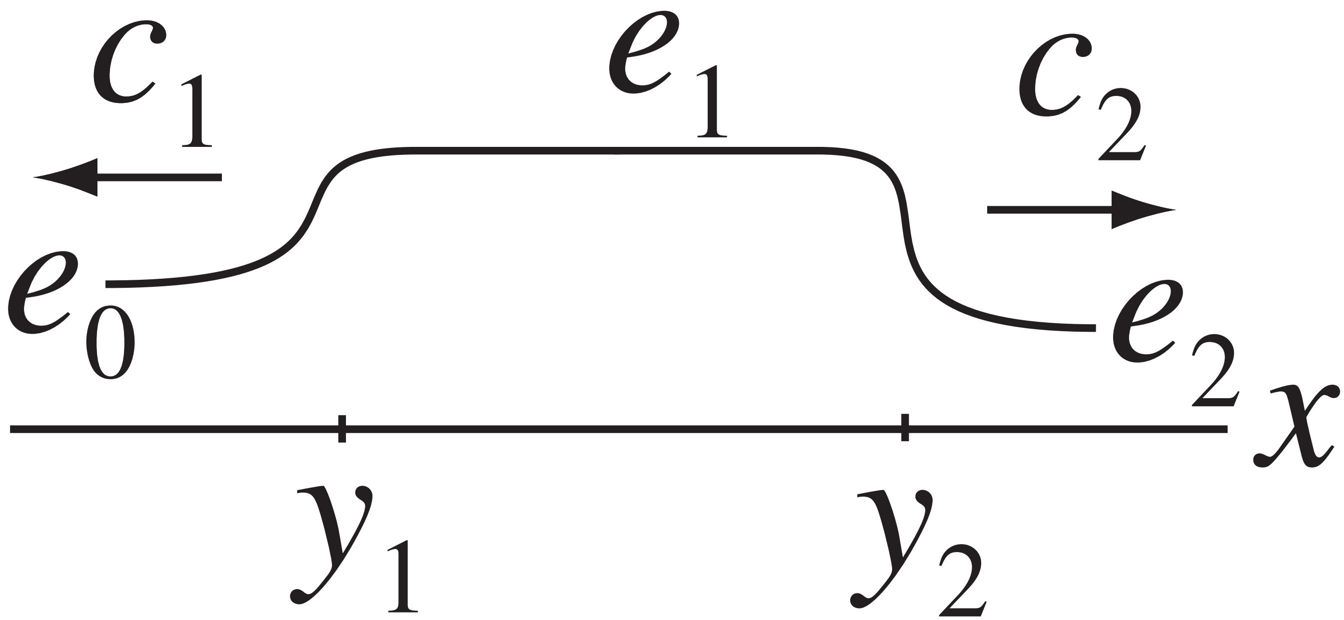
Concatenated Traveling Waves



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I. Traveling waves for reaction-diffusion equations

II. Concatenated traveling waves

References

X.-B. Lin and S., *Stability of concatenated traveling waves*, J. Dynam. Differential Equations, to appear.

X.-B. Lin and S., *Stability of concatenated traveling waves: Alternate approaches*, J. Differential Eqs. **259** (2015), 3144–3177.

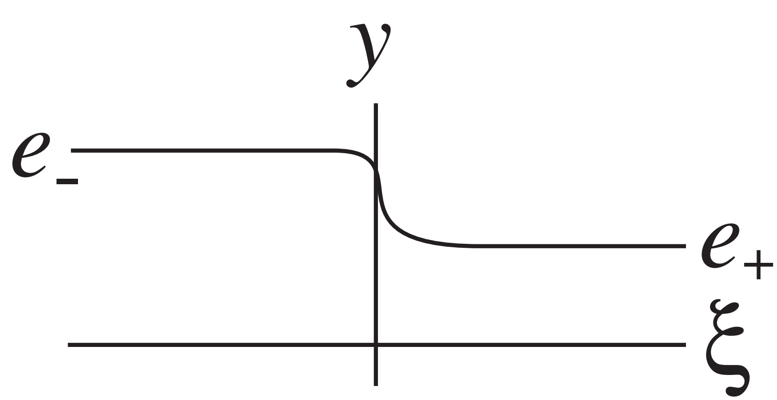
I. Traveling waves for reaction-diffusion equations

Reaction-diffusion equation:

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n$$

Traveling wave:

$$q(\xi), \quad \xi = x - ct, \quad q(\pm\infty) = e_\pm, \quad q(\xi) \to e_\pm \text{ exponentially as } \xi \to \pm\infty$$



Change of variables: $x \to \xi = x - ct$:

$$u_t = u_{\xi\xi} + cu_\xi + f(u)$$

The traveling wave $q(\xi)$ is a stationary solution: it satisfies

$$0 = u_{\xi\xi} + cu_\xi + f(u)$$

To study stability, linearize at the stationary solution $q(\xi)$:

$$U_t = LU = U_{\xi\xi} + cU_\xi + Df(q(\xi))U$$

Regard L as a linear operator on $L^2(\mathbb{R})$ (for example).

Spectrum: $\lambda \in \sigma(L)$ if $L - \lambda I$ does not have a bounded inverse.

The traveling wave is **spectrally stability** if for some $\nu < 0$,

$$\sigma(L) \subset \{\lambda : \mathrm{Re}\,\lambda < \nu\}$$

except for a simple eigenvalue at 0.

(The eigenfunction is q'. Reflects the fact that traveling waves can be shifted.)

Consequences of spectral stability:

(1) **Linearized stability**: Every solution of $U_t = LU$ decays exponentially to a multiple of q'.

(2) **Nonlinear stability**: Every solution of $u_t = u_{\xi\xi} + cu_\xi + f(u)$ that starts near q decays exponentially to a shift of q.

Checking spectral stability

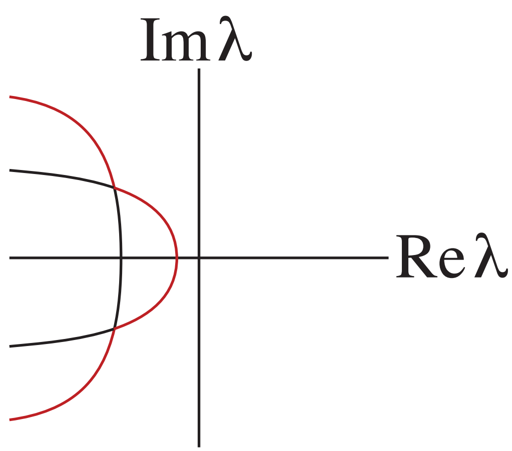
$$U_t = LU = U_{\xi\xi} + cU_\xi + Df(q(\xi))U, \quad q(\xi) \to q_\pm \text{ as } \xi \to \pm\infty.$$

1. Find the **essential spectrum** of L by finding the **dispersion relation** for the constant coefficient equations

$$U_t = L_\pm U = U_{\xi\xi} + cU_\xi + Df(q_\pm))U.$$

Get collection of curves $\lambda = \lambda_i(\mu)$. $L - \lambda I$ is not Fredholm iff λ belongs to one of these curves.

Rightmost boundary $=$ **Fredholm border** of L.



2. **To the right of the Fredholm border** of L, write $(L-\lambda I)U = 0$ as a first-order system:

$$\begin{pmatrix} U \\ V \end{pmatrix}_\xi = \begin{pmatrix} 0 & I \\ \lambda I - Df(q(\xi)) & -cI \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \quad \text{or} \quad X_\xi = A(\xi,\lambda)X$$

$A(-\infty,\lambda)$ and $A(\infty,\lambda)$ are hyperbolic matrices of the same type.

$X_\xi = A(\xi)X$ has an **exponential dichotomy** on an interval if there exist two complementary spaces of solutions

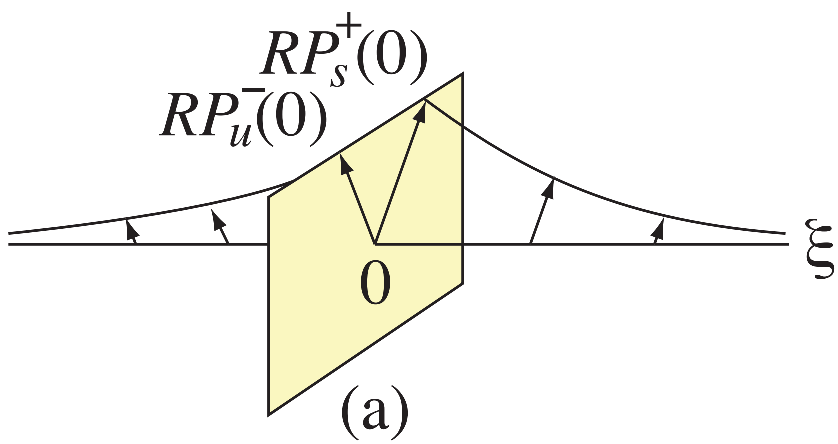
- one consists of solutions that decrease exponentially at the right;
- the other consists of solutions that decrease exponentially at the left.

Important fact: exponential dichotomies persist under perturbation.

For λ to the right of the Fredholm border, $X_\xi = A(\xi,\lambda)X$ has

- exponential dichotomy on $(-\infty,0]$, with projections $P_s^-(\xi)+P_u^-(\xi) = I$;
- exponential dichotomy on $[0,\infty)$, with projections $P_s^+(\xi)+P_u^+(\xi) = I$.

Two cases:



(a) $RP_u^-(0,\lambda)$ **and** $RP_s^+(0,\lambda) = \{0\}$ **are complementary: exponential dichotomy on** $\mathbb{R}$**,** $\lambda \notin \sigma(L)$.

One can use the dichotomy and variation of constants to invert the operator:

$$(L - \lambda I)U = h \quad \Leftrightarrow \quad X_\xi = A(\lambda, u)X_\xi + (0, h) \quad \Leftrightarrow$$

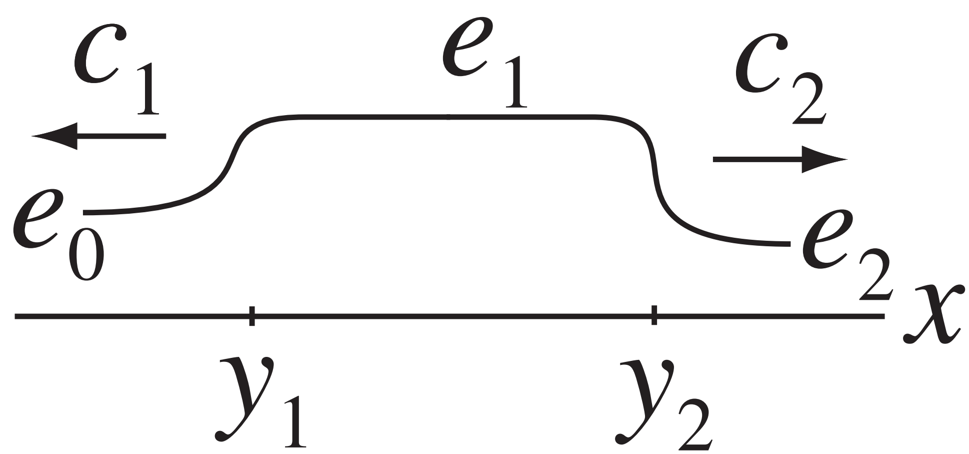
$$X(\xi) = \int_{-\infty}^{\xi} T(\xi, \eta, \lambda) P_s^+(\eta)(0, h(\eta))\, d\eta + \int_{\infty}^{\xi} T(\xi, \eta, \lambda) P_u^-(\eta)(0, h(\eta))\, d\eta.$$

(b) $RP_u^-(0,\lambda)$ **and** $RP_s^+(0,\lambda)$ **intersect: no exponential dichotomy on** $\mathbb{R}$**,** λ **is an eigenvalue of** L.

Case (b) occurs for $\lambda = 0$**.**

II. Concatenated traveling waves

Approximate picture:



Concatenated wave structure with two waves:

(1) $q_1(x - c_1 t)$ connects e_0 to e_1,

(2) $q_2(x - c_2 t)$ connects e_1 to e_2,

(3) $c_1 < c_2$.

Assume each traveling wave is spectrally stable.

Are there solutions that look like the picture? Are they stable?

Answers: yes and yes.

Doug Wright, *Separating dissipative pulses: the exit manifold*, J. Dynam. Differential Equations **21** (2009), 315–328.

Sabrina Selle, *Decomposition and stability of multifronts and multipulses*, thesis, University of Bielefeld, 2009.

Idea:

- Look for solutions near **the sum**

$$u = q_1(x - y_1 - c_1 t) + q_2(x - y_2 - c_2 t) - e_1, \quad y_1 << y_2.$$

We looked for an alternate approach that would not "smear" the the effect of each wave on the other.

We hoped a different approach would be easier to use with less restrictive assumptions.

Our approach is based on concatenated waves and spatial dynamics (Laplace transform and exponential dichotomies).

Previous uses of Laplace transform and exponential dichotomies to study stability of traveling waves:

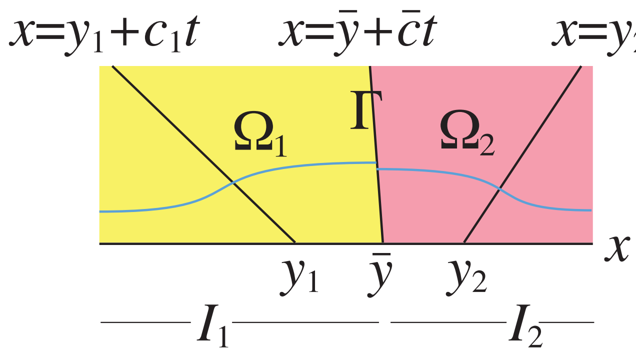
Xiao-Biao Lin, *Local and global existence of multiple waves near formal approximations*, Nonlinear dynamical systems and chaos (Groningen, 1995), 385–404, Progr. Nonlinear Differential Equations Appl. **19**, Birkhauser, Basel, 1996.

Jens Rottmann-Matthes, *Linear stability of traveling waves in first-order hyperbolic PDEs*, J. Dynam. Differential Equations **23** (2011), 365–393.

G. Kreiss, H-O. Kreiss, and N. A. Petersson, *On the convergence of solutions of nonlinear hyperbolic-parabolic systems*, SIAM J. Numer. Anal. **31** (1994), 1577–1604.

Stability of the concatenated wave structure: definitions and results for two waves

Realization of the concatenated wave structure: let $\bar{y} = \frac{1}{2}(y_1 + y_2)$, $\bar{c} = \frac{1}{2}(c_1 + c_2)$.

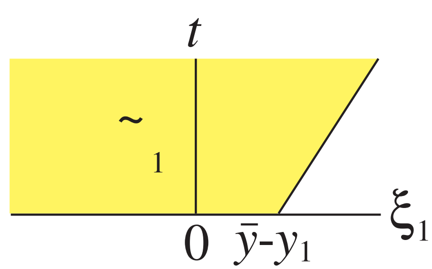


Realization $= q_j(x - y_j - c_j t)$ for $(x,t) \in \Omega_j$.

Discontinuous along Γ, but discontinuity decays exponentially as $t \to \infty$.

In Ω_j it is natural to replace x with the moving coordinate $\xi_j = x - y_j - c_j t$.

In $\xi_1 t$-coordinates, Ω_1 corresponds to



$$u_t = u_{xx} + f(u),$$

Notation:

- Initial condition: $u_0^{ex}(x)$
- Solution: $u^{ex}(x,t)$
- Solution in $\tilde{\Omega}_j$ in $\xi_j t$-coordinates: $\tilde{u}_j^{ex}(\xi_j, t)$

Definition. The concatenated wave structure is **exponentially stable** provided for each $\varepsilon > 0$ there exist $\chi > 0$ and $\delta > 0$ for which the following is true. Suppose $y_2 - y_1 > \chi$ and $\|u_0^{ex}(x) - q_j(x - y_j)\|_{H^1(I_j)} < \delta$ for $j = 1, 2$. Then $u^{ex}(x,t)$ can be written in each $\tilde{\Omega}_j$ as

$$\tilde{u}_j^{ex}(\xi_j, t) = q_j(\xi_j + \beta_j(t)) + Y_j(\xi_j, t),$$

where $\dot{\beta}_j(t)$ and $Y_j(\xi_j, t)$ decay exponentially, and in appropriate function spaces have norms less than ε.

Notice $\beta_j(t)$ **approaches a finite limit.**

Theorem. If each traveling wave (A1) approaches its end states exponentially and (A2) is spectrally stable, then the concatenated wave structure is exponentially stable.

The theorem follows from a linear result

Equation in $\tilde{\Omega}_j$:

$$u_t = u_{\xi\xi} + c_j u_\xi + f(u)$$

Decomposition of solution in $\tilde{\Omega}_j$:

$$\tilde{u}_j^{ex}(\xi,t) = q_j(\xi + \beta_j(t)) + Y_j(\xi,t)$$

Substitute the solution into the equation and expand $q_j(\xi+\beta)$ and $f(u)$ about $q_j(\xi)$

$$q_j'(\xi)\dot{\beta}_j + \partial_t Y_j = \partial_{\xi\xi} Y_j + c_j \partial_\xi Y_j + Df(q_j(\xi))Y_j + F_j(\xi, Y_j, \beta_j, \dot{\beta}_j)$$

Iinitial condition on I_j treated analogously:

$$\tilde{u}_j^{ex}(\xi,0) = q_j(\xi+\beta_j(0)) + Y_j(\xi,0) = q_j(\xi) + \beta_j(0)q_j'(\xi) + G_j(\xi,\beta_j(0)) + Y_j(\xi,0)$$

Jump condition across Γ treated analogously:

$$\begin{aligned} 0 = [\tilde{u}_j^{ex}](\Gamma) &= [q_j(\xi_j + \beta_j)](\Gamma) + [Y_j](\Gamma) \\ &= [q_j(\xi_j)](\Gamma) + [\beta_j q_j'(\xi_j)](\Gamma) + [G_j(\xi_j,\beta_j)](\Gamma) + [Y_j](\Gamma). \end{aligned}$$

$$F_j = O\left(|Y_j|^2 + |Y_j||\beta_j| + |\beta_j||\dot{\beta}_j|\right), \quad G_j = O(\beta_j^2(t))$$

Nonlinear system

(N1) In $\tilde{\Omega}_j$, (Y_j, β_j) satisfies

$$\partial_t Y_j + q_j'(\xi)\dot{\beta}_j = L_j Y_j + F_j(\xi, Y_j, \beta_j, \dot{\beta}_j)$$
$$Y_j(\xi,0) + \beta_j(0) q_j'(\xi) = \tilde{u}_j^{ex}(\xi,0) - q_j(\xi) - G_j(\xi, \beta_j(0))$$

(N2) Along Γ

$$[Y_j](\Gamma) + [\beta_j q_j'(\xi_j)](\Gamma) = -[q_j(\xi_j)](\Gamma) - [G_j(\xi_j, \beta_j)](\Gamma)$$
$$[Y_{j\xi}](\Gamma) + [\beta_j q_j''(\xi_j)](\Gamma) = -[q_j'(\xi_j)](\Gamma) - [G_{j\xi}(\xi_j, \beta_j)](\Gamma).$$

Linear system

(L1) In $\tilde{\Omega}_j$, (Y_j, β_j) satisfies

$$\partial_t Y_j + q_j'(\xi)\dot{\beta}_j = L_j Y_j + h_j(\xi, t)$$
$$Y_j(\xi,0) + \beta_j(0) q_j'(\xi) = w_j(\xi)$$

(L2) Along Γ

$$[(Y_j, Y_{j\xi})](\Gamma) + [(\beta_j q_j'(\xi_j), \beta_j q_j''(\xi_j))] = J_j.$$

Assume the compatibility condition

$$[(w_j, w_{j\xi})](x_j) = J_j(0).$$

Function spaces we use (after Lions and Magenes)

$$\textbf{Solution space: } H^{2,1}(\Omega) = \{u : \Omega \to \mathbb{R}^n \mid u, u_x, u_{xx} \text{ and } u_t \in L^2(\Omega\}$$
$$H^{2,1}(\Omega,\gamma) = \{u : \Omega \to \mathbb{R}^n \mid e^{-\gamma t}u \in H^{2,1}(\Omega)\}$$
$$\textbf{Trace space: } H^{0.75\times 0.25}(\mathbb{R}^+) = H^{0.75}(\mathbb{R}^+) \times H^{0.25}(\mathbb{R}^+)$$
$$H^{0.75\times 0.25}(\mathbb{R}^+,\gamma) = H^{0.75}(\mathbb{R}^+,\gamma) \times H^{0.25}(\mathbb{R}^+,\gamma).$$
$$X^1(\mathbb{R}^+,\gamma) = \{u : \mathbb{R}^+ \to \mathbb{R}^n \mid e^{-\gamma t}\dot{u} \in L^2(\mathbb{R}^+)\}; \ |u|_{X^1(\mathbb{R}^+,\gamma)} = |u(0)| + |e^{-\gamma t}\dot{u}|_{L^2(\mathbb{R}^+)}.$$

$\mathcal{Y}$ = space of "solutions" $(Y_1, \beta_1, Y_2, \beta_2)$, $Y_j \in H^{2,1}(\tilde{\Omega}_j, \gamma)$, $\beta_j \in X^1(\mathbb{R}^+,\gamma)$).

$\mathcal{Z}$ = space of inhomogeneous terms (h_1, h_2, w_1, w_2, J), $h_j \in L^2(\tilde{\Omega}_j, \gamma)$, $w_j \in H^1(I_j)$, $J \in H^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$ such that the compatibility condition is satisfied.

Linear Theorem. Assume (A1)–(A2) and $y_2 >> y_1$. Fix γ, $\eta \leq \gamma < 0$. Then the linear problem with $(h_1, h_2, w_1, w_2, J) \in \mathcal{Z}$ has a solution $(Y_1, \beta_1, Y_2, \beta_2)$ in $\mathcal{Y}$ given by a bounded linear mapping

$$L : \mathcal{Z} \to \mathcal{Y}, \quad L(h_1, h_2, w_1, w_2, J) = (Y_1, \beta_1, Y_2, \beta_2).$$

The bound is independent of y_1, y_2.

Nonlinear result follows from the linear result by a contraction mapping argument, with a further restriction on γ.

Proof of the linear theorem: step 1: ignore the jump

Linear system without the jump condition

(L1) In $\tilde{\Omega}_j$, (Y_j, β_j) satisfies

$$\partial_t Y_j + q_j'(\xi)\dot{\beta}_j = L_j Y_j + h_j(\xi,t)$$
$$Y_j(\xi,0) + \beta_j(0) q_j'(\xi) = w_j(\xi)$$

Proposition. Assume (A1)–(A2). Fix γ, $\eta \leq \gamma < 0$. Then the linear problem (L1) with $(h_1, h_2, w_1, w_2) \in L^2(\tilde{\Omega}_1, \gamma) \times L^2(\tilde{\Omega}_2, \gamma) \times H^1(I_1) \times H^1(I_2)$ has a solution $(Y_1, \beta_1, Y_2, \beta_2) \in \mathcal{Y}$ that is given by a bounded linear mapping

$$L^{(1)}(h_1, h_2, w_1, w_2) = (Y_1, \beta_1, Y_2, \beta_2).$$

The bound is independent of y_1, y_2 provided $y_2 - y_1 \geq \varepsilon > 0$.

Proof. Extend h_j et w_j to $\mathbb{R}^2$ and $\mathbb{R}$, solve using the semigroup e^{tL_j}. Note the effect of the eigenvalue 0.

We use a family of extension operators $L^2(\Omega) \to L^2(\mathbb{R}^2)$ and $H^1(I) \to H^1(\mathbb{R})$ that is uniformly bounded independent of Ω and I.

Proof of the linear theorem: step 2: deal with the jump

Given $\tilde{J} \in H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$ find $(Y_1, \beta_1, Y_2, \beta_2) \in \mathcal{Y}$ such that the functions $U_j(\xi_j, t) = Y_j(\xi_j, t) + \beta_j q_j'(\xi_j)$ satisfy

$$U_t = L_j U, \quad (\xi, t) = (\xi_j, t) \in \tilde{\Omega}_j, \tag{1}$$

$$U(\xi, 0) = 0, \quad \xi = \xi_j \in I_j, \tag{2}$$

$$[(U_j, U_{j\xi})](\Gamma) = \tilde{J}. \tag{3}$$

This linear problem has zero forcing and initial condition zero.

Jump Theorem. Assume (A1)–(A2) and $y_2 >> y_1$. Fix γ, $\eta \leq \gamma < 0$. Then the linear problem (1)–(3) has a solution $(Y_1, \beta_1, Y_2, \beta_2) \in \mathcal{Y}$ that is given by a bounded linear mapping

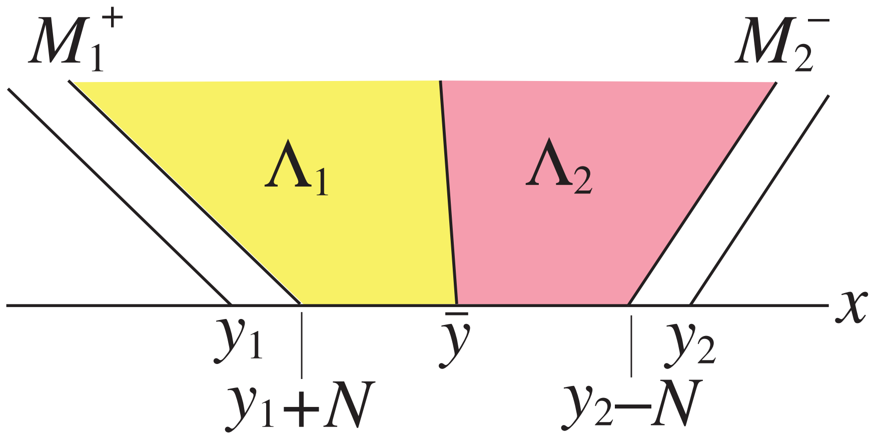
$$L^{(2)}(\tilde{J}) = (Y_1, \beta_1, Y_2, \beta_2).$$

The bound is independent of y_1, y_2.

Proof of the linear theorem: Add the solutions given by the previous proposition and the jump theorem.

Proof of the Jump Theorem: two lemmas

1. Tail Lemma



Let $\phi \in H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$. On $\Lambda_1 \cup \Lambda_2$, we look for $U(x,t)$ that satisfies

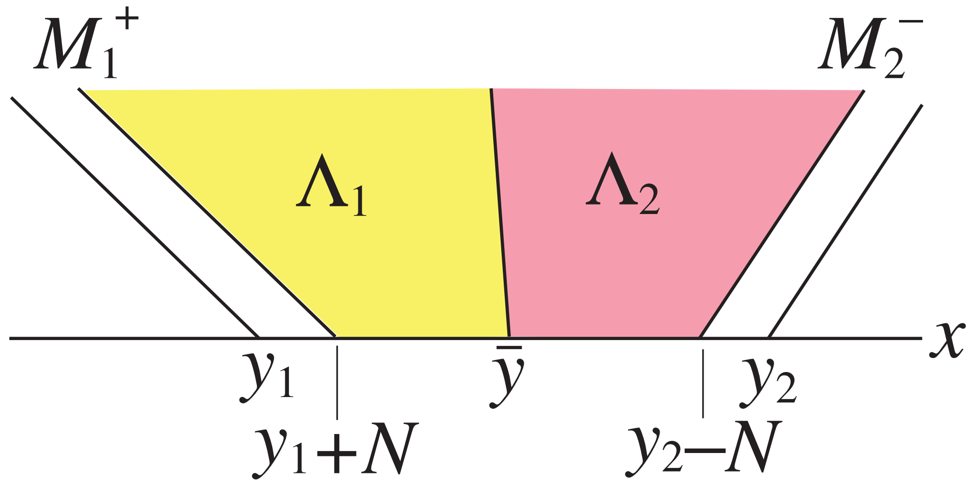
(4) $$U_t = U_{xx} + Df(q_1(x - y_1 - c_1 t))U, \qquad (x,t) \text{ in } \Lambda_1,$$

(5) $$U_t = U_{xx} + Df(q_2(x - y_2 - c_2 t))U, \qquad (x,t) \text{ in } \Lambda_2,$$

(6) $$U(x,0) = 0, \qquad [U, U_x](\Gamma) = \phi.$$

The solution should decay exponentially in t as $t \to \infty$ and in x as (x,t) moves away from Γ.

On $\Lambda_1 \cup \Lambda_2$, independent of y_1 and y_2, $Df(q_1(x - y_1 - c_1 t))$ and $Df(q_2(x - y_2 - c_2 t))$ are close to $Df(e_1)$.



Tail Lemma. Assume (A1)–(A2). Fix γ, $\eta \le \gamma < 0$. Then there is a number $N > 0$ such that if $y_2 - y_1 > 2N$, then the linear problem (4)–(6) has a solution $U(x,t)$ in $H_0^{2,1}(\Lambda_1 \cup \Lambda_2, \gamma)$ that is given by a bounded linear mapping

$$L^{(3)} : H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma) \to H_0^{2,1}(\Lambda_1 \cup \Lambda_2, \gamma).$$

The bound is independent of y_1, y_2. There are numbers $C > 0$ and $\alpha > 0$, independent of y_1, y_2, such that

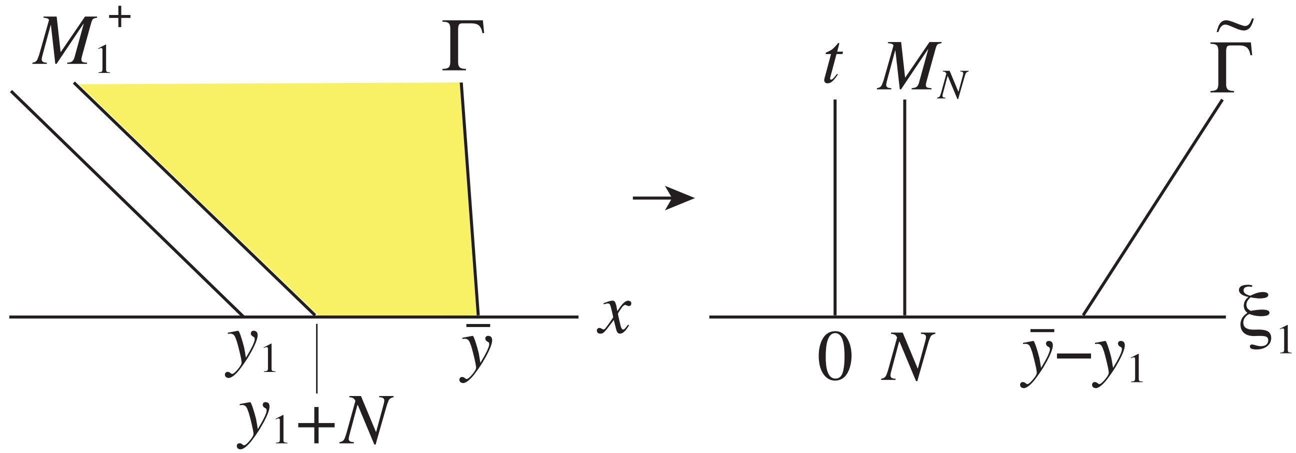
$$\|U|_{M_1^+}\| + \|U|_{M_2^-}\| \le C(e^{-\alpha(\bar{y}-y_1-N)} + e^{-\alpha(y_2-N-\bar{y})})\|\phi\|,$$

where all the norms are in $H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$.

U decays exponentially in t, but we cannot guarantee that the solution continues to decay in x, essentially because of the eigenvalue 0.

The Tail Lemma deals with the discontinuity along Γ but leaves smaller discontinuities along M_1^+ and M_2^-. The Interior Lemma deals with them but leaves even smaller discontinuities along Γ.

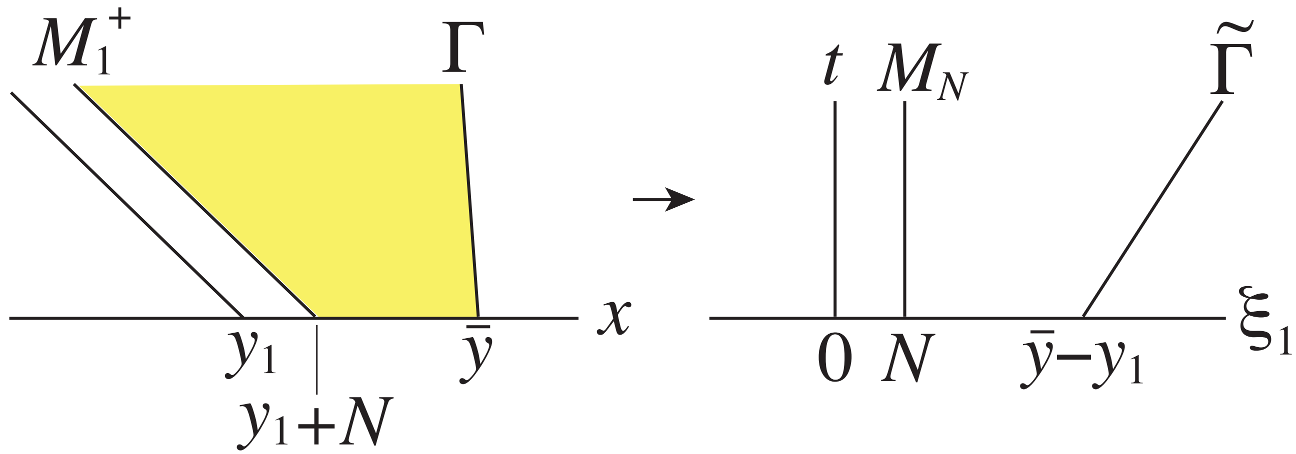
To deal with the jump on M_1^+, we use x_1t-coordinates:



Let $\phi \in H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$. Consider the problem

(7) $$U_t = L_1 U, \quad (\xi,t) \in \mathbb{R}\times\mathbb{R}^+ \setminus M_N,$$

(8) $$U(\xi,0) = 0, \quad [(U,U_\xi)](M_N) = \phi.$$



Interior Lemma. Assume (A1)–(A2). Fix γ, $\eta \leq \gamma < 0$. Assume $y_2 - y_1 > 2N$. Then the linear problem (7)–(8) has a solution $U = Y_1(\xi, t) + \beta_1(t) q_1'(\xi)$ with

(1) $Y_1 \in H_0^{2,1}(\mathbb{R} \times \mathbb{R}^+ \setminus M_N, \gamma)$,

(2) $\beta_1 \in X_0^1(\mathbb{R}^+, \gamma)$.

The solution is given by a bounded linear mapping

$$L^{(4)} : H_0^{0.75 \times 0.25}(\mathbb{R}^+, \gamma) \to H_0^{2,1}(\mathbb{R} \times \mathbb{R}^+ \setminus M_N) \times X_0^1(\mathbb{R}^+, \gamma), \quad L^{(4)}(\phi) = (Y_1, \beta_1).$$

The bound is independent of y_1, y_2. There are numbers $C > 0$ and $\alpha > 0$, independent of y_1, y_2, such that

$$\|\tilde{U}|_\Gamma\| \leq C e^{-\alpha(\bar{y} - y_1 - N)} \|\phi\|,$$

where all the norms are in $H_0^{0.75 \times 0.25}(\mathbb{R}^+, \gamma)$.

U has a part that does not decay in t, essentially because of the eigenvalue 0.

Proof of the jump theorem: Add an infinite series of solutions given by the tail lemma and the interior lemma.

Proving the Lemmas

1. Laplace transform

Consider a second-order linear partial differential equation with zero initial conditions:

$$U_t = U_{\xi\xi} + cU_\xi + A(\xi,t)U, \quad (\xi,t) \in I \times \mathbb{R}^+, \quad U(\xi,0) = 0.$$

Apply Laplace transform $\mathcal{L}$ in t, write $\hat{U}(\xi,s) = \mathcal{L}U(\xi,t)$:

$$s\hat{U} = \hat{U}_{\xi\xi} + c\hat{U}_\xi + \left(\hat{A}(\xi,\cdot) \overset{s}{*} \hat{U}(\xi,\cdot)\right)(s).$$

Convert both equations first-order systems in ξ:

(9) $$U_\xi = V, \quad V_\xi = U_t - cV - A(\xi,t)U, \quad (U,V)(\xi,0) = (0,0)$$

(10) $$\hat{U}_\xi = \hat{V}, \quad \hat{V}_\xi = s\hat{U} - c\hat{V} - \left(\hat{A}(\xi,\cdot) \overset{s}{*} \hat{U}(\xi,\cdot)\right)(s)$$

We regard (9) as a linear differential equation in ξ on the Banach space $H_0^{0.75\times0.25}(\mathbb{R}^+,\gamma)$, a space of functions of t (**spatial dynamics**).

We regard (10) as a linear differential equation in ξ on the Hardy-Lebesgue space that corresponds to it, $\mathcal{H}^{0.75\times0.25}(\gamma)$.

2. Hardy-Lebesgue spaces

$f(s)$ is in the Hardy-Lebesgue space $\mathcal{H}(\gamma), \gamma \in \mathbb{R}$, if

(1) $f(s)$ is analytic in $\Re(s) > \gamma$;
(2) $\sup_{\sigma>\gamma}(\int_{-\infty}^{\infty} |f(\sigma+i\omega)|^2 d\omega)^{1/2} < \infty$.

$\mathcal{H}(\gamma)$ is a Banach space with norm defined by (2).

For $k \geq 0$ and $\gamma \in \mathbb{R}$, define

$$\mathcal{H}^k(\gamma) = \{u(s) : u(s) \text{ and } (s-\gamma)^k u(s) \in \mathcal{H}(\gamma)\}.$$

Paley-Wiener Theorem.

- $u(t) \in L^2(\mathbb{R}^+, \gamma) \Longleftrightarrow \hat{u}(s) \in \mathcal{H}(\gamma)$.
- $u(t) \in H_0^k(\mathbb{R}^+, \gamma) \Longleftrightarrow \hat{u}(s) \in \mathcal{H}^k(\gamma)$.
- $(u, v) \in H_0^{k_1 \times k_2}(\mathbb{R}^+, \gamma) \Longleftrightarrow (\hat{u}, \hat{v}) \in \mathcal{H}^{k_1 \times k_2}(\gamma)$.

In each case, the mapping $u \to \hat{u}$ is a Banach space isomorphism

The Hardy-Lebesgue space that corresponds to $H_0^{0.75\times 0.25}(\mathbb{R}^+, \gamma)$ is $\mathcal{H}^{0.75\times 0.25}(\gamma)$.

3. Exponential dichtomies in $H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$ corespond to exponential dichotomies in $\mathcal{H}^{0.75\times 0.25}(\gamma)$

$$U_\xi = V, \quad V_\xi = U_t - cV - A(\xi,t)U, \quad (U,V)(\xi,0) = (0,0) \tag{9}$$

$$\hat{U}_\xi = \hat{V}, \quad \hat{V}_\xi = s\hat{U} - c\hat{V} - \left(\hat{A}(\xi,\cdot) \overset{s}{*} \hat{U}(\xi,\cdot)\right)(s) \tag{10}$$

Lemma. Assume (10) has an exponential dichotomy on $\mathcal{H}^{0.75\times 0.25}(\gamma)$ for $\xi \in I$. Then (9) has an exponential dichotomy on $H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$ for $\xi \in I$, with the same constants K, α. The projections and solution maps are related by Laplace transform. In addition, $u(\xi,t) \in H_0^{2,1}(I \times \mathbb{R}_+,\gamma)$.

4. Proving exponential dichotomies on Hardy-Lebesgue spaces if $A(\xi,t)$ is independent of t

If $A(\xi,t) = A(\xi)$ is independent of time t, then (10) simplifies to

$$\hat{U}_\xi = \hat{V}, \quad \hat{V}_\xi = s\hat{U} - c\hat{V} - A(\xi)\hat{U}. \tag{11}$$

We can regard (11) **as a family of ordinary differential equations in** $\xi \in I$ **on** $\mathbb{C}^n$**, with** s **as a parameter** in a set $\mathcal{S} \subset \mathbb{C}$, with solution operator $T(\xi,\zeta,s)$.

Let $|u|$ denote the usual norm on $\mathbb{C}^n$. Let $E^{k_1 \times k_2}(s)$ denote $\mathbb{C}^n \times \mathbb{C}^n$ with the norm

$$|(u,v)|_{E^{k_1 \times k_2}(s)} = (1+|s|^{k_1})|u| + (1+|s|^{k_2})|v|.$$

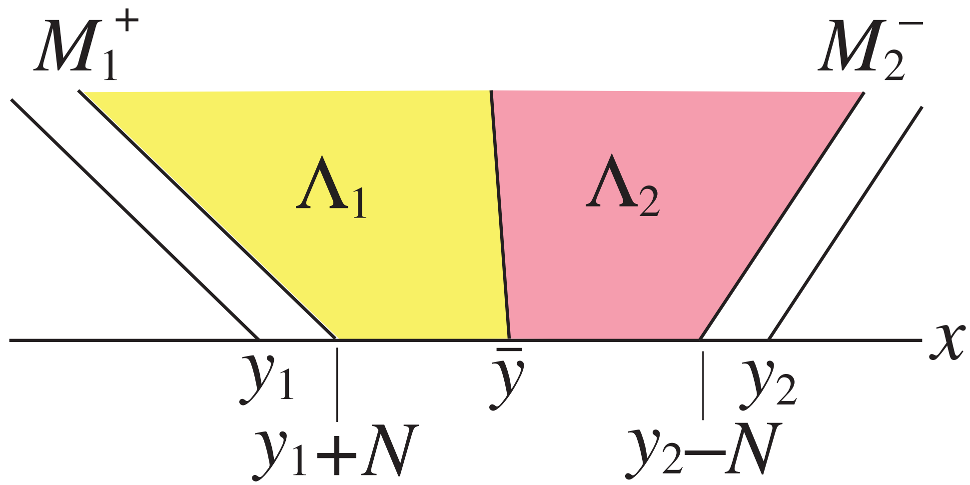
We say that (11) has a **uniform exponential dichotomy** on the spaces $E^{0.75 \times 0.25}(s)$ for $s \in \mathcal{S}$ and $\xi \in I$ if it has an exponential dichotomy for each s; the projections $P_j(\xi,s)$, $j = s,u$, are analytic in s for fixed ξ; and there are constants $K, \alpha > 0$ such that, **when norms in the spaces** $E^{0.75 \times 0.25}(s)$ **are used**,

(1) each $K(s) \leq K$, and
(2) $\rho(s) = \alpha(1+|s|^{0.5})$.

(11) $$\hat{U}_\xi = \hat{V}, \quad \hat{V}_\xi = s\hat{U} - c\hat{V} - A(\xi)\hat{U}.$$

Lemma. Suppose (11) has a uniform exponential dichotomy on the spaces $E^{0.75\times0.25}(s)$ for $\Re(s) \geq \gamma$ and $\xi \in I$. Then (11) has an exponential dichotomy on $\mathcal{H}^{0.75\times0.25}(\gamma)$ for $\xi \in I$ with projections derived from those in $E^{0.75\times0.25}(s)$, multiplicative constant K, and exponent α.

5. Proof of the Tail Lemma



Let $\phi \in H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$. On $\Lambda_1 \cup \Lambda_2$, we look for $U(x,t)$ that satisfies

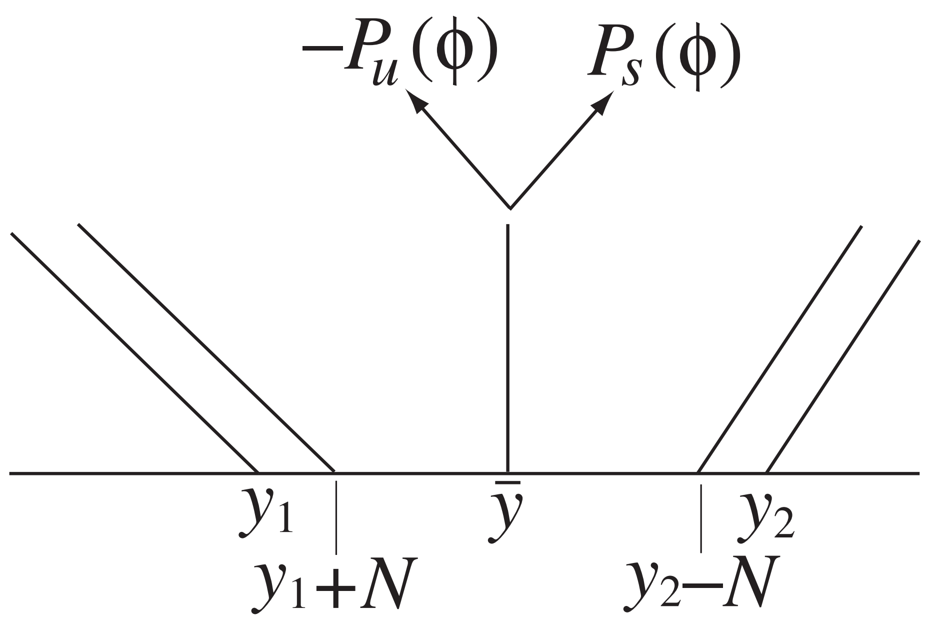
$$\begin{aligned}
&U_t = U_{xx} + Df(q_1(x-y_1-c_1t))U, && (x,t) \text{ in } \Lambda_1,\\
&U_t = U_{xx} + Df(q_2(x-y_2-c_2t))U, && (x,t) \text{ in } \Lambda_2,\\
&U(x,0) = 0, \qquad [U,U_x](\Gamma) = \phi.
\end{aligned}$$

The solution should decay exponentially in t as $t \to \infty$ and in x as (x,t) moves away from Γ.

For large N, independent of y_1 and y_2, on $\Lambda_1 \cup \Lambda_2$, $Df(q_1(x-y_1-c_1t))$ and $Df(q_2(x-y_2-c_2t))$ are both near $Df(e_1)$.

$$
\begin{aligned}
&U_t = U_{xx} + Df(q_1(x - y_1 - c_1 t))U, && (x,t) \text{ in } \Lambda_1,\\
&U_t = U_{xx} + Df(q_2(x - y_2 - c_2 t))U, && (x,t) \text{ in } \Lambda_2,\\
&U(x,0) = 0, \qquad [U, U_x](\Gamma) = \phi. &&
\end{aligned}
$$

Make Γ vertical. **The exponential dichotomy for** $U_t = U_{xx} + Df(e_1)U$ **on** $H_0^{0.75\times 0.25}(\mathbb{R}^+,\gamma)$ **persists.** Decompose the jump.



6. Proof of the Interior Lemma

Step 1: $|s| \geq \varepsilon$**.** System:

$$U_t = L_1 U, \quad (\xi, t) \in \mathbb{R} \times \mathbb{R}^+ \setminus M_N,$$
$$U(\xi, 0) = 0, \quad [(U, U_\xi)](M_N) = \phi.$$

Laplace transform:

$$0 = (L_1 - sI)\hat{U}, \quad [(\hat{U}, \hat{U}_\xi)](M_N) = \hat{\phi}(s). \tag{12}$$

Write as a first-order system:

$$\hat{U}_\xi = \hat{V},\ \hat{V}_\xi = (sI - Df(q_j(\xi)))\hat{U} - c_j\hat{V}, \quad [(\hat{U}, \hat{V})](M_N) = \hat{\phi}(s). \tag{13}$$

$\hat{\phi}(s)$ is in $\mathcal{H}^{0.75\times 0.25}(\gamma)$. Look for solutions of (13) that decay to zero as ξ moves away from M_N.

Let $\varepsilon > 0$. For $\Re(s) \geq \eta$ and $|s| \geq \varepsilon$, system (13) has a unique solution $(\hat{U}, \hat{V})(\xi, s)$ that decays exponentially as $\xi \to \pm\infty$. The solution depends analytically on s, and there are constants $C_1(\varepsilon) > 0$ and $\alpha_1(\varepsilon) > 0$ such that for $\Re(s) \geq \eta$, $|s| \geq \varepsilon$, and $\rho_1(\varepsilon) = \alpha_1(\varepsilon)(1 + |s|^{0.5})$, the solution satisfies

$$\|(\hat{U}, \hat{V})(\xi, s)\|_{E^{0.72\times 0.25}(s)} \leq C_1(\varepsilon) e^{-\rho_1(\varepsilon)|\xi - N|} |\hat{\phi}(s)|_{E^{0.72\times 0.25}(s)}.$$

Step 2: $|s| \leq \varepsilon$.

$$U_t = L_1 U, \quad (\xi, t) \in \mathbb{R} \times \mathbb{R}^+ \setminus M_N,$$
$$U(\xi, 0) = 0, \quad [(U, U_\xi)](M_N) = \phi.$$

Let $P_j =$ spectral projection of $L^2(\mathbb{R})$ onto $< q_j' >$. Let $U(\xi,t) = Y(\xi,t) + \beta(t) q'(\xi)$ with $P_j Y(\cdot, t) = 0$:

$$Y_t + \dot{\beta}(t) q_j'(\xi) = L_1 Y, \quad Y(\xi, 0) = 0, \, \beta(0) = 0, \quad [(Y, Y_\xi)](M_N) = \hat{\phi}(s).$$

Write $h(t) = \dot{\beta}(t)$. Take Laplace transform:

(14) $$(L_1 - sI)\hat{Y} = \hat{h}(s) q'(\xi), \quad [(\hat{Y}, \hat{Y}_\xi)](N) = \hat{\phi}(s),$$

Write as a first order system:

(15) $$(\hat{Y}, \hat{Z})_\xi = (\hat{Z}, (sI - Df(q_j(\xi)))\hat{Y} - c_j \hat{Z}) + (0, \hat{h}(s) q_j'(\xi)), \; [(\hat{Y}, \hat{Z})](N) = \hat{\phi}(s).$$

There exists $\varepsilon > 0$ such that for $|s| \leq \varepsilon$, (15) has a unique solution $((\hat{Y}, \hat{Z})(\xi, s), \hat{h}(s))$ such that $P_j \hat{Y}(\cdot, s) = 0$ and $(\hat{Y}, \hat{Z})(\xi, s)$ decays exponentially as $\xi \to \pm\infty$. The solution depends analytically on s, and there are constants $C_2 > 0$ and $\alpha_2 > 0$ such that for $|s| \leq \varepsilon$ and $\rho_2 = \alpha_2(1 + |s|^{0.5})$, the solution satisfies

$$\|(\hat{Y}, \hat{Z})(\xi, s)\|_{E^{0.72 \times 0.25}(s)} \leq C_2 e^{-\rho_2 |\xi - N|} |\hat{\phi}(s)|_{E^{0.72 \times 0.25}(s)}.$$

Proof: For each small s, there exist two exponential dichotomies for (15), for $\xi \le N$ and for $\xi \ge N$. Denote the projections by $P_s^-(\xi,s)+P_u^-(\xi,s)=I$ for $\xi \le N$ and $P_s^+(\xi,s)+P_u^+(\xi,s)=I$ for $\xi \ge N$.

Bounded solutions of (15) can be expressed as:

$$\text{for } \xi \le N, \quad (\hat{Y},\hat{Z})(\xi,s) = T(\xi,N,s)P_u^-(N,s)(\hat{Y},\hat{Z})(N-,s)$$
$$+\int_{-\infty}^{\xi} T(\xi,\zeta,s)P_s^-(\zeta,s)(0,\hat{h}(s)q_j'(\zeta))d\zeta + \int_N^{\xi} T(\xi,\zeta,s)P_u^-(\zeta,s)(0,\hat{h}(s)q_j'(\zeta))d\zeta;$$
$$\text{for } \xi \ge N, \quad (\hat{Y},\hat{Z})(\xi,s) = T(\xi,N,s)P_s^+(N,s)(\hat{Y},\hat{Z})(N+,s)$$
$$+\int_{N}^{\xi} T(\xi,\zeta,s)P_s^+(\zeta,s)(0,\hat{h}(s)q_j'(\zeta))d\zeta + \int_{\infty}^{\xi} T(\xi,\zeta,s)P_u^+(\zeta,s)(0,\hat{h}(s)q_j'(\zeta))d\zeta.$$

Let

$$\mu_u^-(s) = P_u^-(N,s)(\hat{Y},\hat{Z})(N-,s), \quad \mu_s^+(s) = P_s^+(N,s)(\hat{Y},\hat{Z})(N+,s),$$
$$v(s) = \int_{-\infty}^{N} T(N,\zeta,s)P_s^-(\zeta,s)(0,q_j'(\zeta))d\zeta + \int_N^{\infty} T(N,\zeta,s)P_u^+(\zeta,s)(0,q_j'(\zeta))d\zeta.$$

The jump condition at $\xi = N$ is satisfied provided

$$\mu_s^+(s) - \mu_u^-(s) - \hat{h}(s)v(s) = \hat{\phi}(s).$$

$$\mu_s^+(s) - \mu_u^-(s) - \hat{h}(s)v(s) = \hat{\phi}(s).$$

For each s this is $2n$ equations in the $2n+1$ unknowns (μ_u^-, μ_s^+, h).

One more equation: $P_1\hat{Y} = 0$.

Show invertibility at $s = 0$.

Step 3:

Express the solution from step 1 as $\hat{U}(\xi, s) = \hat{Y}(\xi, s) + \hat{h}(s)q'(\xi)$.

Combine the solutions from steps 1 and 2.

Invert the Laplace transform.