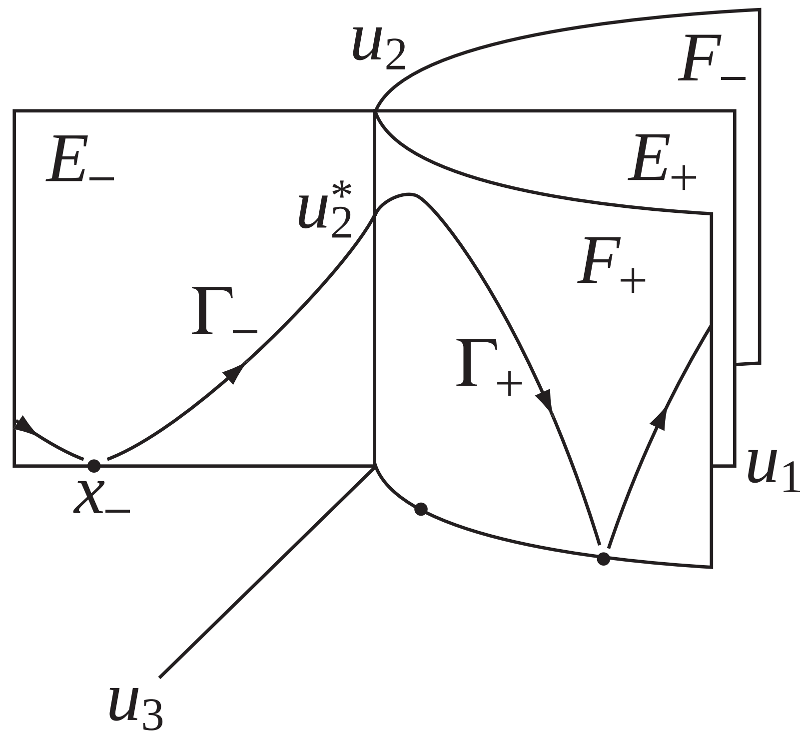
Heteroclinic solutions of a singularly perturbed Hamiltonian system



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Motivation for work

Sourdis and Fife, *Existence of heteroclinic orbits for a corner layer problem in anisotropic interfaces*, Advances in Differential Equations **12** (2007), 623–668:

The physical motivation comes from a multi-order-parameter phase field model, developed by Braun et al. for the description of crystalline interphase boundaries. The smallness of ε is related to large anisotropy. [The heteroclinic orbit represents a moving interface between ordered and disordered states.] The mathematical interest stems from the fact that the smoothness and normal hyperbolicity of the critical manifold fails at certain points. Thus the well-developed geometric singular perturbation theory does not apply. The existence of such a heteroclinic, and its dependence on ε, is proved via a functional analytic approach.

Motivation for talk

Show how the blow-up technique of geometric singular perturbation theory (Dumortier, Roussarie, Szmolyan, Krupa, . . .) can help with such problems.

Help is: geometric matching of outer and inner solutions.

Second-order system

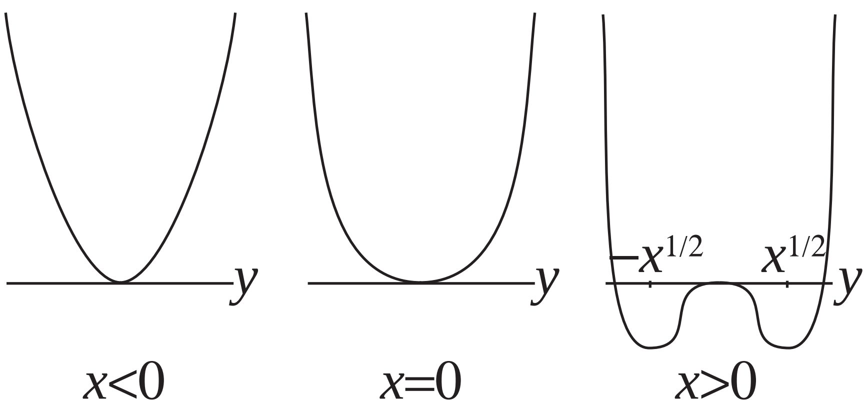
We consider

(1) $$x_{\tau\tau} = g_x(x,y),$$

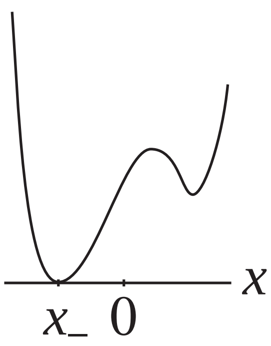
(2) $$\varepsilon^2 y_{\tau\tau} = g_y(x,y),$$

where

(3) $$g(x,y) = \frac{1}{4}y^4 - \frac{1}{2}xy^2 + h(x).$$



Graph of $(1/4)y^4-(1/2)xy^2$



Graph of $h(x)$

First-order system

Write (1)–(2) as a first-order system (the slow system) with $u_1 = x$, $u_3 = y$:

(4) $$u_{1\tau} = u_2,$$

(5) $$u_{2\tau} = g_x(u_1, u_3) = -\frac{1}{2}u_3^2 + h'(u_1),$$

(6) $$\varepsilon u_{3\tau} = u_4,$$

(7) $$\varepsilon u_{4\tau} = g_y(u_1, u_3) = u_3^3 - u_1 u_3.$$

In (4)–(7) let $\tau = \varepsilon\sigma$. We obtain the fast system:

(8) $$u_{1\sigma} = \varepsilon u_2,$$

(9) $$u_{2\sigma} = \varepsilon g_x(u_1, u_3) = \varepsilon\left(-\frac{1}{2}u_3^2 + h'(u_1)\right),$$

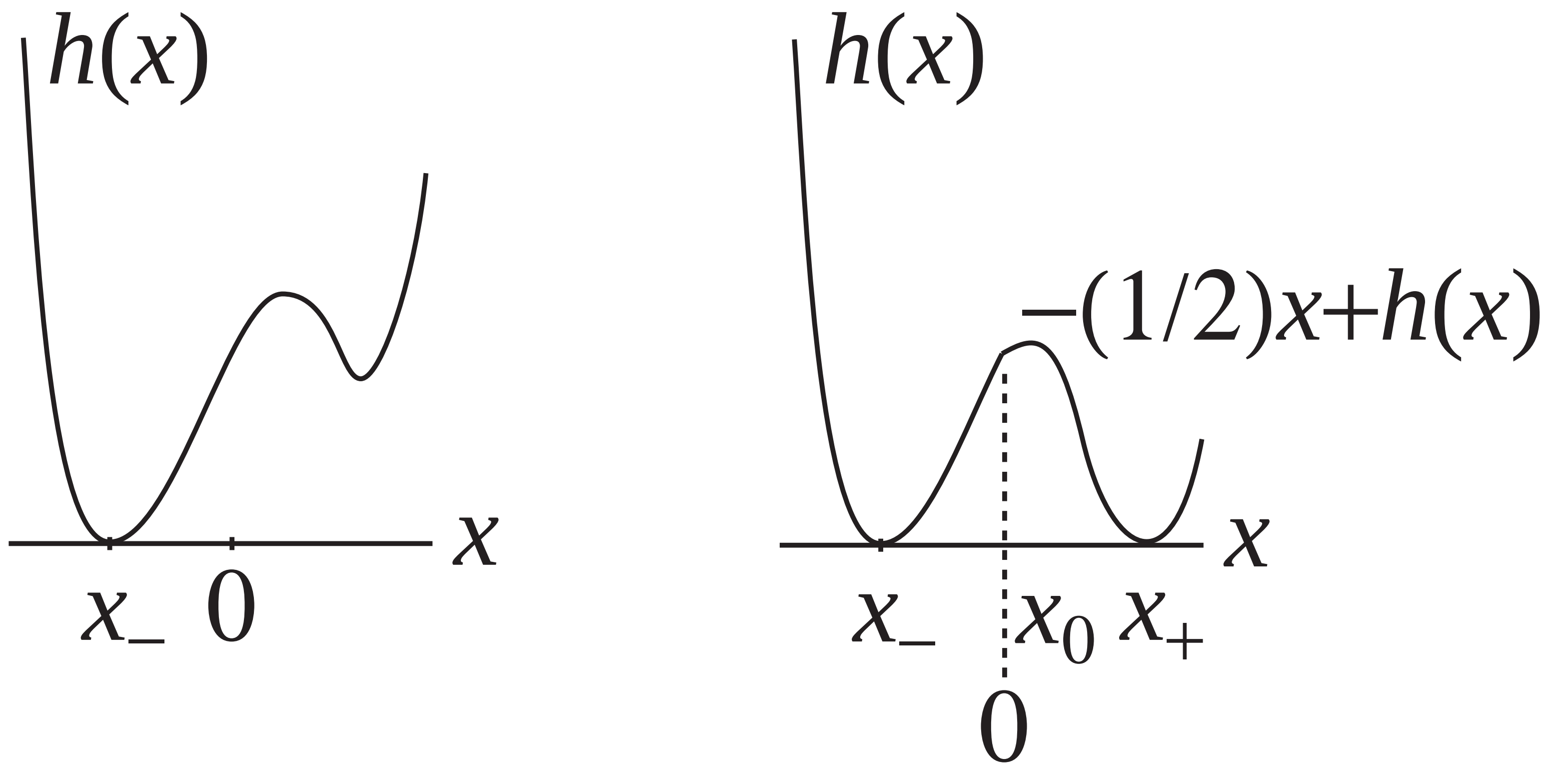
(10) $$u_{3\sigma} = u_4,$$

(11) $$u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3 = u_3(u_3^2 - u_1).$$

Equilibria of the fast system for $\varepsilon > 0$:

$$(u_1, 0, 0, 0) \text{ with } h'(u_1) = 0, \quad (u_1, 0, \pm u_1^{\frac{1}{2}}, 0) \text{ with } -\frac{1}{2}u_1 + h'(u_1) = 0.$$

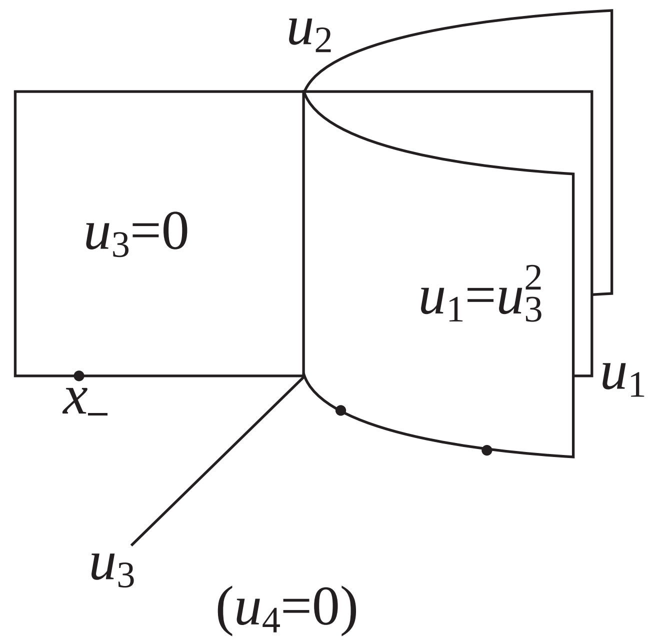
Assumptions on h:



Equilibria of the fast system

$$u_{1\sigma} = \varepsilon u_2,$$
$$u_{2\sigma} = \varepsilon g_x(u_1, u_3) = \varepsilon\left(-\frac{1}{2}u_3^2 + h'(u_1)\right),$$
$$u_{3\sigma} = u_4,$$
$$u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3 = u_3(u_3^2 - u_1)$$

for $\varepsilon > 0$:



$$(x_-, 0, 0, 0), \quad (x_0, 0, \pm x_0^{\frac{1}{2}}, 0), \quad (x_+, 0, \pm x_+^{\frac{1}{2}}, 0).$$

For each ε, the fast system has the first integral

$$H(u_1,u_2,u_3,u_4) = \frac{1}{2}u_2^2 + \frac{1}{2}u_4^2 - g(u_1,u_3).$$

Note:

$$H(x_-,0,0,0) = H(x_+,0,x_+^{\frac{1}{2}},0) = 0.$$

Goal: show that for small $\varepsilon > 0$, there is a heteroclinic solution of the fast system from $(x_-,0,0,0)$ to $(x_+,0,x_+^{\frac{1}{2}},0)$.

For $\varepsilon > 0$, $(x_-,0,0,0)$ and $(x_+,0,x_+^{\frac{1}{2}},0)$ are hyperbolic equilibria of the fast system with two negative eigenvalues and two positive eigenvalues.

The heteroclinic solution will correspond to an intersection of the 2-dimensional manifolds $W^u_\varepsilon(x_-,0,0,0)$ and $W^s_\varepsilon(x_+,0,x_+^{\frac{1}{2}},0)$ that is transverse within the 3-dimensional manifold $H^{-1}(0)$ (which is indeed a manifold away from equilibria).

Fast limit and slow systems

Set $\varepsilon = 0$ in the fast system to obtain the fast limit system:

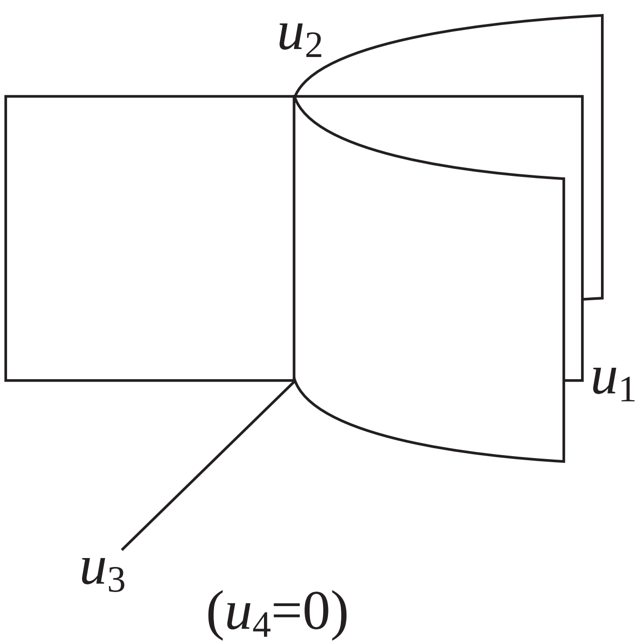
$$u_{1\sigma} = 0, \quad (12)$$

$$u_{2\sigma} = 0, \quad (13)$$

$$u_{3\sigma} = u_4, \quad (14)$$

$$u_{4\sigma} = g_y(u_1, u_3) = u_3(u_3^2 - u_1). \quad (15)$$

Equilibria (slow manifold):

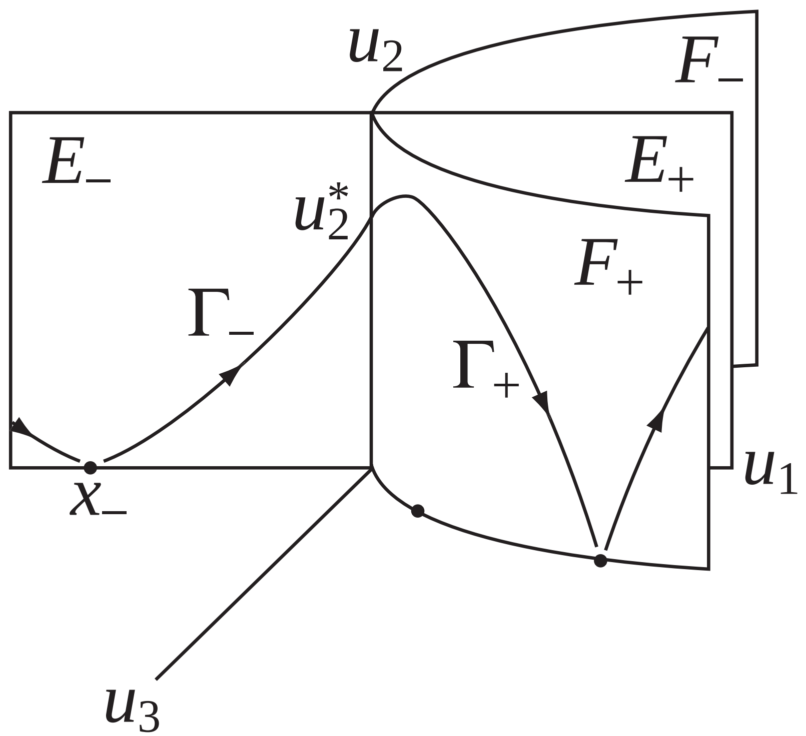


Three manifolds of normally hyperbolic equilibria:

$$E_- = \{(u_1, u_2, 0, 0) : u_1 < 0 \text{ and } u_2 \text{ arbitrary}\},$$

$$F_- = \{(u_1, u_2, -u_1^{\frac{1}{2}}, 0) : u_1 > 0 \text{ and } u_2 \text{ arbitrary}\},$$

$$F_+ = \{(u_1, u_2, u_1^{\frac{1}{2}}, 0) : u_1 > 0 \text{ and } u_2 \text{ arbitrary}\}.$$



Each has one positive eigenvalue and one negative eigenvalue. (On E_+ there are two pure imaginary eigenvalues. On the u_2-axis all eigenvalues are 0.)

Set $\varepsilon = 0$ in the slow system to obtain the slow limit system:

$$u_{1\tau} = u_2, \tag{16}$$

$$u_{2\tau} = g_x(u_1, u_3) = -\frac{1}{2}u_3^2 + h'(u_1), \tag{17}$$

$$0 = u_4, \tag{18}$$

$$0 = g_y(u_1, u_3) = u_3(u_3^2 - u_1). \tag{19}$$

$E_\pm$, $F_\pm$ are manifolds of solutions of (18)–(19). Equations (16)–(17) give the slow system on these manifolds.

Slow system on E_- ($u_1 < 0$, u_2 arbitrary):

$$u_{1\tau} = u_2, \tag{20}$$

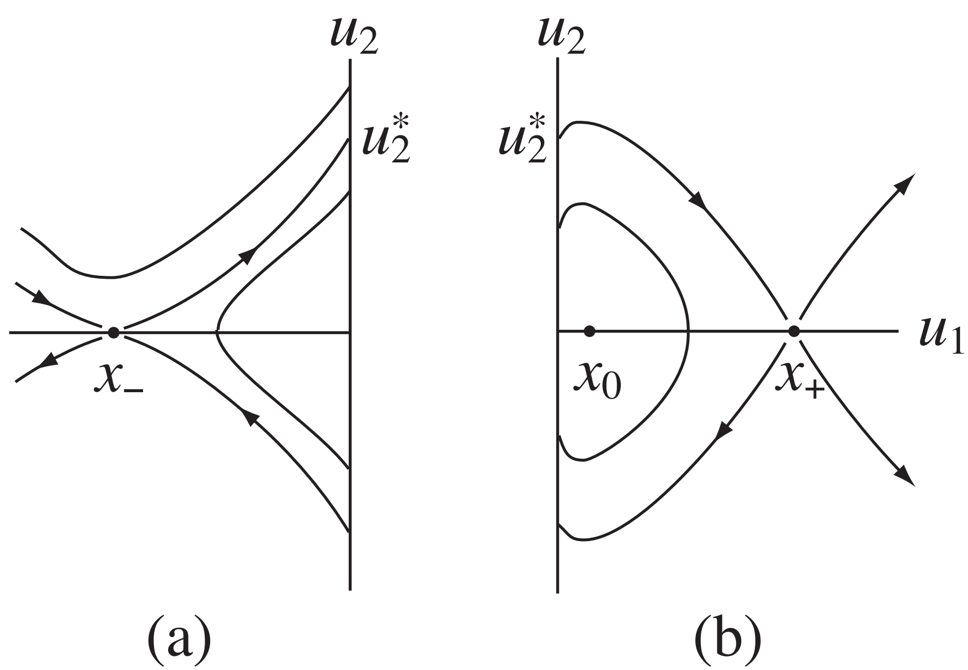
$$u_{2\tau} = g_x(u_1, 0) = h'(u_1). \tag{21}$$

Slow system on F_+ ($u_1 > 0$, u_2 arbitrary):

$$u_{1\tau} = u_2, \tag{22}$$

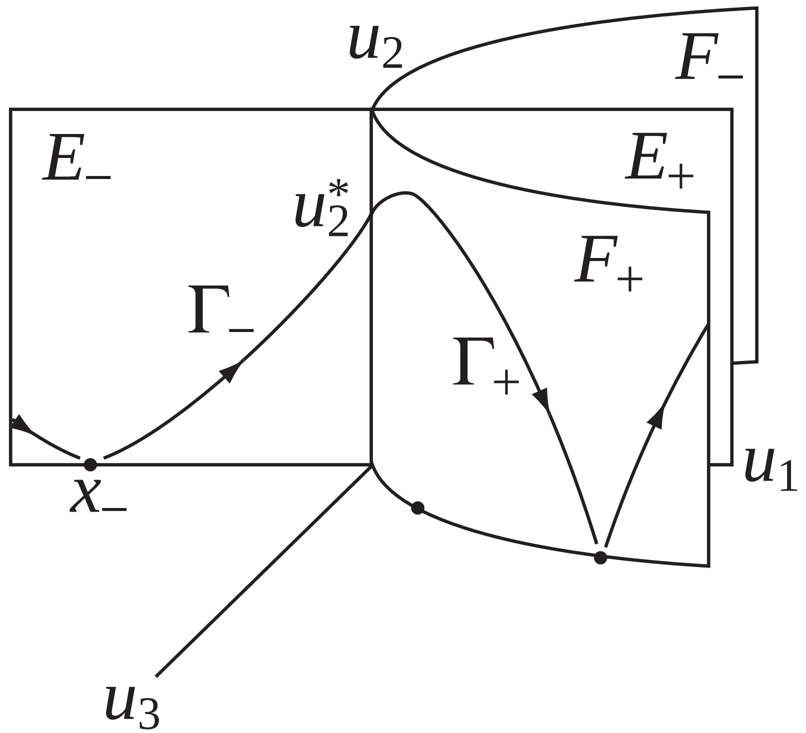
$$u_{2\tau} = g_x(u_1, u_1^{\frac{1}{2}}) = -\frac{1}{2}u_1 + h'(u_1). \tag{23}$$

Phase portraits of slow system on E_- and F_+ in $u_1 u_2$-coordinates, both extended to $u_1 = 0$:



- In (a), $(x_-, 0)$ is a hyperbolic saddle, and a branch of its unstable manifold meets the u_2 axis at a point $(0, u_2^*)$.
- In (b), $(x_+, 0)$ is a hyperbolic saddle, and a branch of its stable manifold meets the u_2 axis at the same point $(0, u_2^*)$.

Slow limit system on E_- and F_+:



Theorem 1. For small $\varepsilon > 0$, there is a heteroclinic solution of the fast system from $(x_-, 0, 0, 0)$ to $(x_+, 0, x_+^{\frac{1}{2}}, 0)$ that is close to $\Gamma_- \cup \Gamma_+$.

Blow-up

To the fast system append the equation $\varepsilon_\sigma = 0$:

$$u_{1\sigma} = \varepsilon u_2, \tag{24}$$

$$u_{2\sigma} = \varepsilon g_x(u_1, u_3) = \varepsilon(-\frac{1}{2}u_3^2 + h'(u_1)), \tag{25}$$

$$u_{3\sigma} = u_4, \tag{26}$$

$$u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3, \tag{27}$$

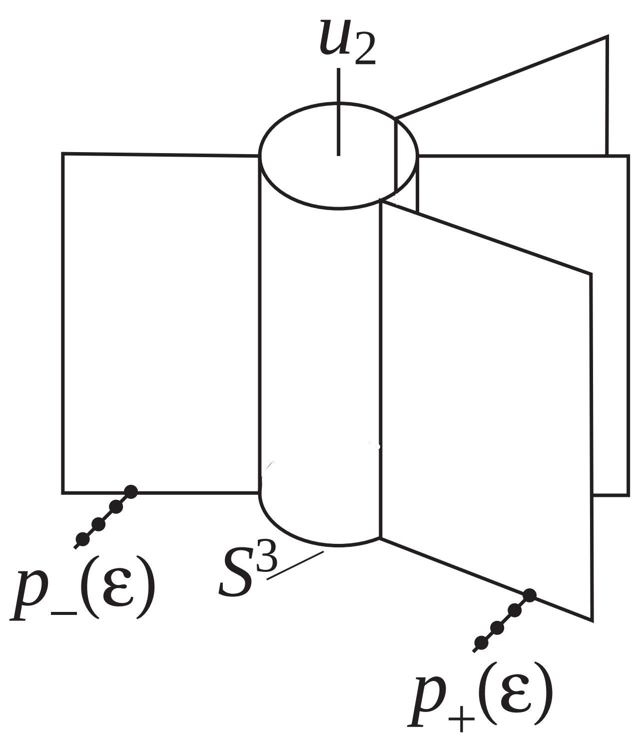
$$\varepsilon_\sigma = 0. \tag{28}$$

The u_2-axis consists of equilibria of (24)–(27) with $\varepsilon = 0$ that are not normally hyperbolic within $u_1u_2u_3u_4$-space

In $u_1u_2u_3u_4\varepsilon$-space, we shall it blow up to the product of the u_2-axis with a 3-sphere. The 3-sphere is a blow-up of the origin in $u_1u_3u_4\varepsilon$-space.

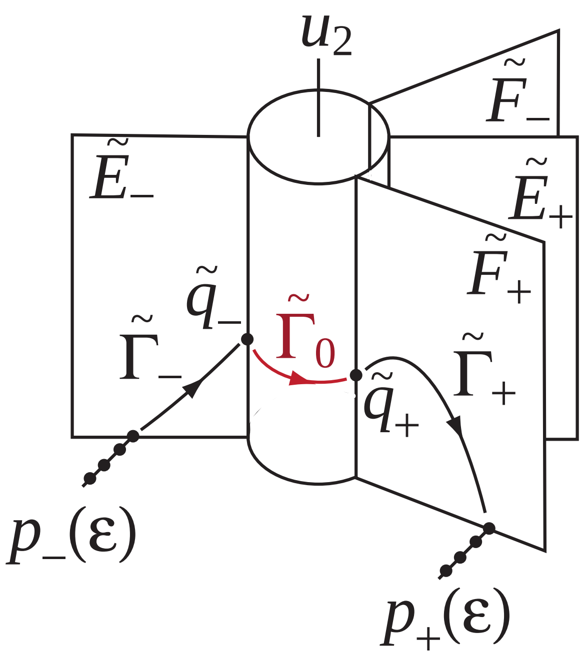
The blowup transformation is a map from $\mathbb{R} \times S^3 \times [0, \infty)$ to $u_1u_2u_3u_4\varepsilon$-space. Let $(u_2, (\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\varepsilon}), \bar{r})$ be a point of $\mathbb{R} \times S^3 \times [0, \infty)$; we have $\bar{u}_1{}^2 + \bar{u}_3{}^2 + \bar{u}_4{}^2 + \bar{\varepsilon}^2 = 1$. Then

$$u_1 = \bar{r}^2\bar{u}_1, \quad u_2 = u_2, \quad u_3 = \overline{ru}_3, \quad u_4 = \bar{r}^2\bar{u}_4, \quad \varepsilon = \bar{r}^3\bar{\varepsilon}. \tag{29}$$



Under this transformation (24)–(28) pulls back to a vector field X on $\mathbb{R}\times S^3\times[0,\infty)$ for which the cylinder $\bar{r}=0$ consists entirely of equilibria. The vector field we shall study is $\tilde{X}=\bar{r}^{-1}X$. Division by $\bar{r}$ desingularizes the vector field on the cylinder $\bar{r}=0$ but leaves it invariant.

Let $p_-(\varepsilon)$ (respectively $p_+(\varepsilon)$) be the unique point in $\mathbb{R}\times S^3\times[0,\infty)$ that corresponds to $(x_-,0,0,0,\varepsilon)$ (respectively $(x_+,0,x_+^{\frac{1}{2}},0,\varepsilon)$). We wish to show that for small $\varepsilon>0$ there is an integral curve of X from $p_-(\varepsilon)$ to $p_+(\varepsilon)$. Equivalently, we shall show that for small $\varepsilon>0$ there is an integral curve of $\tilde{X}$ from $p_-(\varepsilon)$ to $p_+(\varepsilon)$.



In blow-up space:

- $\tilde{\Gamma}_-$ corresponds to Γ_- and approaches a point $\tilde{q}_- = (u_2^*, \hat{q}_-, 0)$ on the blow-up cylinder.
- $\tilde{\Gamma}_+$ corresponds to Γ_+ and approaches a point $\tilde{q}_+ = (u_2^*, \hat{q}_+, 0)$ on the blow-up cylinder.
- On the blow-up cylinder, each 3-sphere $u_2 =$ constant is invariant.

Proposition 2. There is an integral curve $\tilde{\Gamma}_0$ of $\tilde{X}$ from $\tilde{q}_-$ to $\tilde{q}_+$ that lies in the 3-dimensional hemisphere given by $u_2 = u_2^*$, $\bar{r} = 0$, $\bar{\varepsilon} > 0$.

Theorem 3. For small $\varepsilon > 0$ there is an integral curve $\tilde{\Gamma}(\varepsilon)$ of $\tilde{X}$ from $p_-(\varepsilon)$ to $p_+(\varepsilon)$. As $\varepsilon \to 0$, $\tilde{\Gamma}(\varepsilon) \to \tilde{\Gamma}_- \cup \tilde{\Gamma}_0 \cup \tilde{\Gamma}_+$.

We shall need three charts on blow-up space:

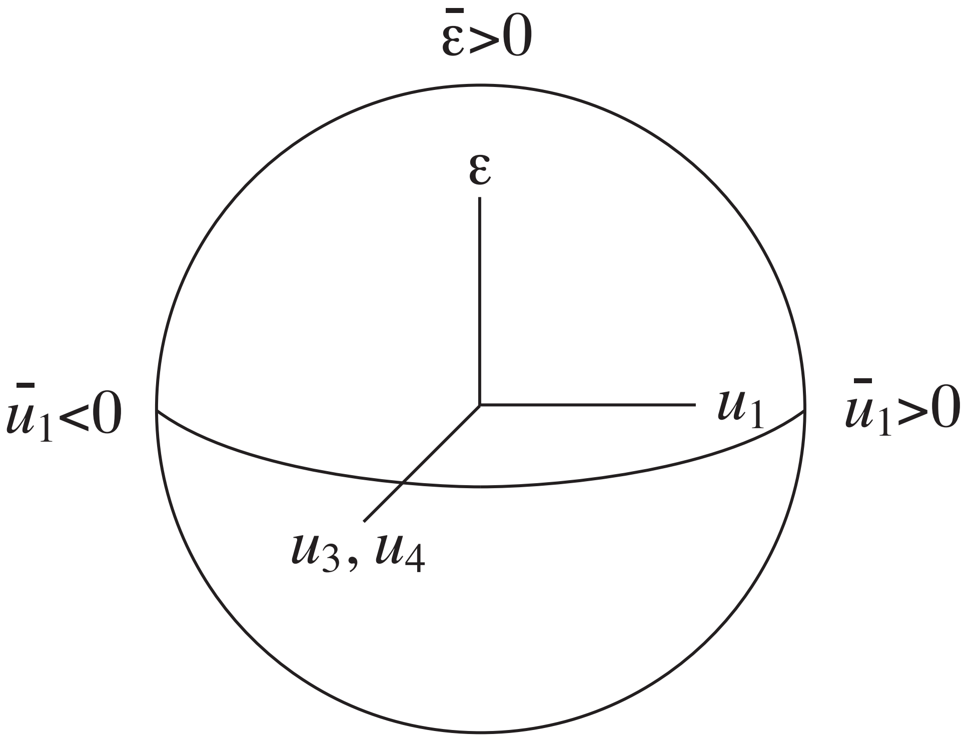


Chart for $\bar{\varepsilon} > 0$

On the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\bar{\varepsilon} > 0$, let

$$u_1 = r^2 b_1, \quad u_2 = u_2, \quad u_3 = r b_3, \quad u_4 = r^2 b_4, \quad \varepsilon = r^3, \tag{30}$$

with $r \geq 0$. After division by r, (24)–(28) becomes

$$b_{1s} = u_2, \tag{31}$$

$$u_{2s} = r^2\left(-\frac{1}{2} r^2 b_3^2 + h'(r^2 b_1)\right), \tag{32}$$

$$b_{3s} = b_4, \tag{33}$$

$$b_{4s} = b_3^3 - b_1 b_3, \tag{34}$$

$$r_s = 0. \tag{35}$$

Note 1: $r = 0$ implies $u_{2s} = 0$.

Note 2: $b_1 = \bar{u}_1 \bar{\varepsilon}^{-\frac{2}{3}}$, u_2, $b_3 = \bar{u}_3 \bar{\varepsilon}^{-\frac{1}{3}}$, $b_4 = \bar{u}_4 \bar{\varepsilon}^{-\frac{2}{3}}$, and $r = \bar{r}\bar{\varepsilon}^{\frac{1}{3}}$.

Note 3: (31)–(35) actually represents the vector field

$$r^{-1} X = \bar{r}^{-1} \bar{\varepsilon}^{-\frac{1}{3}} X = \bar{\varepsilon}^{-\frac{1}{3}} \tilde{X}$$

Chart for $\overline{u_1} < 0$

On the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\overline{u_1} < 0$, let

$$u_1 = -v^2, \quad u_2 = u_2, \quad u_3 = v a_3, \quad u_4 = v^2 a_4, \quad \varepsilon = v^3 \delta, \tag{36}$$

with $v \geq 0$. After division by v, (24)–(28) becomes

$$v_t = -\frac{1}{2} v \delta u_2, \tag{37}$$

$$u_{2t} = v^2 \delta(-\frac{1}{2} v^2 a_3^2 + h'(-v^2)), \tag{38}$$

$$a_{3t} = a_4 + \frac{1}{2} \delta u_2 a_3, \tag{39}$$

$$a_{4t} = a_3^3 + a_3 + \delta u_2 a_4, \tag{40}$$

$$\delta_t = \frac{3}{2} \delta^2 u_2. \tag{41}$$

Note 1: $v = 0$ implies $u_{2t} = 0$.

Note 2: $v = \overline{r}(-\overline{u_1})^{\frac{1}{2}}$, u_2, $a_3 = \overline{u_3}(-\overline{u_1})^{-\frac{1}{2}}$, $a_4 = -\overline{u_4}\,\overline{u_1}^{-1}$, and $\delta = \overline{\varepsilon}(-\overline{u_1})^{-\frac{3}{2}}$.

Note 3: (37)–(41) actually represents the vector field

$$v^{-1} X = \overline{r}^{-1}(-\overline{u_1})^{-\frac{1}{2}} X = (-\overline{u_1})^{-\frac{1}{2}} \tilde{X}$$

Chart for $\overline{u_1} > 0$

On the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\overline{u_1} > 0$, let

$$u_1 = w^2, \quad u_2 = u_2, \quad u_3 = wc_3, \quad u_4 = w^2 c_4, \quad \varepsilon = w^3\gamma. \tag{42}$$

with $w \geq 0$. After division by w, (24)–(28) becomes

$$w_t = \frac{1}{2} w\gamma u_2, \tag{43}$$

$$u_{2t} = w^2\gamma(-\frac{1}{2}w^2c_3^2 + h'(w^2)), \tag{44}$$

$$c_{3t} = c_4 - \frac{1}{2}\gamma u_2 c_3, \tag{45}$$

$$c_{4t} = c_3^3 - c_3 - \gamma u_2 c_4, \tag{46}$$

$$\gamma_t = -\frac{3}{2}\gamma^2 u_2. \tag{47}$$

Note 1: $w = 0$ implies $u_{2t} = 0$.

Note 2: $w = \overline{r}\overline{u_1}^{\frac{1}{2}}$, u_2, $c_3 = \overline{u_3}\,\overline{u_1}^{-\frac{1}{2}}$, $c_4 = \overline{u_4}\,\overline{u_1}^{-1}$, and $\gamma = \overline{\varepsilon}\,\overline{u_1}^{-\frac{3}{2}}$.

Note 3: (43)–(47) actually represents the vector field

$$w^{-1}X = \overline{r}^{-1}\overline{u_1}^{-\frac{1}{2}}X = \overline{u_1}^{-\frac{1}{2}}\tilde{X}$$

Construction of the inner solution $\tilde{\Gamma}_0$

Let $\hat{X}$ denote the restriction of the vector field $\tilde{X}$ to the invariant 3-sphere $M = \{u_2^*\} \times S^3 \times \{0\}$, $S^3 = \{(\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\varepsilon}) : \bar{u}_1^2 + \bar{u}_3^2 + \bar{u}_4^2 + \bar{\varepsilon}^2 = 1\}$.

Chart on the open subset of M with $\bar{u}_1 < 0$: $a_3 = \bar{u}_3(-\bar{u}_1)^{-\frac{1}{2}}$, $a_4 = -\bar{u}_4\bar{u}_1^{-1}$, $\delta = \bar{\varepsilon}(-\bar{u}_1)^{-\frac{3}{2}}$. In this chart, the vector field $(-\bar{u}_1)^{-\frac{1}{2}}\hat{X}$ is

(48) $$a_{3t} = a_4 + \frac{1}{2}\delta u_2^* a_3,$$

(49) $$a_{4t} = a_3^3 + a_3 + \delta u_2^* a_4,$$

(50) $$\delta_t = \frac{3}{2}\delta^2 u_2^*.$$

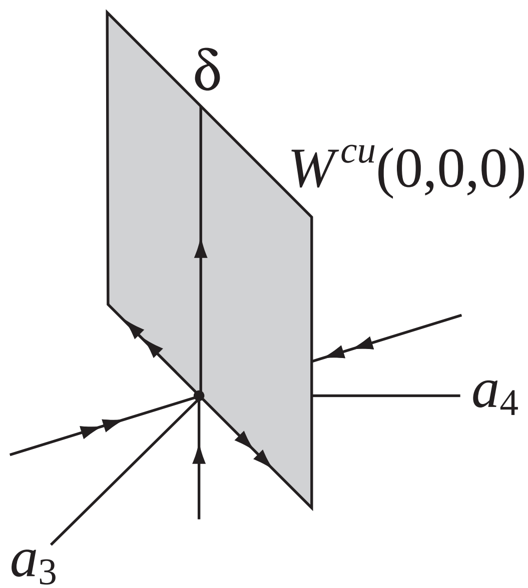


Chart on the open subset of M with $\bar{u}_1 > 0$: $c_3 = \bar{u}_3\bar{u}_1^{-\frac{1}{2}}$, $c_4 = \bar{u}_4\bar{u}_1^{-1}$, $\gamma = \bar{\varepsilon}\bar{u}_1^{-\frac{3}{2}}$. In this chart, the vector field $\bar{u}_1^{-\frac{1}{2}}\hat{X}$ is

(51) $$c_{3t} = c_4 - \frac{1}{2}\gamma u_2^* c_3,$$

(52) $$c_{4t} = c_3^3 - c_3 - \gamma u_2^* c_4,$$

(53) $$\gamma_t = -\frac{3}{2}\gamma^2 u_2^*.$$

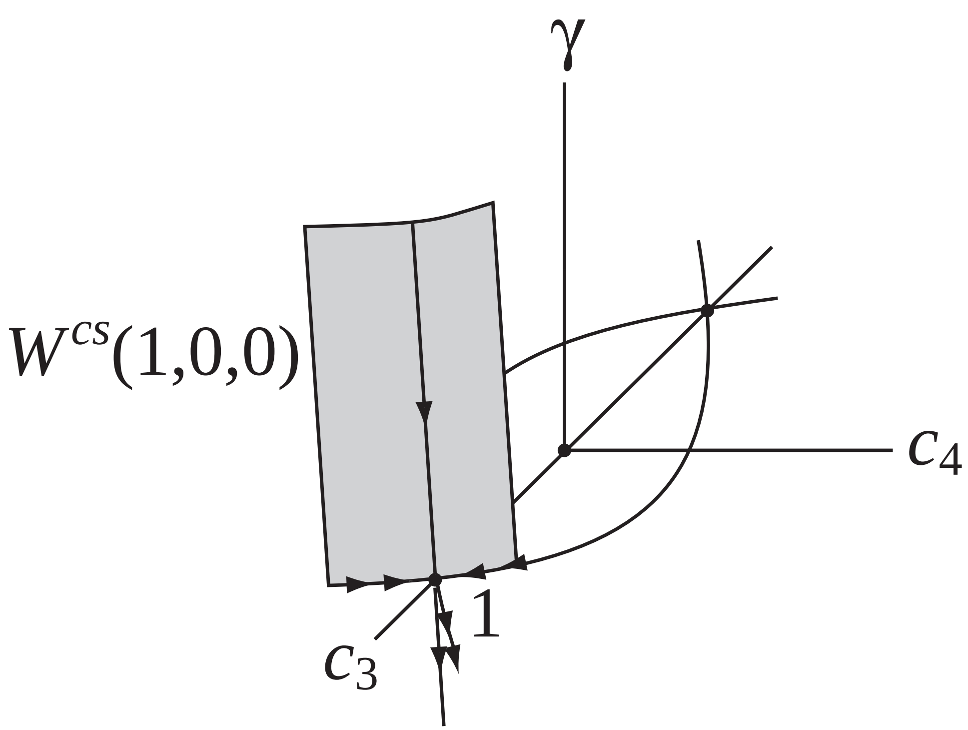


Chart on the open subset of M with $\bar{\varepsilon} > 0$: $b_1 = \bar{u}_1\bar{\varepsilon}^{-\frac{2}{3}}$, $b_3 = \bar{u}_3\bar{\varepsilon}^{-\frac{1}{3}}$, $b_4 = \bar{u}_4\bar{\varepsilon}^{-\frac{2}{3}}$. In this chart, the vector field $\bar{\varepsilon}^{-\frac{1}{3}}\hat{X}$ is

(54) $$b_{1s} = u_2^*,$$
(55) $$b_{3s} = b_4,$$
(56) $$b_{4s} = b_3^3 - b_1 b_3 = b_3(b_3^2 - b_1).$$

The solution of (54) with $b_1(0) = 0$ is $b_1 = u_2^* s$. Substitute into (56) and combining (55) and (56) into a second-order equation:

(57) $$b_{3ss} = b_3(b_3^2 - u_2^* s)$$

By Sourdis and Fife, (57) has a solution $b_3(s)$ with $b_{3s} > 0$ such that

(S1) $b_3(s) = O\left(|s|^{-\frac{1}{4}} e^{-\frac{2}{3}(u_2^*)^{\frac{1}{2}}|s|^{\frac{3}{2}}}\right)$ as $s \to -\infty$,
(S2) $b_3(s) = (u_2^* s)^{\frac{1}{2}} + O(s^{-\frac{5}{2}})$ as $s \to \infty$,
(S3) $b_{3s}(s) \leq C|s|^{-\frac{1}{2}}$, $s \neq 0$.

$(u_2^* s, b_3(s), b_{3s}(s))$ is a solution of (54)–(56). It represents an intersection of $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ in the 3-sphere M.

Transversality

$W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ are 2-dimensional submanifolds of the 3-sphere M.

Let $\tilde{\Gamma}_0 = (u_2^*, \hat{\Gamma}_0, 0)$. They intersect along $\hat{\Gamma}_0$.

Proposition 4. $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ intersect transversally within M along $\hat{\Gamma}_0$.

Proof. The linearization of

$$\begin{aligned} b_{1s} &= u_2^*, \\ b_{3s} &= b_4, \\ b_{4s} &= b_3^3 - b_1 b_3 \end{aligned}$$

along $(u_2^* s, b_3(s), b_{3s}(s))$ is

$$\begin{pmatrix} B_{1s} \\ B_{3s} \\ B_{4s} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b_3(s) & 3b_3(s)^2 - u_2^* s & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_3 \\ B_4 \end{pmatrix}. \tag{58}$$

We must show there are no solutions with appropriate behavior at $s = \pm\infty$ other than multiples of (u_2^*, b_{3s}, b_{3ss}).

There is a complementary 2-dimensional space of solutions of (58) with $B_1(s) = 0$ and $(B_3(s), B_4(s))$ a solution of

(59)
$$\begin{pmatrix} B_{3s} \\ B_{4s} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3b_3(s)^2 - u_2^* s & 0 \end{pmatrix} \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}$$

We must show that no nontrivial solution has appropriate behavior at $s = \pm\infty$.

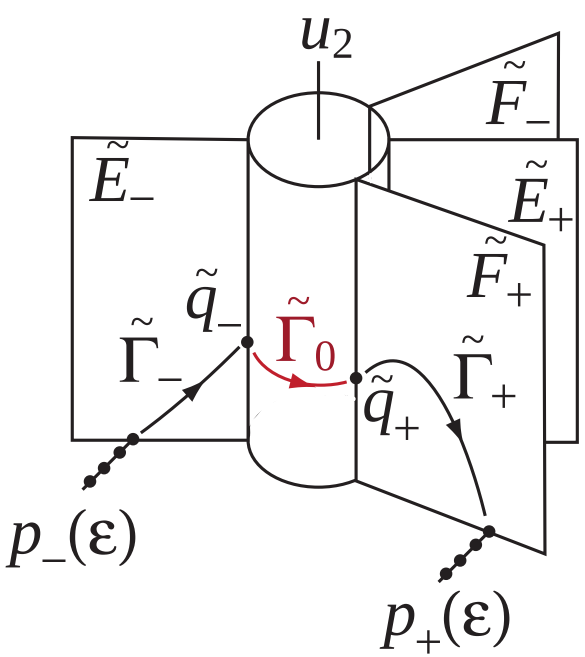
(59) is equivalent to the second order linear system

(60)
$$B_{3ss} = (3b_3(s)^2 - u_2^* s)B_3.$$

By Alikakos, Bates, Cahn, Fife, Fusco, and Tanoglu, *Analysis of the corner layer problem in anisotropy*, Discrete Contin. Dyn. Syst. **6** (2006), 237–255, (60) has no nontrivial solutions in L^2, hence no solution with the correct asymptotic behavior.

Proof of Theorem 3

Theorem 3. For small $\varepsilon > 0$ there is an integral curve $\tilde{\Gamma}(\varepsilon)$ of $\tilde{X}$ from $p_-(\varepsilon)$ to $p_+(\varepsilon)$. As $\varepsilon \to 0$, $\tilde{\Gamma}(\varepsilon) \to \tilde{\Gamma}_- \cup \tilde{\Gamma}_0 \cup \tilde{\Gamma}_+$.

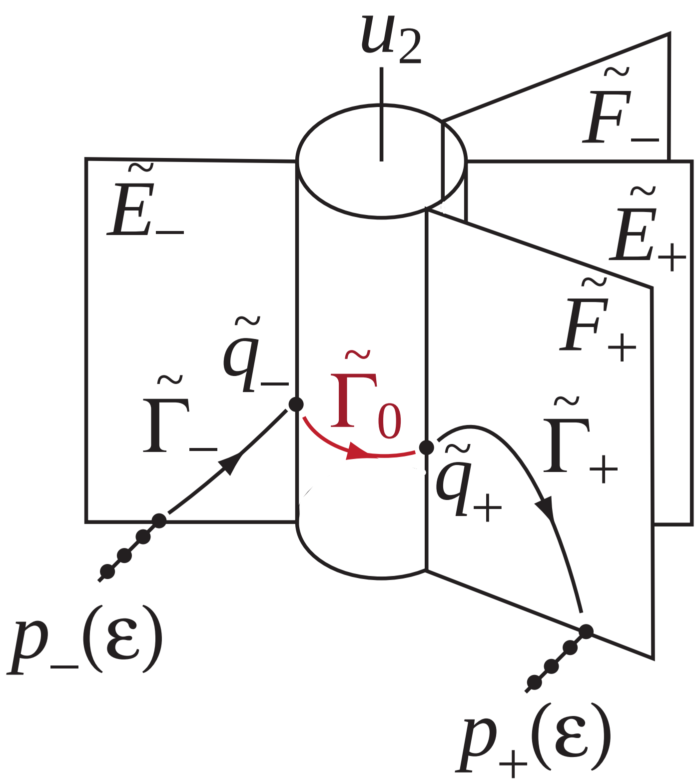


Recall: for each ε, the fast system has the first integral

$$H(u_1,u_2,u_3,u_4) = \frac{1}{2}u_2^2 + \frac{1}{2}u_4^2 - \left(\frac{1}{4}u_3^4 - \frac{1}{2}u_1u_3^2 + h(u_1)\right).$$

H gives rise to a first integral for $\tilde{H}$ on blow-up space:

$$\tilde{H}(u_2,(\bar{u}_1,\bar{u}_3,\bar{u}_4,\bar{\varepsilon}),\bar{r}) = \frac{1}{2}u_2^2 + \bar{r}^4\left(\frac{1}{2}\bar{u}_4^2 - \frac{1}{4}\bar{u}_3^4 + \frac{1}{2}\bar{u}_1\bar{u}_3^2\right) - h(\bar{r}^2\bar{u}_1).$$



Let N_ε denote the set of points in blow-up space at which $\tilde{H} = 0$ and $\bar{r}^3\bar{\varepsilon} = \varepsilon$.

Away from equilibria of $\tilde{X}$, each N_ε is a manifold of dimension 3.

For the vector field $\tilde{X}$ and $\varepsilon > 0$, the equilibria $p_-(\varepsilon)$ and $p_+(\varepsilon)$ have 2-dimensional unstable and stable manifolds.

We will prove the theorem by showing that for small $\varepsilon > 0$, $W^u(p_-(\varepsilon))$ and $W^s(p_+(\varepsilon))$ have a nonempty intersection that is transverse within N_ε.

Chart for $\overline{u}_1 < 0$:

$$\begin{aligned} v_t &= -\frac{1}{2}v\delta u_2, \\ u_{2t} &= v^2\delta(-\frac{1}{2}v^2a_3^2 + h'(-v^2)), \\ a_{3t} &= a_4 + \frac{1}{2}\delta u_2 a_3, \\ a_{4t} &= a_3^3 + a_3 + \delta u_2 a_4, \\ \delta_t &= \frac{3}{2}\delta^2 u_2. \end{aligned}$$

The 3-dimensional space $a_3 = a_4 = 0$ is invariant, and is normally hyperbolic near the plane of equilibria $a_3 = a_4 = \delta = 0$. One eigenvalue is positive, one is negative.

The plane of equilibria corresponds to E_-. Normal hyperbolicity within $\delta = 0$ is *not* lost at $v = 0$, which corresponds to $u_1 = 0$.

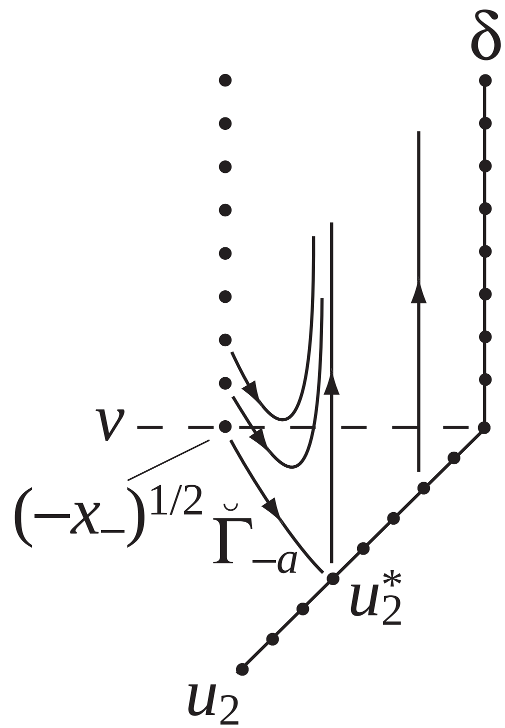
Restrict to $a_3 = a_4 = 0$ and divide by δ:

(61) $$\dot{v} = -\frac{1}{2}vu_2,$$

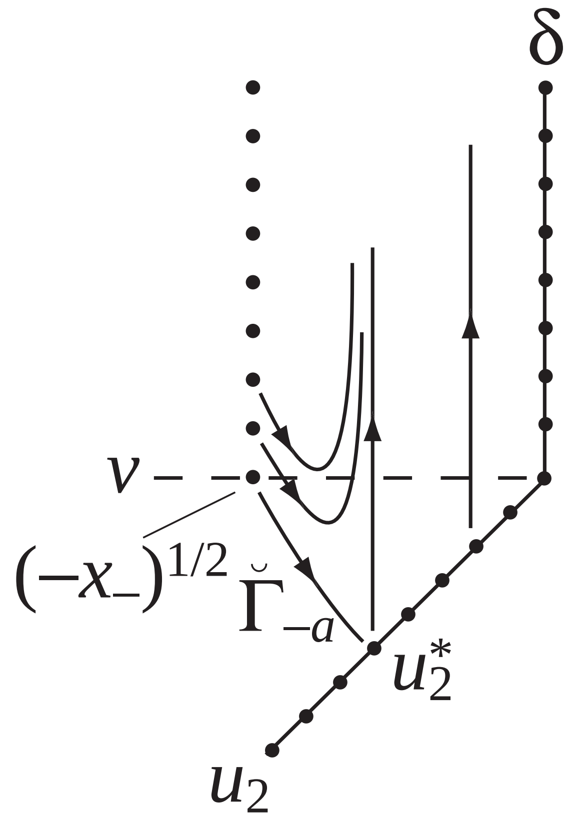
(62) $$\dot{u}_2 = v^2 h'(-v^2),$$

(63) $$\dot{\delta} = \frac{3}{2}\delta u_2.$$

$$\dot{v} = -\frac{1}{2} v u_2,$$
$$\dot{u}_2 = v^2 h'(-v^2),$$
$$\dot{\delta} = \frac{3}{2}\delta u_2.$$



Equilibria on the lines $\{(v,u_2,\delta) : v = (-x_-)^{\frac{1}{2}}, u_2 = 0\}$ and $\{(v,u_2,\delta) : v = \delta = 0, u_2 \neq 0\}$ are normally hyperbolic, with one positive eigenvalue and one negative eigenvalue.



Lemma 4. As $\delta_0 \to 0+$, $W^u((-x_-)^{\frac{1}{2}}, 0, \delta_0)$ approaches $W^u(0, u_2^*, 0)$ in the C^1 topology. (Both have dimension 1.)

Lemma 5. In the chart for $\bar{u}_1 < 0$, as $\delta_0 \to 0+$, $W^u((-x_-)^{\frac{1}{2}}, 0, 0, 0, \delta_0)$ approaches the manifold of unstable fibers over $W^u(0, u_2^*, 0)$ in the C^1 topology. (Both have dimension 2.)

The latter corresponds to $W^{cu}(\hat{q}_1)$ in $M = \{u_2^*\} \times S^3 \times \{0\}$.

Chart for $\bar{u}_1 > 0$:

$$\begin{aligned}
w_t &= \frac{1}{2} w \gamma u_2, \\
u_{2t} &= w^2 \gamma (-\frac{1}{2} w^2 c_3^2 + h'(w^2)), \\
c_{3t} &= c_4 - \frac{1}{2} \gamma u_2 c_3, \\
c_{4t} &= c_3^3 - c_3 - \gamma u_2 c_4, \\
\gamma_t &= -\frac{3}{2} \gamma^2 u_2.
\end{aligned}$$

The equilibria of the plane $c_3 = 1$, $c_4 = \gamma = 0$ have, transverse to the plane, one positive eigenvalue, one negative eigenvalue, one zero eigenvalue.

Therefore this plane is part of a 3-dimensional normally hyperbolic invariant manifold S_2, with equations

$$c_3 = 1 + \gamma^2 \tilde{c}_3(w, u_2, \gamma), \quad c_4 = \gamma \tilde{c}_4(w, u_2, \gamma).$$

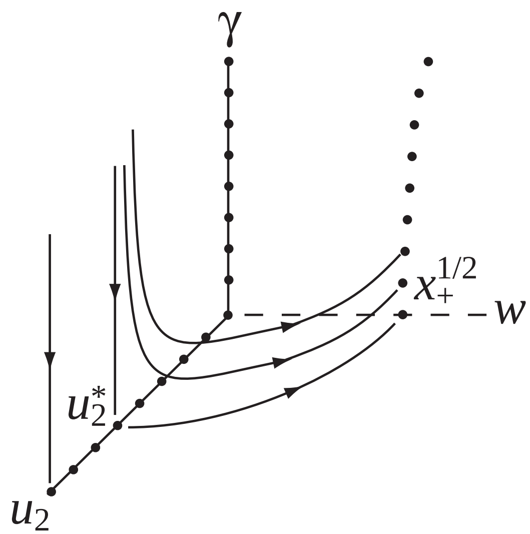
The plane of equilibria corresponds to F_+. Normal hyperbolicity within $\gamma = 0$ is *not* lost at $w = 0$, which corresponds to $u_1 = 0$.

Restrict to S_2 and divide by γ:

$$w_t = \frac{1}{2} w u_2, \tag{64}$$

$$u_{2t} = w^2\left(-\frac{1}{2} w^2 (1+\gamma^2 \tilde{c}_3)^2 + h'(w^2)\right), \tag{65}$$

$$\gamma_t = -\frac{3}{2}\gamma u_2. \tag{66}$$



Lemma 6. As $\gamma_0 \to 0+$, $W^s(x_+^{\frac{1}{2}}, 0, \gamma_0)$ approaches $W^s(0, u_2^*, 0)$ in the C^1 topology. (Both have dimension 1.)

Lemma 7. In the chart for $\overline{u_1} > 0$, as $\gamma_0 \to 0+$, $W^s(x_+^{\frac{1}{2}}, 0, 1, 0, \gamma_0)$ approaches the manifold of stable fibers over $W^s(0, u_2^*, 0)$ in the C^1 topology. (Both have dim 2.)

The latter corresponds to $W^{cs}(\hat{q}_+)$ in $M = \{u_2^*\} \times S^3 \times \{0\}$.

In blow-up space:

Lemma 8. As $\varepsilon \to 0+$, $W^u(p_-(\varepsilon))$ approaches $W^{cu}(\hat{q}_-)$ in the C^1 topology.

Lemma 9. As $\varepsilon \to 0+$, $W^s(p_+(\varepsilon))$ approaches $W^{cs}(\hat{q}_+)$ in the C^1 topology.

By Proposition 4: $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ meet transversally within the 3-sphere $\bar{r} = 0$, $u_2 = u_2^*$, which is N_0.

In the chart for $\bar{\varepsilon} > 0$, H corresponds to

$$H_b(b_1, u_2, b_3, b_4, r) = \frac{1}{2}u_2^2 + r^4(\frac{1}{2}b_4^2 - \frac{1}{4}b_3^4 + \frac{1}{2}b_1 b_3^2) + h(r^2 b_1).$$

N_0 corresponds to the set of (b_1, u_2, b_3, b_4, r) such that $H_b = 0$ and $r = 0$. The functions H_b and r have linearly independent gradients provided $u_2 \neq 0$. Therefore, where $u_2 \neq 0$, the sets $N_{\varepsilon^{\frac{1}{3}}} = N_r$ depend smoothly on r. Since $W^c u(\hat{q}_-)$ and $W^c s(\hat{q}_+)$ meet transversally within N_0, it follows that $W^u(p_-(\varepsilon))$ and $W^s(p_+(\varepsilon))$ meet transversally within N_ε for ε small.