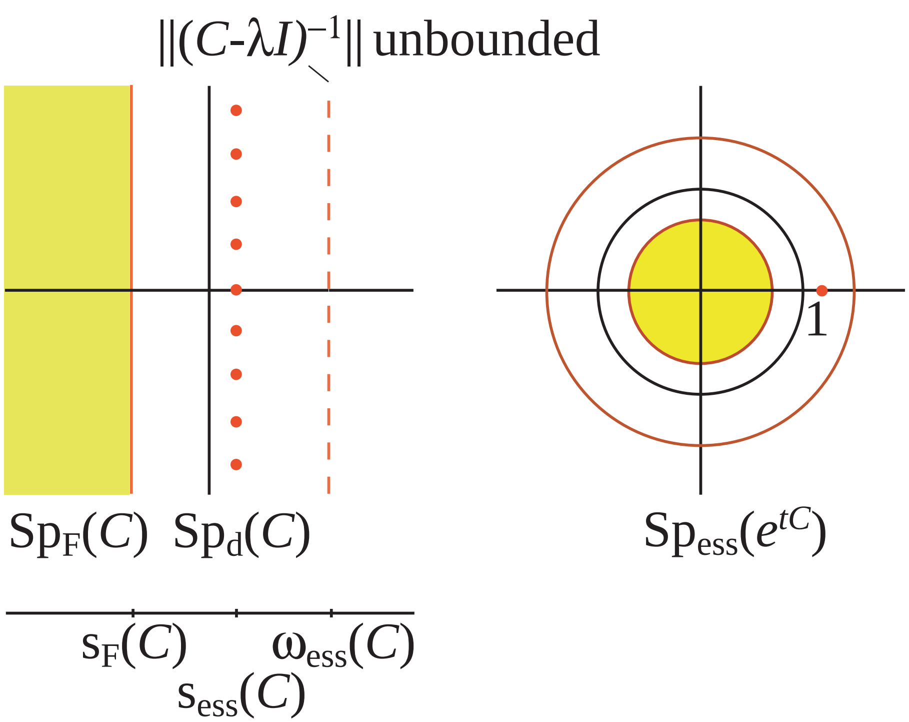
- The *spectral radius* of $\mathcal{T}$, the supremum of $\{|\lambda| : \lambda \in \mathrm{Sp}(\mathcal{T})\}$.
- The *essential spectral radius* of $\mathcal{T}$, the supremum of $\{|\lambda| : \lambda \in \mathrm{Sp}_{\mathrm{ess}}(\mathcal{T})\}$.

If $\mathcal{C}$ generates a C_0-semigroup, we can define:

- The *growth bound*: the real number ω such that $e^{t\omega(\mathcal{C})}$ is the spectral radius of $e^{t\mathcal{C}}$.
- The *essential growth bound*: the real number $\omega_{\text{ess}}(\mathcal{C})$ such that $e^{t\omega_{\text{ess}}(\mathcal{C})}$ is the essential spectral radius of $e^{t\mathcal{C}}$.

Fact. If $\mathcal{C} : \mathcal{X} \to \mathcal{X}$ generates a C_0-semigroup, then:

$$s_F(\mathcal{C}) \le s_{\text{ess}}(\mathcal{C}) \le \omega_{\text{ess}}(\mathcal{C}).$$



Fact. Let $\mathcal{C}$ be any of the constant-coefficient operators $\mathcal{A}^{\pm}$, $\mathcal{G}^{\pm}$, $\mathcal{L}^{\pm}$. In $L^2(\mathbb{R})$, $\mathrm{Sp}(\mathcal{C})$, and hence $\mathrm{s}(\mathcal{C})$ can be computed by Fourier transform; $\mathrm{Sp}(\mathcal{C})$ and hence $\mathrm{s}(\mathcal{C})$ are the same in $L^1(\mathbb{R})$, $H^1(\mathbb{R})$, and $BUC(\mathbb{R})$. Morover, $\mathrm{s}(\mathcal{C}) = \mathrm{s}_{\mathrm{F}}(\mathcal{C}) = \mathrm{s}_{\mathrm{ess}}(\mathcal{C})$.

Fact. Suppose $\mathcal{C} : \mathcal{X} \to \mathcal{X}$ generates the C_0-semigroup $e^{t\mathcal{C}}$, $t \geq 0$. Let $\omega > \omega_{\mathrm{ess}}(\mathcal{C})$ be a number such that no element of $\mathrm{Sp}(\mathcal{C})$ has real part ω. Then there is a finite set $\{\lambda_1, \ldots, \lambda_k\} \subset \mathbb{C}$ such that

$$\mathrm{Sp}(\mathcal{C}) \cap \{\lambda : \mathrm{Re}\,\lambda > \omega\} = \mathrm{Sp}_{\mathrm{d}}(\mathcal{C}) \cap \{\lambda : \mathrm{Re}\,\lambda > \omega\} = \{\lambda_1, \ldots, \lambda_k\}.$$

Let $E_1, \ldots, E_k$ be the generalized eigenspaces of $\lambda_1, \ldots, \lambda_k$ respectively; they are finite-dimensional. Then there is a closed subspace E_0 of $\mathcal{X}$ such that $\mathcal{X} = E_0 \times E_1 \times \cdots \times E_k$ and E_0 is invariant under $\mathcal{C}$. Moreover, there is a number $K > 0$ such that $\|e^{t\mathcal{C}}|E_0\| \leq Ke^{\omega t}$.

Linear Lemma. $\mathrm{s}_{\mathrm{F}}(\mathcal{L}) = \mathrm{s}_{\mathrm{ess}}(\mathcal{L}) = \omega_{\mathrm{ess}}(\mathcal{L})$.

Linear Theorem Again. Assume (1) $\mathrm{s}_{\mathrm{F}}(\mathcal{L}) < 0$ and (2) $\{\lambda : \mathrm{Re}\,\lambda \geq 0\}$ is contained in the resolvent set of $\mathcal{L}$ except possibly for an eigenvalue 0 with generalized null space $\mathcal{N}$. Let $\mathcal{P}$ be the Riesz spectral projection for $\mathcal{L}$ whose kernel is equal to $\mathcal{N}$. Then there are positive numbers K and ν such that $\|e^{t\mathcal{L}\mathcal{P}}\| \leq Ke^{-\nu t}$.

Proof. Choose $\nu > 0$ such that $\mathrm{s}_{\mathrm{F}}(\mathcal{L}) < -\nu$ and the only element of $\mathrm{Sp}_{\mathrm{d}}(\mathcal{C})$ with real part $\geq -\nu$ is 0.

Proof of the Linear Lemma is based on:

(1) Palmer's Theorem
(2) properties of $\mathcal{A}$ and $\mathcal{G}$
(3) a triangular factorization of $\mathcal{L}$
(4) Gearhart-Prüss or Greiner Spectral Mapping Theorem

Proposition. Let $\mathcal{C}$ be any of $\mathcal{A}$, $\mathcal{G}$, or $\mathcal{L}$. Then $s_F(\mathcal{C}) = \max\{s_F(\mathcal{C}^-), s_F(\mathcal{C}^+)\}$.

Follows from Palmer's Theorem and an argument of Sandstede and Scheel.

Fact.

(1) $\mathcal{A}$ is *sectorial*.
(2) There is a constant c and a sector Σ such that $\sigma(\mathcal{A}) \subset \Sigma$, and, for each $z = x + iy \notin \Sigma$,

$$\|(\mathcal{A} - zI)^{-1}\| \leq \frac{c}{|y|}.$$

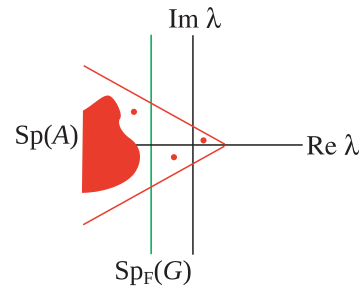
(3) $s_F(\mathcal{A}) = s_{ess}(\mathcal{A}) = \omega_{ess}(\mathcal{A})$.

Fact.

(1) $\mathrm{Sp}(\mathcal{G})$ consists of vertical lines.
(2) If $z \in \rho(\mathcal{G})$, then $z + i\alpha \in \rho(\mathcal{G})$ for all $\alpha \in \mathbb{R}$, and, moreover,

$$\|(\mathcal{G} - zI)^{-1}\| = \|(\mathcal{G} - (z + i\alpha)I)^{-1}\| = \|(\mathcal{G} - \mathrm{Re}\, zI)^{-1}\|.$$

(3) $s(\mathcal{G}) = s_F(\mathcal{G}) = s_{ess}(\mathcal{G}) = \omega_{ess}(\mathcal{G}) = \omega(\mathcal{G})$.



Lemma.

(1) $s_F(\mathcal{G}^{\pm}) \leq s_F(\mathcal{L}^{\pm})$ and $s_F(\mathcal{G}) \leq s_F(\mathcal{L})$.

(2) There is a continuous nonnegative function $r(x)$ defined for $x > s_F(\mathcal{G})$ such that if $|y| \geq r(x)$ then $z = x + iy \in \rho(\mathcal{L})$. Moreover, for each $x > s_F(\mathcal{G})$, $\sup_{|y| \geq r(x)} \|(\mathcal{L} - (x+iy)I)^{-1}\| < \infty$.

Proof is by triangular factorizations. Let

$$\mathcal{H}(z) = \mathcal{G} - zI - B_{21}(\mathcal{A} - zI)^{-1}B_{12}.$$

Then

$$\mathcal{L} - zI = \begin{pmatrix} \mathcal{A} - zI & 0 \\ B_{21} & I \end{pmatrix} \begin{pmatrix} I & (\mathcal{A} - zI)^{-1}B_{12} \\ 0 & \mathcal{H}(z) \end{pmatrix}$$

Moreover, $\mathcal{G} - zI = \mathcal{H}(z)(I - \mathcal{F}(z))$ with $\mathcal{F}(z) = -\mathcal{H}(z)^{-1}B_{21}(\mathcal{A} - zI)^{-1}B_{12}$.

Corollary. $s_F(\mathcal{L}) = s_{ess}(\mathcal{L})$.

To relate to $\omega_{\text{ess}}(\mathcal{L})$, the tools are:

Gearhart-Prüss Spectral Mapping Theorem. Suppose $\mathcal{C}$ is the generator of a C_0-semigroup on a Hilbert space $\mathcal{X}$, and let $z \in \mathbb{C}$. Then the following are equivalent:

(i) $e^z \in \rho(e^{\mathcal{C}})$.
(ii) $z + 2\pi i k \in \rho(\mathcal{C})$ for all $k \in \mathbb{Z}$ and $\sup_{k\in\mathbb{Z}} \|(\mathcal{C} - (z+2\pi i k)I)^{-1}\| < \infty$,

Greiner's Spectral Mapping Theorem. Suppose $\mathcal{C}$ is the generator of a C_0-semigroup $\{e^{t\mathcal{C}}\}_{t\geq 0}$ on a Banach space $\mathcal{X}$, and let $z \in \mathbb{C}$. Then the following are equivalent:

(i) $e^z \in \rho(e^{\mathcal{C}})$.
(ii) $z + 2\pi i k \in \rho(\mathcal{C})$ for all $k \in \mathbb{Z}$, and for each $w \in \mathcal{X}$ the series $\sum_{k\in\mathbb{Z}}(\mathcal{C} - (z + 2\pi i k)I)^{-1} w$ is Cesaro summable.

A series $\sum_{k\in\mathbb{Z}} w_k$, with $w_k \in \mathcal{X}$, is called Cesaro summable if the following limit exists in $\mathcal{X}$:

(6) $$(C,1)\sum_{k\in\mathbb{Z}} w_k := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \sum_{|k|\leq m} w_k = \lim_{n\to\infty} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) w_k.$$

Also need a generalization where for a finite number of k, $z + 2\pi i k \in \mathrm{Sp}_{\mathrm{d}}(\mathcal{C})$.

Lemma. For each z with $\mathrm{Re}\, z > s_F(\mathcal{G})$, there is an integer $K \geq 0$ such that $z + 2\pi ik \in \rho(\mathcal{L})$ for all $k \in \mathbb{Z}$ with $|k| \geq K$. Moreover,

for each $w \in \mathcal{X}$, the series $\sum_{|k|\geq K} (\mathcal{L} - (z + 2\pi ik)I)^{-1} w$ is Cesaro summable.

Proof uses (1) the expression for $(\mathcal{L} - zI)^{-1}$ we get from the triangular factorization of $\mathcal{L} - zI$ and (2) Fejer's summability kernel

$$K_n(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt}, \quad t \in \mathbb{T},$$

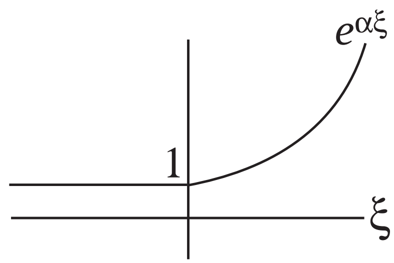
whose convolution property is

$$\|K_n * f - f\|_{L^1(\mathbb{T};\mathcal{X})} \to 0 \text{ as } n \to \infty \text{ for every } f \in L^1(\mathbb{T};\mathcal{X}).$$

IV. Proof of Nonlinear Theorem

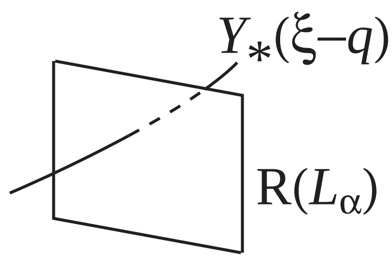
$$Y_t = DY_{\xi\xi} + cY_\xi + R(Y).$$

Work in $\mathcal{E}_0 \cap \mathcal{E}_\alpha$, $\|Y\| = \max(\|Y\|_0, \|Y\|_\alpha) =$ norm with respect to weight function



For $Y \in \mathcal{E}_0 \cap \mathcal{E}_\alpha$ near Y_* write

$$Y(\xi) = Y_*(\xi - q) + \tilde{Y}(\xi), \quad \tilde{Y} \in \mathrm{R}(\mathcal{L}_\alpha) \cap \mathcal{E}_0,$$



or

$$\begin{pmatrix} U(\xi) \\ V(\xi) \end{pmatrix} = \begin{pmatrix} U_*(\xi - q) \\ V_*(\xi - q) \end{pmatrix} + \begin{pmatrix} \tilde{U}(\xi) \\ \tilde{V}(\xi) \end{pmatrix}.$$

Let

$$Y_q = Y_*(\xi - q).$$

Then

$$-Y_*'(\xi - q(t))q'(t) + \tilde{Y}_t = D\tilde{Y}_{\xi\xi} + c\tilde{Y}_\xi + R(Y_q + \tilde{Y}) - R(Y_q).$$

Write $R(Y+\tilde{Y}) - R(Y) - DR(Y)\tilde{Y} = N(Y,\tilde{Y})\tilde{Y}$ and rewrite the equation as

$$\tilde{Y}_t = L\tilde{Y} + (DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q,\tilde{Y})\tilde{Y} + Y_*'(\xi - q(t))q'(t).$$

Apply $\mathcal{P}_\alpha^s$ and $\mathcal{P}_\alpha^c = I - \mathcal{P}_\alpha^s$:

$$\tilde{Y}_t = L\tilde{Y} + \mathcal{P}_\alpha^s\big(DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q,\tilde{Y})\tilde{Y} + Y_*'(\xi - q(t))q'(t)\big),$$
$$-q'(t)\mathcal{P}_\alpha^c Y_*'(\xi - q(t)) = \mathcal{P}_\alpha^c\big((DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q,\tilde{Y})\tilde{Y}\big).$$

Rewrite the second equation as

$$-q'(t)\pi_\alpha Y_*'(\xi - q(t)) = \pi_\alpha\big((DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q,\tilde{Y})\tilde{Y}\big)$$

where $\pi_\alpha(\cdot)$ is a number.

Then for $q(t)$ small, $q'(t) = -\big(\pi_\alpha Y_*'(\xi - q(t))\big)^{-1}\pi_\alpha\big((DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q,\tilde{Y})\tilde{Y}\big).$

$$\tilde{Y}_t = L\tilde{Y} + DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q, \tilde{Y})\tilde{Y} + Y_*'(\xi - q(t))q'(t)),$$
$$q'(t) = -\left(\pi_\alpha Y_*'(\xi - q(t))\right)^{-1}\pi_\alpha\left((DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q, \tilde{Y})\tilde{Y}\right)$$

Let

$$G(\tilde{Y}, q) = (DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q, \tilde{Y})\tilde{Y},$$
$$\kappa(\tilde{Y}, q) = -(\pi_\alpha Y_*'(\xi - q))^{-1}\pi_\alpha G(\tilde{Y}, q).$$

Then

(7) $$\partial_t \tilde{Y} = L\tilde{Y} + G(\tilde{Y}, q) + \kappa(\tilde{Y}, q)Y_*'(\xi - q),$$
(8) $$\dot{q} = \kappa(\tilde{Y}, q).$$

Proposition. The formulas for $G(\tilde{Y}, q)$ and $\kappa(\tilde{Y}, q)$ define mappings from $(\mathcal{E}_0 \cap \mathcal{E}_\alpha) \times \mathbb{R}$ to $\mathcal{E}_0 \cap \mathcal{E}_\alpha$ and to $\mathbb{R}$ respectively. On any bounded neighborhood of $(0,0)$ in $(\mathcal{E}_0 \cap \mathcal{E}_\alpha) \times \mathbb{R}$, the mappings are Lipschitz, and there is a constant C such that:

(1) $\|G(\tilde{Y}, q)\|_\alpha \leq C(|q| + \|V\|_0)\|V\|_\alpha$.

(2) $|\kappa(\tilde{Y}, q)| \leq C(|q| + \|V\|_0)\|V\|_\alpha$.

Study of the system on $(\mathrm{R}(\mathcal{L}_\alpha)\cap\mathcal{E}_0)\times\mathbb{R}$

1. Existence of solutions on $(\mathrm{R}(\mathcal{L}_\alpha)\cap\mathcal{E}_0)\times\mathbb{R}$ and *a priori* bound

$$\begin{aligned}\partial_t\tilde{Y} &= L\tilde{Y}+G(\tilde{Y},q)+\kappa(\tilde{Y},q)Y_*'(\xi-q),\\ \dot{q} &= \kappa(\tilde{Y},q).\end{aligned}$$

Proposition 1. For each $\delta>0$, if $0<\gamma<\delta$, then there exists T, with $0<T\leq\infty$, such that the following is true: if $(\tilde{Y}^0,q^0)\in(\mathrm{R}(\mathcal{L}_\alpha)\cap\mathcal{E}_0)\times\mathbb{R}$ satisfies

$$\|(\tilde{Y}^0,q^0)\| = \|\tilde{Y}^0\|+|q^0|\leq\gamma \tag{9}$$

and $0\leq t<T$, then $(\tilde{Y},q)(t,\tilde{Y}^0,q^0)\in\mathrm{R}(\mathcal{L}_\alpha)\cap\mathcal{E}_0$ is defined and satisfies

$$\|\tilde{Y}(t,\tilde{Y}^0,q^0)\|+|q(t,\tilde{Y}^0,q^0)|\leq\delta. \tag{10}$$

Let $T_{\max}(\delta,\gamma)$ denote the supremum of all T such that (10) holds for all $0\leq t<T$ whenever (9) is satisfied.

2.Decay of $\|\tilde{Y}(t)\|_\alpha$

Proposition 2. Consider the solution given by Proposition 1. Then there are numbers $\nu > 0$, $C > 0$, and $K_\alpha > 0$ such, that if $\delta > 0$ is sufficiently small and $0 < \gamma < \delta$, then following is true. Let $(\tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{L}_\alpha) \cap \mathcal{E}_0$ satisfy (9), so that $(\tilde{Y}, q)(t, \tilde{Y}^0, q^0)$ satisfies (10) for $0 \le t < T_{\max}(\delta, \gamma)$. Then:

(11) $$\|\tilde{Y}(t)\|_\alpha \le K_\alpha e^{-\nu t}\|\tilde{Y}^0\|_\alpha \text{ and } |q(t) - q^0| \le C\|\tilde{Y}^0\|_\alpha \text{ for } 0 \le t < T_{\max}(\delta, \gamma).$$

Moreover, if $T_{\max}(\delta, \gamma) = \infty$, then there is $q^* \in \mathbb{R}$ such that

(12) $$|q(t) - q^*| \le Ce^{-\nu t}\|\tilde{Y}^0\|_\alpha \text{ for all } t \ge 0.$$

- Idea–**bound the solution in a uniform norm in order to prove convergence in a weighted norm**–comes from R. Pego and M. Weinstein, *Asymptotic stability of solitary waves*, Comm. Math. Phys. **164** (1994), 305–349.
- In Pego and Weinstein, boundedness in the uniform norm follows from a Hamiltonian structure.
- In other papers, it is related to the stability of the bifurcating patterns that are connected by the front.

3. Bounds for $\|\tilde{Y}(t)\|_0$

Proposition 3. Consider the solution given by Proposition 1. Then there is a number $C > 0$ such that if (1) δ is sufficiently small and (2) $0 < \gamma < \delta$, then the following is true. Let $(\tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{L}_\alpha) \cap \mathcal{E}_0$ satisfy (9), so that $(\tilde{Y}, q)(t, \tilde{Y}^0, q^0)$ satisfies (10) for $0 \le t < T_{\mathrm{max}}(\delta, \gamma)$, and:

$$\|\tilde{U}(t)\|_0 \le C\|\tilde{Y}^0\|, \tag{13}$$

$$\|\tilde{V}(t)\|_0 \le C\|\tilde{Y}^0\|e^{-\rho t}. \tag{14}$$

Proof is completed by bootstrapping.