JOURNAL OF DIFFERENTIAL EQUATIONS **126**, 303–354 (1996)
ARTICLE NO. 0053

Structurally Stable Riemann Solutions*

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Received August 5, 1994

We study the structure of solutions of Riemann problems for systems of two conservation laws. Such a solution comprises a sequence of elementary waves, *viz*., rarefaction and shock waves of various types; shock waves are required to have viscous profiles. We construct a Riemann solution by solving a system of equations characterizing its component waves. A Riemann solution is "structurally stable" if the number and types of its component waves are preserved when the initial data and the flux function are perturbed.

Under the assumption that rarefaction waves and shock states lie in the stricly hyperbolic region, we characterize Riemann solutions for which the definition equations have maximal rank and we prove that such solutions are structurally stable. Structurally stable Riemann solutions cannot contain overcompressive shock waves, but they can contain transitional shock waves, including doubly sonic transitional shock waves, including doubly sonic transitional shock waves that have not been observed before. © 1996 Academic Press, Inc.

* This work was supported in part by: the Conselho Nacional de Desenvolvimento Tecnológico e Científico under Grant CNPq/NSF 910087/92-0; the Financiadora de Estudos e Projectos under Grant 65920311-00; the National Science Foundation under Grant INT-9216357; the U.S. Army Research Office under Grant DAAL03-91-C-0027 to the Mathematical Sciences Institute of Cornell University, through subcontract to Stony Brook; the Applied Mathematics Subprogram of the U.S. Department of Energy under Grant DE-FG02-90ER25084; and the National Science Foundation under Grant DMS-9205535.

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0022-0396/96 $18.00

1. Introduction

We consider systems of conservation laws in one space dimension. These are partial differential equations of the form

$$U_t + F(U)_x = 0 \tag{1.1}$$

with $t \in \mathbb{R}^+$, $x \in \mathbb{R}$, $U(x, t) \in \mathbb{R}^N$, and $F\colon \mathbb{R}^N \to \mathbb{R}^N$ a smooth map. Such equations arise in the modeling of many physical systems, such as gas dynamics [4, 18], three-phase flow in a porous medium [3, 31, 1, 27, 11], elastic strings [19, 13, 25], plasticity [20, 32], magnetohydrodynamics [34, 5, 2], chromatography [21], and phase transitions [12, 28, 24]. A good general reference is Ref. [29].

For both theoretical and numerical purposes, the most basic initial-value problem for Eq. (1.1) is the *Riemann problem*, in which the initial data are piecewise constant with a single jump at $x = 0$:

$$U(x, 0) = \begin{cases} U_L & \text{for} \quad x < 0, \\ U_R & \text{for} \quad x > 0. \end{cases} \tag{1.2}$$

Riemann solutions have a rich wave structure. In this work, we propose a systematic program to study this structure and we carry out the first step of the program.

We seek piecewise continuous weak solutions of Riemann problems in the scale-invariant form $U(x, t) = \hat{U}(x/t)$ consisting of a finite number of constant parts, continuously changing parts (rarefaction waves), and jump discontinuities (shock waves). Shock waves occur when

$$\lim_{\xi \to s-} \hat{U}(\xi) = U_- \neq U_+ = \lim_{\xi \to s+} \hat{U}(\xi). \tag{1.3}$$

To have a weak solution of Eq. (1.1), the Rankine–Hugoniot condition

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0 \tag{1.4}$$

must hold. It is well known that this requirement alone allows multiple solutions of Riemann problems, including ones that are clearly not physical.

Various shock admissibility criteria are used to remedy this situation. Perhaps the most widely accepted is the *viscous profile criterion.* Suppose that Eq. (1.1) arises by ignoring the small viscous term in the parabolic equation

$$U_t + F(U)_x = \varepsilon\, [D(U)\, U_x]_x. \tag{1.5}$$

(Here $D(U)$ is a positive matrix, dictated by the physical application, and $\varepsilon > 0$ is small.) Then the viscous profile criterion states that the discontinuity (1.3) is *admissible* in solutions of Eqs. (1.1) and (1.2) if and only if the parabolic equation (1.5) has a traveling wave solution $U(x, t) = \tilde{U}((x - st)/\varepsilon)$ with

$$\lim_{\xi \to \pm\infty} \tilde{U}(\xi) = U_\pm, \qquad \lim_{\xi \to \pm\infty} \tilde{U}'(\xi) = 0. \tag{1.6}$$

This amounts to requiring that the ordinary differential equation

$$D(U)\,\dot{U} = F(U) - F(U_-) - s(U - U_-) \tag{1.7}$$

have an orbit from the equilibrium U_- to a second equilibrium U_+.

In simple cases, the viscous profiles criterion coincides with the more easily-used admissibility criterion of Lax [14] and with its generalization due to Liu [15]. However, the viscous profile criterion allows, for example, transitional (or undercompressive) shock waves that correspond to saddle-to-saddle connections of Eq. (1.7), which fail to satisfy the Lax and Liu criteria. Recent work strongly supports admitting these nonclassical shock waves: they are sometimes needed to solve Riemann problems [27, 26, 10, 23]; they arise, apparently stably, in numerical calculations [35]; and they can sometimes be proved to be time-asymptotically stable solutions of Eq. (1.5) [16]. We shall therefore adopt the viscous profile shock admissibility criterion in this paper.

In the current work, we will restrict our attention to systems of two conservation laws, i.e., $N = 2$. We shall make the further simplification $D(U) \equiv I$, despite that this is physically unrealistic and that the solutions of Riemann problems generally depend on the viscosity matrix [9]. Our results actually hold in somewhat greater generality, but further work is needed to address the case of general viscosity matrices.

In the literature, Riemann solutions are usually pictured by fixing U_L and drawing the U_R-plane, which is divided into regions in which different types of solutions occur. The classical work of Lax [14], which treats U_R close to U_L, leads to Fig. 1.1, in which we have used notation that will be used throughout this paper. If $U_R = U_L$ (the dot at the center of the picture), the solution is constant. If U_R lies on one of the curves drawn throughout U_L, the solution contains a single wave: a 1- or 2-rarefaction wave (denoted R_1 or R_2), a 1-shock wave (denoted $R \cdot S$ because the shock is represented by a repeller-to-saddle connection of Eq. (1.7)), or a 2-shock wave (denoted $S \cdot A$ because the shock is represented by a saddle-to-attractor connection of Eq. (1.7)). If U_R lies in one of the open regions separated by the curves, the Riemann solution has two waves, as indicated in Fig. 1.1.

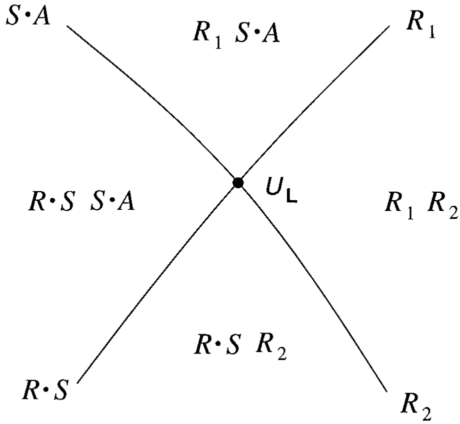

FIG. 1.1. Different Riemann solutions for fixed U_L in a neighborhood of $U_R = U_L$ in the U_R-plane.

Figure 1.1 is the starting point for the literature on Riemann problems. There are various approaches to generalizations: (a) extend the wave curves (i.e., the codimension-one bifurcation curves in Fig. 1.1) through various subsequent codimension-two bifurcations [33, 6]; (b) identify classes of flux functions F for which the slow and fast wave curves are transverse, as in the Lax construction [30, 15]; (c) study the failure of the two basic hypotheses of Lax, genuine nonlinearity and strict hyperbolicity [15, 27]. Wendroff [33] and Liu [15] used wave curves to construct more general Riemann solutions, assuming technical hypotheses that imply the global transversality of wave curves. Furtado [6] demonstrated the structural stability of wave curves assuming that shock waves satisfy the Lax admissibility criterion. Studies of physical models that are not strictly hyperbolic have demonstrated the importance of transitional waves.

These generalizations lead to diagrams that are far more complicated than Fig. 1.1, and there is, at present, a desire among workers in the field for organizing principles that will bring some order to the profusion of examples. In this paper we propose an approach to Riemann problems that we believe organizes the subject in a comprehensible way.

Our approach can be explained in the context of Fig. 1.1. This figure can be viewed as a bifurcation diagram. If U_R lies in one of the open regions, the Riemann solution is structurally stable, in the sense that if we vary U_L, U_R, and F a little, the Riemann solution is a sequence of the same number of waves with the same types. (Structural stability is, in general, distinct from stability of the Cauchy problem and from time-asymptotic stability.) Points U_R on the curves through $U_R = U_L$ in Fig. 1.1 correspond to codimension-one bifurcations of the Riemann solution. At the point $U_R = U_L$ there is a codimension-two bifurcation.

In bifurcation theory and singularity theory, one normally analyzes first the structurally stable problems, then the codimension-one problems, *etc.* From this point of view, the Lax construction, which is based on the codimension-two Riemann problem $U_R = U_L$, should not be taken as the starting point for a systematic approach to solving Riemann problems. Instead we propose to start the study of Riemann solutions with the structurally stable solutions. This is the first step of a program that has an obvious continuation: to analyze how the Riemann solution bifurcates when exactly one of the assumptions that lead to structural stability is violated. This program provides an organized approach to understanding codimension-one Riemann solutions, such as the one-wave solutions in Fig. 1.1.

Here is a brief summary of the contents of the paper. Let

$$\mathcal{U}_F = \{ U \in \mathbb{R}^2 \colon DF(U) \text{ has distinct real eigenvalues} \} \tag{1.8}$$

be the *strictly hyperbolic region.* We restrict our attention to "classical" rarefaction waves, i.e., those such that $\hat{U}(\xi) \in \mathcal{U}_F$ for all ξ, and shock waves (1.3) with $U_\pm \in \mathcal{U}_F$. (This rules out transitional rarefaction waves [9]. We believe that these are the only new waves that occur in the study of structurally stable Riemann problems when the hypothesis of strict hyperbolicity is relaxed to nonstrict hyperbolicity.)

Starting at the left state U_L, a Riemann solution can be constructed by appending successive elementary waves until an open region of states U_R is attained. Each appended wave w introduces a certain number of degrees of freedom; this number is called the *Riemann number* $\rho(w)$ of the wave. It is not difficult to verify that $\rho(w)$ is an integer between -2 and 1, determined by the wave type of w. For example, the Riemann number of a rarefaction wave or an $R \cdot S$ or $S \cdot A$ shock wave is 1, while the Riemann number of a shock wave of type $S \cdot S$ (saddle-to-saddle) is 0. We argue that for a Riemann solution to be structurally stable, the sum of the Riemann numbers of its component waves should be 2. We then identify precisely the class of finite wave sequences that have this property. Finally, we show that, given certain nondegeneracy conditions, all the wave sequences in this class are in fact structurally stable. As a side benefit, this analysis identifies one type of wave (the doubly sonic transitional wave) that occurs in structurally stable Riemann problems but has not yet been observed in case studies.

The remainder of the paper is organized as follows. In Sec. 2 we establish notation and terminology. Then we state our principal results, the Wave Structure Theorem and the Structural Stability Theorem. Proofs are in Secs. 3–7. Further discussion of our results is in Sec. 8.

2. Definitions and Results

We consider the system (1.1) with $t \in \mathbb{R}^+$, $x \in \mathbb{R}$, $U(x, t) \in \mathbb{R}^2$, and $F: \mathbb{R}^2 \to \mathbb{R}^2$ a C^2 map. Let $\mathcal{U}_F$ be defined by Eq. (1.8). For $U \in \mathcal{U}_F$, let $\lambda_1(U) < \lambda_2(U)$ denote the eigenvalues of $DF(U)$, and let $l_i(U)$ and $r_i(U)$, $i = 1, 2$, denote corresponding left and right eigenvectors with $l_i(U)\, r_j(U) = \delta_{ij}$.

A *rarefaction wave* of *type* R_i is a differentiable map $\hat{U}: [a, b] \to \mathcal{U}_F$, where $a < b$, such that $\hat{U}'(\xi)$ is a multiple of $r_i(\hat{U}(\xi))$ and $\xi = \lambda_i(\hat{U}(\xi))$ for each $\xi \in [a, b]$. The states $U = \hat{U}(\xi)$ with $\xi \in [a, b]$ constitute the *rarefaction curve* $\bar{\Gamma}$. The definition of rarefaction wave implies that if $U \in \bar{\Gamma}$, then

$$D\lambda_i(U)\, r_i(U) = l_i(U)\, D^2F(U)(r_i(U), r_i(U)) \neq 0. \tag{2.1}$$

Condition (2.1) is *genuine nonlinearity* of the ith characteristic line field at U. The definition also implies that $\lambda_i(U_-) < \lambda_i(U_+)$, where $U_- = \hat{U}(a)$ and $U_+ = \hat{U}(b)$ are the *left* and *right states* of the rarefaction wave, respectively. We will find it convenient to associate a specific *speed* s to a rarefaction wave: for a rarefaction wave of type R_1, $s = \lambda_1(U_+)$; for a rarefaction wave of type R_2, $s = \lambda_2(U_-)$. (Of course, this definition is appropriate only in our present context of two-component conservation laws.)

Remark. Even in the context of rarefaction waves that lie within $\mathcal{U}_F$, our definition of rarefaction wave is not the most general one. In particular, it excludes points along the rarefaction curve where genuine nonlinearity fails. However, work of Liu [15] and Furtado [6] strongly suggests that rarefaction waves for which genuine nonlinearity fails at some point do not occur in structurally stable Riemann solutions.

A *shock wave* consists of a *left state* U_-, a *right state* U_+, a *speed* s, and a *connecting orbit* Γ, *i.e.*, a orbit of the ordinary differential equation

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \tag{2.2}$$

from the equilibrium U_- to the equilibrium U_+. (Recall that we are taking $D(U) \equiv I$ in Eqs. (1.5) and (1.7).) In particular, the speed and the left and right states of a shock wave are related by the Rankine–Hugoniot condition (1.4), which states that U_+ is an equilibrium for Eq. (2.2). Equivalently, we could require that the ordinary differential equation

$$\dot{U} = F(U) - F(U_+) - s(U - U_+) \tag{2.3}$$

have an equilibrium at U_- and an orbit from U_- to U_+. Notice that, in general, there might be more than one orbit from U_- to U_+; according to

TABLE 2.1

Types of Equilibria

Name	Symbol	Eigenvalues
Repeller	R	$+ \quad +$
Repeller-Saddle	RS	$0 \quad +$
Saddle	S	$- \quad +$
Saddle-Attractor	SA	$- \quad 0$
Attractor	A	$- \quad -$

our definition, a particular connecting orbit Γ must be chosen in order to specify a shock wave.

For any equilibrium $U \in \mathscr{U}_F$ of Eq. (2.2), the eigenvalues of the linearization at U are $\lambda_i(U) - s$, $i = 1, 2$. We shall use the terminology defined in Table 2.1 for such an equilibrium.

Remark. Our name for an equilibrium accounts only for the signs of the eigenvalues; it does not necessarily reflect the topological type of the phase portrait if there is a zero eigenvalue. For instance, an equilibrium with one positive and one zero eigenvalue (which we call a repeller-saddle) has, in the nondegenerate case, the topological type of a repeller-saddle; but it can have a degenerate topological type, such as that of a weak saddle. Figures 2.1–2.4 show the phase portraits in the nondegenerate situation. These are the phase portraits that occur in the structurally stable Riemann solutions that we construct.

If w is a shock wave, its *type* is determined by the equilibrium types of its left and right states. (For example, w is of type $R \cdot S$ if its connecting orbit joins a repeller to a saddle.) The sixteen types are listed in Figs. 2.1–2.4; they are grouped into four sets of four: *slow*, *fast*, *overcompressive*, and *transitional* shock waves. Slow and fast waves are called *classical* shock waves. For a classical or transitional shock wave there are at most two possible choices for its connecting orbit Γ; for an overcompressive shock wave there is, in general, an infinite number of possible choices for Γ. As we shall see, overcompressive waves do not occur in Riemann solutions for which the sum of Riemann numbers is 2.

An *elementary wave* w is either a rarefaction wave or a shock wave. We write

$$w\colon U_- \xrightarrow{s} U_+ \tag{2.4}$$

if w has left state U_-, right state U_+, and speed s. Note that an elementary wave also has a *type* T, as defined above.

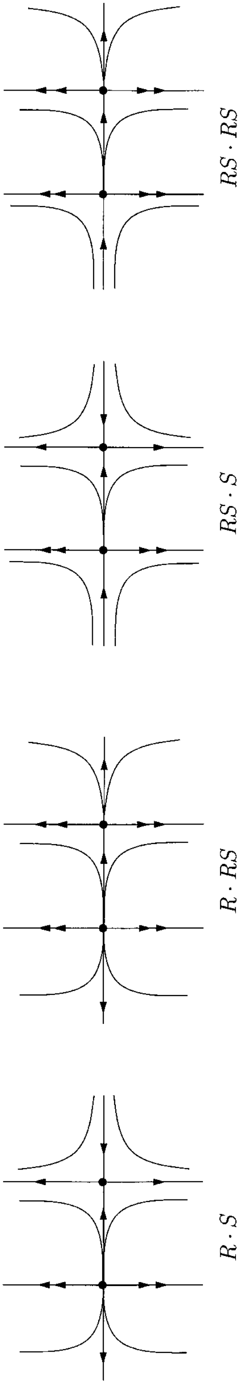

FIG. 2.1. The four types of slow shock waves.

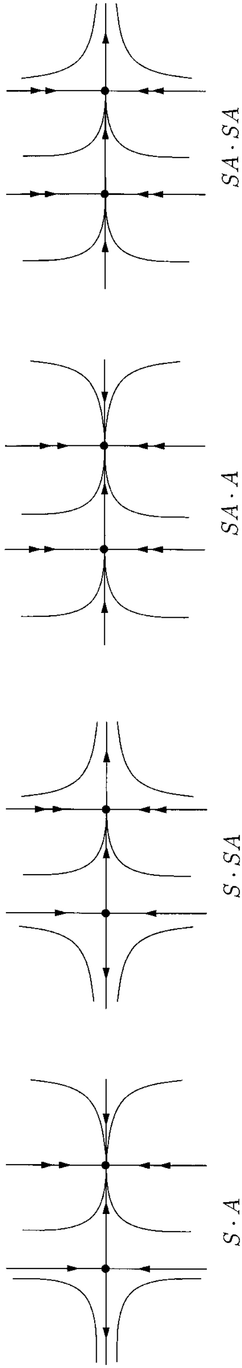

FIG. 2.2. The four types of fast shock waves.

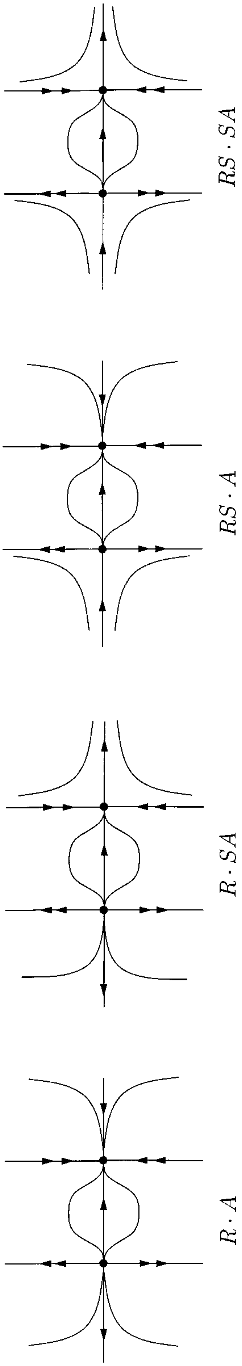

FIG. 2.3. The four types of overcompressive shock waves.

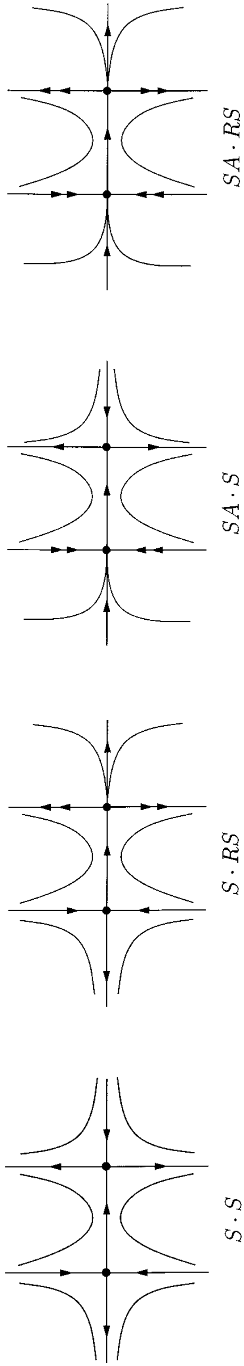

FIG. 2.4. The four types of transitional shock waves.

Associated with each elementary wave is a *speed interval* σ: for a rarefaction wave of type R_i, $\sigma = [\lambda_i(U_-), \lambda_i(U_+)]$, whereas for a shock wave of speed s, $\sigma = [s, s]$. If σ_1 and σ_2 are speed intervals, we write $\sigma_1 \leqslant \sigma_2$ if $s_1 \leqslant s_2$ for every $s_1 \in \sigma_1$ and $s_2 \in \sigma_2$.

Also associated with each elementary wave is the set $\bar{\Gamma}$: if w is a rarefaction wave, $\bar{\Gamma}$ denotes its rarefaction curve; if w is a shock wave, then $\bar{\Gamma}$ denotes the closure of its connecting orbit. We shall say that an open set $\mathcal{N} \subseteq \mathbb{R}^2$ is a *neighborhood* of the elementary wave w: $U_- \xrightarrow{s} U_+$ if $\bar{\Gamma} \subset \mathcal{N}$.

Sequences of elementary waves can be used to construct solutions of Riemann problems. A wave sequence $(w_1, w_2, ..., w_n)$ is said to be *allowed* if:

1. for each $i = 1, ..., n-1$, the right state of w_i coincides with the left state of w_{i+1};
2. the speed intervals σ_i for w_i satisfy
$$\sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_n; \tag{2.5}$$
3. no two successive waves are rarefaction waves of the same type.

For such a wave sequence we write
$$(w_1, w_2, ..., w_n)\colon U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n. \tag{2.6}$$

If $U_0 = U_L$ and $U_n = U_R$, then associated with an allowed wave sequence $(w_1, w_2, ..., w_n)$ is a piecewise continuous weak solution $U(x, t) = \hat{U}(x, t)$ of the Riemann initial-value problem (1.1)–(1.2). In this solution, a discontinuity along the ray $x = st$ arises from one or more admissible shock waves with speed s. Conversely, if a solution can be regarded as being composed of a finite sequence of elementary waves, separated by constant states, then this sequence is allowed. Therefore we shall often refer to an allowed wave sequence as a *Riemann solution.*

Let
$$(w_1^*, w_2^*, ..., w_n^*)\colon U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{2.7}$$
be a Riemann solution for $U_t + F^*(U)_x = 0$. Fix a compact set $K \subset \mathbb{R}^2$ such that Int K is a neighborhood of w_i^* for $i = 1, ..., n$. Let $\mathcal{B}$ denote the Banach space of C^2 functions $F\colon K \to \mathbb{R}^2$, equipped with the C^2 norm. In the following we will regard the flux function F as an element of $\mathcal{B}$, but the results will not depend on the choice of K. Also, let $\mathcal{H}(\text{Int } K)$ denote the set of nonempty, closed subsets of Int K, which we equip with the Hausdorff metric.

Definition 2.1. We shall say that the Riemann solution (2.7) is *structurally stable* if there are neighborhoods $\mathcal{U}_i$ of U_i^*, $\mathcal{I}_i$ of s_i^*, and $\mathcal{F}$ of F^* and a C^1 map
$$G\colon \mathcal{U}_0 \times \mathcal{I}_1 \times \mathcal{U}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_n \times \mathcal{U}_n \times \mathcal{F} \to \mathbb{R}^{3n-2} \tag{2.8}$$

with $G(U_0^*, s_1^*, U_1^*, s_2^*, ..., s_n^*, U_n^*, F^*)=0$ such that:

(P1) $G(U_0, s_1, U_1, s_2, ..., s_n, U_n, F)=0$ implies that there exists a Riemann solution

$$(w_1, w_2, ..., w_n) \colon U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n \tag{2.9}$$

for $U_t + F(U)_x = 0$ with successive waves of the same types as those of the wave sequence (2.7) and with each w_i contained in Int K;

(P2) $DG(U_0^*, s_1^*, U_1^*, s_2^*, ..., s_n^*, U_n^*, F^*)$, restricted to the $(3n-2)$-dimensional space of vectors $\{(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, ..., \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = 0 = \dot{U}_n, \dot{F}=0\}$, is an isomorphism onto $\mathbb{R}^{3n-2}$.

Condition (P2) implies, by the implicit function theorem, that $G^{-1}(0)$ is a graph over $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$. Therefore for each wave w_i we can define a map $\bar{\Gamma}_i \colon \mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F} \to \mathscr{H}(\text{Int } K)$; namely, $\bar{\Gamma}_i(U_0, U_n, F)$ is the rarefaction curve or the closure of the connecting orbit of the wave w_i. We further require that

(P3) $(w_1, w_2, ..., w_n)$ can be chosen so that $\bar{\Gamma}_i(U_0^*, U_n^*, F^*) = \bar{\Gamma}_i^*$ and each map $\bar{\Gamma}_i$ is continuous.

The map G will be said to *exhibit* the structural stability of the Riemann solution (2.7).

Remark. By condition (P2), $(s_1, U_1, ..., U_{n-1}, s_n)$ is determined by (U_0, U_n, F). However, if w_i^* is a shock wave, then the connecting orbit Γ_i (and hence w_i itself) might not be uniquely determined by the data (U_{i-1}, s_i, U_i). Condition (P3) asserts that each connecting orbit Γ_i can be chosen in a continuous way.

In Sec. 4 we shall explicitly give *local defining maps* for each type of elementary wave, with which we will construct maps G that exhibit structural stability. Let $w^* \colon U_-^* \xrightarrow{s^*} U_+^*$ be an elementary wave of type T for $U_t + F^*(U)_x = 0$. The local defining map G_T has as its domain a set of the form $\mathscr{U}_- \times \mathscr{I} \times \mathscr{U}_+ \times \mathscr{F}$ (with $\mathscr{U}_\pm$ being neighborhoods of $U_\pm^*$, $\mathscr{I}$ a neighborhood of s^*, and $\mathscr{F}$ a neighborhood of F^*). The range is some $\mathbb{R}^e$; the number e depends only on the wave type T. The local defining map is such that $G_T(U_-^*, s^*, U_+^*, F^*)=0$. Moreover, if certain *wave non-degeneracy conditions* are satisfied at (U_-^*, s^*, U_+^*, F^*), then there is a neighborhood $\mathscr{N}$ of w^* such that:

(D1) $G_T(U_-, s, U_+, F)=0$ if and only if there exists an elementary wave $w \colon U_- \xrightarrow{s} U_+$ of type T for $U_t + F(U)_x = 0$ contained in $\mathscr{N}$;

(D2) $DG_T(U_-^*, s^*, U_+^*, F^*)$, restricted to the space $\{(\dot{U}_-, \dot{s}, \dot{U}_+, \dot{F}) : \dot{F}=0\}$, is surjective.

Condition (D2) implies, by the implicit function theorem, that $G_T^{-1}(0)$ is a manifold of codimension e. Therefore we can define a map $\bar{\Gamma}$ from this manifold to $\mathscr{H}(\operatorname{Int} K)$ (just as above). We shall establish further that

(D3) w can be chosen so that $\bar{\Gamma}(U_-^*, s^*, U_+^*, F^*) = \bar{\Gamma}^*$ and $\bar{\Gamma}$ is continuous.

For the Riemann solution (2.7), let w_i^* have type T_i and local defining map G_{T_i}, with range $\mathbb{R}^{e_i}$. For an appropriate neighborhoods $\mathscr{U}_i$ of U_i^*, $\mathscr{I}_i$ of s_i^*, $\mathscr{F}$ of F^*, and $\mathscr{N}_i$ of w_i^*, we can define a map $G\colon \mathscr{U}_0 \times \mathscr{I}_1 \times \cdots \times \mathscr{I}_n \times \mathscr{U}_n \times \mathscr{F} \to \mathbb{R}^{e_1 + \cdots + e_n}$ by $G = (G_1, ..., G_n)$, where

$$G_i(U_0, s_1, ..., s_n, U_n, F) = G_{T_i}(U_{i-1}, s_i, U_i, F). \tag{2.10}$$

The map G is called the *local defining map* of the wave sequence (2.7). Assuming the wave nondegeneracy conditions, if $G(U_0, s_1, ..., s_n, U_n, F) = 0$, then for each $i = 1, ..., n$, there is an elementary wave $w_i\colon U_{i-1} \xrightarrow{s_i} U_i$ of type T_i for $U_t + F(U)_x = 0$ contained in $\mathscr{N}_i$, for which $\bar{\Gamma}_i$ is continuous. In this paper we shall study Riemann solutions (2.7) whose structural stability is exhibited by the local defining map of the solution.

In view of the requirement that the local defining map have range $\mathbb{R}^{3n-2}$, a necessary condition for $G = (G_1, ..., G_n)$ to exhibit the structural stability of the wave sequence (2.7) is that

$$\sum_{i=1}^{n} e_i = 3n - 2, \tag{2.11}$$

i.e.,

$$\sum_{i=1}^{n} (3 - e_i) = 2. \tag{2.12}$$

We are therefore led to define the *Riemann number* of an elementary wave type T to be

$$\rho(T) = 3 - e(T), \tag{2.13}$$

where $e(T)$ is the number of defining equations for a wave of type T. For convenience, if w is an elementary wave of type T, we shall write $\rho(w)$ instead of $\rho(T)$. Because of Eq. (2.12) we concentrate our attention on allowed sequences of elementary waves $(w_1, ..., w_n)$ with $\sum_{i=1}^{n} \rho(w_i) = 2$.

In Sec. 4 we shall show that the Riemann number for a rarefaction wave is 1, whereas the Riemann numbers of shock waves are as given in Table 2.2. The essence of the argument is the following: the Rankine–Hugoniot condition gives two of the defining equations for a shock wave; one further condition holds for each repeller-saddle and saddle-attractor equilibrium; and

TABLE 2.2

Riemann Numbers of Shock Waves

	U_+			
U_-	RS	S	SA	A
R	0	1	0	1
RS	-1	0	-1	0
S	-1	0	0	1
SA	-2	-1	-1	0

one further condition holds for transitional waves because the connecting orbit is a double separatrix.

Because of the inequalities (2.5) on speed intervals, an allowed sequence of elementary waves can contain only the wave type successions given in Table 2.3, as can be verified easily by comparing shock and characteristic speeds.

Some of these wave type successions do not occur in Riemann solutions for which the sum of Riemann numbers is 2. The wave type successions in Table 2.4 are termed *good.*

We can now state

THEOREM 2.2. *Let $(w_1, ..., w_n)$ be an allowed sequence of elementary waves. Then*

1. $\sum_{i=1}^{n} \rho(w_i) \leqslant 2$;
2. $\sum_{i=1}^{n} \rho(w_i) = 2$ *if and only if*

 (a) *all wave type successions are good*;

 (b) *w_1 is of type $R \cdot RS$, $R \cdot S$ or R_1; and w_n is of type $SA \cdot A$, $S \cdot A$, or R_2.*

TABLE 2.3

Wave Type Successions in Allowed Sequences of Elementary Waves

	T_{i+1}				
T_i	R_1	$RS \cdot *$	$S \cdot *$	$SA \cdot *$	R_2
R_1		$\surd$	$\surd$	$\surd$	$\surd$
$* \cdot RS$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$* \cdot S$			$\surd$	$\surd$	$\surd$
$* \cdot SA$				$\surd$	$\surd$
R_2				$\surd$	

TABLE 2.4

Good Wave Type Succions

T_i	T_{i+1}
R_1	$RS \cdot RS$, $RS \cdot S$, $S \cdot *$, R_2
$* \cdot RS$	R_1
$S \cdot SA$, $SA \cdot SA$	R_2
$* \cdot S$	$S \cdot *$, R_2
R_2	$SA \cdot *$

A proof by induction on n, using nothing more than Tables 2.2, 2.3, and 2.4, is given in Sec. 3. One consequence of this theorem is that overcompressive waves do not occur in Riemann solutions for which the sum of Riemann numbers is 2.

We now give a more conceptual description of the allowed sequences of elementary waves with $\sum_{i=1}^{n} \rho(w_i) = 2$. First we state some more definitions.

A 1-*wave group* is either a single $R \cdot S$ wave or an allowed sequence of elementary waves of the form

$$(R \cdot RS)(R_1 RS \cdot RS) \cdots (R_1 RS \cdot RS)\, R_1 (RS \cdot S), \tag{2.14}$$

where the terms in parentheses are optional. If any of the terms in parentheses are present, the group is termed *composite*.

A *transitional wave group* is either a single $S \cdot S$ wave or an allowed sequence of elementary waves of the form

$$S \cdot RS \quad (R_1 RS \cdot RS) \cdots (R_1 RS \cdot RS)\, R_1 \quad (RS \cdot S) \tag{2.15}$$

or

$$(S \cdot SA)\, R_2 \quad (SA \cdot SA \quad R_2) \cdots (SA \cdot SA \quad R_2)\, SA \cdot S, \tag{2.16}$$

the terms in parentheses being optional. In cases (2.15) and (2.16), the group is termed *composite*.

A 2-*wave group* is either a single $S \cdot A$ wave or an allowed sequence of elementary waves of the form

$$(S \cdot SA)\, R_2 \quad (SA \cdot SA \quad R_2) \cdots (SA \cdot SA \quad R_2)(SA \cdot A), \tag{2.17}$$

where again the terms in parentheses are optional. If any of the terms in parentheses are present, the group is termed *composite*.

An $SA \cdot RS$ wave will be called a *doubly sonic transitional wave*.

Remark. In the wave groups (2.14)–(2.17), the right endpoint of the speed interval σ_i coincides with the left endpoint of σ_{i+1} for all i. In other words, there are no constant states embedded within a composite wave group. Also notice that the sum of the Riemann numbers is 1 for slow and fast wave groups, 0 for transitional wave groups, and -2 for doubly sonic transitional waves.

The reader should note a symmetry between the wave groups (2.14) and (2.17), as well as between the groups (2.15) and (2.16). The wave groups $R \cdot S$, (2.14), and (2.15) are termed *slow*; the wave groups $S \cdot A$, (2.17), and (2.16) are termed *fast*. A solution U for the equation $U_t + F(U)_x = 0$ that consists of a fast wave group corresponds to a solution $\tilde{U}$ for the equation $\tilde{U}_t - F(\tilde{U})_x = 0$ that consists of a slow wave group; the correspondence is

$$\tilde{U}(x, t) = U(-x, t). \tag{2.18}$$

This symmetry will be exploited throughout this paper to shorten the treatment. For example, it motivates the definition of the speed of a rarefaction.

With these definitions, we have

THEOREM 2.3 (Wave Structure). *Let the allowed sequence of elementary waves* (2.7) *have* $\sum_{i=1}^{n} \rho(w_i^*) = 2$.

(1) *Suppose that the wave sequence* (2.7) *includes no* $SA \cdot RS$ *waves. Then it consists of one* 1-*wave group, followed by an arbitrary number of transitional wave groups* (*in any order*), *followed by one* 2-*wave group.*

(2) *Suppose that the wave sequence* (2.7) *includes* $m \geqslant 1$ *waves of type* $SA \cdot RS$. *Then these waves separate* $m+1$ *wave sequences* $g_0, ..., g_m$. *Each* g_i *is exactly as in* (1) *with the restrictions that*:

(a) *if* $i < m$, *the last wave in the group has type* R_2;

(b) *if* $i > 0$, *the first wave in the group has type* R_1.

Remark. A transitional composite wave group arises in modeling three-phase flow in a porous medium [10, 17]. To our knowledge, $SA \cdot RS$ waves do not appear in the literature.

The condition $\sum_{i=1}^{n} \rho(w_i^*) = 2$ simply ensures that the range of the local defining map G of the wave sequence (2.7) is $\mathbb{R}^{3n-2}$. In order to ensure that G also satisfies conditions (P1)–(P3), we impose three additional types of conditions:

1. on each wave we impose the wave nondegeneracy conditions mentioned earlier; they are given precisely in Tables 4.1–4.4.

2. in the absence of $SA \cdot RS$ waves, we impose one *wave group interaction condition* on how the different wave groups are related. If there are $m \geqslant 1$ waves of type $SA \cdot RS$, we impose $m+1$ wave group interaction

conditions, one on each of the $m+1$ wave sequences $g_0, ..., g_m$. Roughly speaking, these conditions say that certain wave curves are transverse.

3. if w_i^* is a $*\cdot S$ wave and w_{i+1}^* is an $S\cdot *$ wave, we require that $s_i^* < s_{i+1}^*$.

We shall prove the following result.

THEOREM 2.4 (Structural Stability). *Suppose that the allowed sequence of elementary waves* (2.7) *has* $\sum_{i=1}^n \rho(w_i^*)=2$. *Assume that*:

(H1) *each wave satisfies the appropriate wave nondegeneracy conditions*;

(H2) *the wave group interaction conditions, as stated precisely in Theorems* 5.5, 6.1, *and* 7.2, *are satisfied*;

(H3) *if* w_i^* *is a* $*\cdot S$ *wave and* w_{i+1}^* *is an* $S\cdot *$ *wave, then* $s_i^* < s_{i+1}^*$.

Then the wave sequence (2.7) *is structurally stable.*

In fact, more can be concluded: not only can the connecting orbit Γ_i of the perturbed shock wave w_i be chosen to vary continuously, but also there is a neighborhood $\mathcal{N}_i$ such that if $\Gamma_i \subset \mathcal{N}_i$, then it is unique.

The remainder of the paper is organized as follows. In Sec. 3 we prove Theorem 2.2 and the Wave Structure Theorem. This section is independent of the rest of the paper. In Sec. 4 we give local defining maps and wave nondegeneracy conditions for each wave type. In Sec. 5 we prove the Structural Stability Theorem in the absence of transitional wave groups and doubly sonic transitional waves. In Sec. 6 we extend the proof to wave sequences containing transitional wave groups, and in Sec. 7 to wave sequences also containing doubly sonic transitional waves. We have included in Sec. 6 and 7 some discussion of the geometric role of transitional wave groups and doubly sonic transitional waves in the solution of Riemann problems. Further discussion of our results is in Sec. 8.

3. PROOFS OF THEOREM 2.2 AND THE WAVE STRUCTURE THEOREM

We divide the proof of Theorem 2.2 into two lemmas.

LEMMA 3.1. *Let* $(T_1, ..., T_n)$ *be a sequence of wave types allowed by Table* 2.3. *Then*

(1) $\sum_{i=1}^n \rho(T_i) \leqslant 2$;

(2) *if* $\sum_{i=1}^n \rho(T_i)=2$, *then*

  (a) *all wave type successions are good*;

  (b) T_1 *is* $R\cdot RS$, $R\cdot S$, *or* R_1; *and* T_n *is* $SA\cdot A$, $S\cdot A$, *or* R_2.

Proof. The proof is by induction on n.

For $n=2$, statement (1) follows from Table 2.2; moreover, $\sum_{i=1}^{n} \rho(T_i)=2$ if and only if T_1 is $R \cdot S$ or R_1 and T_2 is $S \cdot A$ or R_2, so that statement (2) holds, as seen from Table 2.3.

Suppose that the lemma is true for some $n \geqslant 2$. Let $(T_1, ..., T_{n+1})$ be an allowed sequence of elementary wave types.

We prove statement (1) by contradiction. Suppose that $\sum_{i=1}^{n+1} \rho(T_i) \geqslant 3$. By induction, $\sum_{i=1}^{n} \rho(T_i) \leqslant 2$, so that we must have $\sum_{i=1}^{n} \rho(T_i)=2$ and $\rho(T_{n+1})=1$. The induction hypothesis implies that T_n is $SA \cdot A$, $S \cdot A$, or R_2. But $SA \cdot A$ and $S \cdot A$ waves cannot have successors, so that T_n is R_2. From Table 2.3, T_{n+1} is $SA \cdot *$, so that from Table 2.2, $\rho(T_{n+1}) \leqslant 0$. This is a contradiction.

Next we prove statement (2). Suppose that $\sum_{i=1}^{n+1} \rho(T_i)=2$. We consider the different possibilities for T_{n+1}.

Case 1. It cannot happen that $\rho(T_{n+1})<0$, since $\sum_{i=1}^{n} \rho(T_i) \leqslant 2$ by induction, so that $\rho(T_{n+1})<0$ would imply $\sum_{i=1}^{n+1} \rho(T_i)<2$.

Case 2. Suppose that $\rho(T_{n+1})=0$. Then $\sum_{i=1}^{n} \rho(T_i)=2$, so that by induction $(T_1, ..., T_n)$ satisfies (a) and (b). Therefore T_n is R_2 (since an $SA \cdot A$ or $S \cdot A$ wave cannot have a successor), and from Tables 2.3 and 2.2, T_{n+1} is $SA \cdot A$. Thus $(T_1, ..., T_{n+1})$ satisfies both (a) and (b).

Case 3. Suppose that $\rho(T_{n+1})=1$. We consider the different possibilities for T_n.

Case 3.1. If $\rho(T_n)=-2$, then $\sum_{i=1}^{n+1} \rho(T_i)=\sum_{i=1}^{n-1} \rho(T_i)+\rho(T_n)+\rho(T_{n+1}) \leqslant 2-2+1$. Thus $\rho(T_n)=-2$ cannot occur.

Case 3.2. Suppose that $\rho(T_n)=-1$. Then $\sum_{i=1}^{n-1} \rho(T_i)=2$, so that $(T_1, ..., T_{n-1})$ satisfies (a) and (b). Therefore T_{n-1} is R_2 and T_n is $SA \cdot S$ or $SA \cdot SA$. If T_n is $SA \cdot S$, T_{n+1} must be $S \cdot A$ or R_2; if T_n is $SA \cdot SA$, T_{n+1} must be R_2. Thus $(T_1, ..., T_{n+1})$ satisfies (a) and (b).

Case 3.3. Suppose that $\rho(T_n)=0$. We note that T_n cannot be $RS \cdot A$ or $SA \cdot A$, since these have no successor; nor can it be $R \cdot RS$ or $R \cdot SA$, since these have no predecessor. Therefore T_n is $S \cdot SA$, $S \cdot S$, or $RS \cdot S$.

Case 3.1.1. *If* T_n is $S \cdot SA$, then T_{n+1} is R_2, and thus (b) holds. To verify (a), let $\tilde{T}_1=T_1, ..., \tilde{T}_{n-1}=T_{n-1}$, $\tilde{T}_n=S \cdot A$. Then $(\tilde{T}_1, ..., \tilde{T}_n)$ is an allowed sequence of elementary wave types and $\sum_{i=1}^{n} \rho(\tilde{T}_i)=2$. By induction $(\tilde{T}_1, ..., \tilde{T}_n)$ satisfies (a). From Table 2.4, $\tilde{T}_{n-1}=T_{n-1}$ is R_1 or $* \cdot S$. It follows that $(T_1, ..., T_{n+1})$ satisfies (a).

Case 3.3.2. If T_n is $S \cdot S$, then T_{n+1} is $S \cdot A$ or R_2, and thus (b) holds. The argument to verify (a) is like that when T_n is $S \cdot SA$.

Case 3.3.3. If T_n is $RS \cdot S$, then T_{n-1} is R_1 or $* \cdot RS$.

Case 3.3.3.1. Suppose that $\rho(T_{n-1})<0$. Then

$$2=\sum_{i=1}^{n-2}\rho(T_i)+\rho(T_{n-1})+\rho(T_n)+\rho(T_{n+1})<\sum_{i=1}^{n-2}\rho(T_i)+0+0+1, \quad (3.1)$$

so that by induction $\sum_{i=1}^{n-2}\rho(T_i)=2$, and therefore $\rho(T_{n-1})=-1$. But by induction T_{n-2} is R_2, $SA\cdot A$, or $S\cdot A$; since it has a successor, it is R_2, so that T_{n-1} must be $SA\cdot RS$. This is a contradiction, since an $SA\cdot RS$ wave has $\rho=-2$.

Case 3.3.3.2. Therefore T_{n-1} is R_1 or $R\cdot RS$. This first possibility is handled by an argument like that when T_n is $S\cdot SA$. The second possibility implies that $n=2$ (since an $R\cdot RS$ wave cannot have a predecessor), but then yields a Riemann sum of just 1.

Case 3.4. Suppose that $\rho(T_n)=1$. Then T_n is R_1 (the only wave type with Riemann number 1 that can have both a predecessor and a successor with Riemann number 1), T_{n+1} is $S\cdot A$ or R_2, and T_{n-1} is $*\cdot RS$. Thus (b) holds. To verify (a), we consider the following cases.

Case 3.4.1. If T_{n-1} is $R\cdot RS$, then $n=2$ and (a) is satisfied.

Case 3.4.2. If T_{n-1} is $S\cdot RS$, then $\sum_{i=1}^{n-2}\rho(T_i)=1$. Let $\tilde{T}_1=T_1, ..., \tilde{T}_{n-2}=T_{n-2}$, $\tilde{T}_{n-1}=S\cdot A$. Then $(\tilde{T}_1, ..., \tilde{T}_{n-1})$ is an allowed sequence of elementary wave types with $\sum_{i=1}^{n-1}\rho(\tilde{T}_i)=2$, so that by induction, $(\tilde{T}_1, ..., \tilde{T}_{n-1})$ satisfies (a) and (b). Therefore $\tilde{T}_{n-2}=T_{n-2}$ is R_1 or $*\cdot S$, so that $(T_1, ..., T_{n+1})$ satisfies (a).

Case 3.4.3. If T_{n-1} is $SA\cdot RS$, then $\sum_{i=1}^{n-2}\rho(T_i)=2$, so that $(T_1, ..., T_{n-2})$ satisfies (a) and (b). Therefore T_{n-2} is R_2. Thus $(T_1, ..., T_{n+1})$ satisfies (a).

Case 3.4.4. Finally, if T_{n-1} is $RS\cdot RS$, then T_{n-2} is $*\cdot RS$ or R_1. The following possibilities occur.

Case 3.4.4.1. We cannot have $T_{n-2}=R\cdot RS$, since that would imply $n=3$ and $\sum_{i=1}^{n+1}\rho(T_i)=1$.

Case 3.4.4.2. If T_{n-2} is $RS\cdot RS$ or $S\cdot RS$, then $\sum_{i=1}^{n-3}\rho(T_i)=2$, so T_{n-3} is R_2; this contradicts T_{n-2} being $RS\cdot RS$ or $S\cdot RS$.

Case 3.4.4.3. If T_{n-2} is $SA\cdot RS$, then $\sum_{i=1}^{n-3}\rho(T_i)=3$, which is impossible.

Case 3.4.4.4. If T_{n-2} is R_1, let $\tilde{T}_1=T_1, ..., \tilde{T}_{n-2}=T_{n-2}$, and $\tilde{T}_{n-1}=T_{n+1}$. Then $\sum_{i=1}^{n-1}\rho(\tilde{T}_i)=2$, so that $(\tilde{T}_1, ..., \tilde{T}_{n-1})$ satisfies (a). It follows easily that $(T_1, ..., T_{n+1})$ does too. ∎

LEMMA 3.2. *Let* $(T_1, ..., T_n)$ *be a sequence of wave types allowed by Table* 2.3. *Suppose that it satisfies* (a) *and* (b) *of Lemma* 3.1. *Then* $\sum_{i=1}^{n} \rho(T_i) = 2$.

Proof. We shall drop certain terms from the sequence $(T_1, ..., T_n)$, obtaining a shorter sequence with the same Riemann sum that still satisfies (a) and (b).

Step 1. If some T_i is $RS \cdot RS$, then T_{i-1} is R_1. We drop both. Since $\rho(T_{i-1}) + \rho(T_i) = 1 - 1 = 0$, the shorter sequence has the same Riemann sum. From Table 2.4, since T_{i-1} is R_1, either $i = 2$ or T_{i-2} is $* \cdot RS$; and sice T_i is $RS \cdot RS$, T_{i+1} is R_1. Therefore the sequence obtained by dropping T_{i-1} and T_i still satisfies (a) and (b).

Similar arguments justify the next four steps in the proof, but we omit them.

Step 2. If some T_i is $SA \cdot SA$, then T_{i+1} is R_2. We drop both.

Step 3. If some T_i is $S \cdot S$, we drop it.

Let us call the remaining sequence $\tilde{T}_1, ..., \tilde{T}_m$. It has no $RS \cdot RS$, $SA \cdot SA$, or $S \cdot S$ waves.

Step 4. If some $\tilde{T}_i$ is $S \cdot RS$, then $\tilde{T}_{i+1}$ is R_1, and $\tilde{T}_{i+2}$ may or may not be $RS \cdot S$. In the first case we drop $\tilde{T}_i$, $\tilde{T}_{i+1}$, and $\tilde{T}_{i+2}$; in the second case we drop $\tilde{T}_i$ and $\tilde{T}_{i+1}$.

Step 5. If $\tilde{T}_i$ is $SA \cdot S$, then $\tilde{T}_{i-1}$ is R_2, and $\tilde{T}_{i-2}$ may or may not be $S \cdot SA$. In the first case we drop $\tilde{T}_{i-2}$, $\tilde{T}_{i-1}$, and $\tilde{T}_i$; in the second case we drop $\tilde{T}_{i-1}$ and $\tilde{T}_i$.

Let us call the remaining sequence $\hat{T}_1, ..., \hat{T}_p$. It has no $RS \cdot RS$, $SA \cdot SA$, $S \cdot S$, $S \cdot RS$, or $SA \cdot S$ waves, and it satisfies (a) and (b).

Step 6. Suppose no $\hat{T}_i$ is $SA \cdot RS$. Then from the last comment it follows easily that the sequence $\hat{T}_1, ..., \hat{T}_p$ is just a 1-wave group followed by a 2-wave group. The 1-wave group is $R \cdot S$ or $(R \cdot RS)\, R_1(RS \cdot S)$; the two-wave group is $S \cdot A$ or $(S \cdot SA)\, R_2(SA \cdot A)$. The waves in parentheses are optional. Thus $\sum_{i=1}^{p} \rho(\hat{T}_i) = 2$.

Step 7. Suppose that the sequence $\hat{T}_1, ..., \hat{T}_p$ contains $k \geqslant 1$ waves $SA \cdot RS$ (with $\rho = -2$). These k waves separate $k+1$ wave groups $g_0, ..., g_k$. The sum of the Riemann numbers of the elementary waves in each g_i is at most 2. Thus we must have each wave group exactly as described in Step 6, except that: (i) in $g_0, ..., g_{k-1}$, the last wave is R_2 (so that it can have a successor); (ii) in $g_1, ..., g_k$, the first wave is R_1 (so that it can have a predecessor). ∎

Lemmas 3.1 and 3.2 together prove Theorem 2.2. Putting back the waves discarded during the proof of Lemma 3.2, we see that the wave sequences with $\sum_{i=1}^{n} \rho(T_i) = 2$ are exactly as stated in the Wave Structure Theorem.

4. Local Defining Maps and Wave Nondegeneracy Conditions

In this section we shall first give the local defining maps and nondegeneracy conditions that we shall use for elementary waves of each type. We recall from Sec. 2 that if $w^*\colon U^* \xrightarrow{s^*} U^*_+$ is an elementary wave of type T for $U_t + F^*(U)_x = 0$, the local defining map G_T is a map from $\mathscr{U}_- \times \mathscr{I} \times \mathscr{U}_+ \times \mathscr{F}$ to $\mathbb{R}^e$ ($\mathscr{U}_\pm$ being neighborhoods of $U^*_\pm$, $\mathscr{I}$ a neighborhood of s^*, and $\mathscr{F}$ a neighborhood of F^*), and the number e depends only on the wave type T. After giving the local defining maps and wave nondegeneracy conditions, we shall show that if the nondegeneracy conditions for waves of type T are satisfied at (U^*_-, s^*, U^*_+, F^*), then properties (D1)–(D3) hold.

To simplify the exposition, for most of this section we will suppress the dependence of G_T on F. Also, we will refer to the system of equations $G_T(U_-, s, U_+, F) = 0$ as the *local defining equations* for waves of type T.

In order to treat rarefaction waves, we define open subsets $\mathscr{U}_i$, $i = 1, 2$, of the U-plane by

$$\mathscr{U}_i = \{U \in \mathscr{U} : D\lambda_i(U)\, r_i(U) \neq 0\}. \tag{4.1}$$

In $\mathscr{U}_i$ we can normalize $r_i(U)$ to obtain a vector field $\tilde{r}_i(U)$ such that $D\lambda_i(U)\, \tilde{r}_i(U) \equiv 1$. For each $U_- \in \mathscr{U}_1$, define ψ_1 to be the solution of

$$\frac{\partial \psi_1}{\partial s}(U_-, s) = \tilde{r}_1(\psi_1(U_-, s)), \tag{4.2}$$

$$\psi_1(U_-, \lambda_1(U_-)) = U_-. \tag{4.3}$$

Then there is a rarefaction wave of type R_1 for $U_t + F(U)_x = 0$ from U_- to U_+ with speed s if and only if

$$U_+ - \psi_1(U_-, s) = 0 \qquad \text{with} \quad s = \lambda_1(U_+) > \lambda_1(U_-). \tag{4.4}$$

Similarly, for $U_+ \in \mathscr{U}_2$, define ψ_2 to be the solution of

$$\frac{\partial \psi_2}{\partial s}(s, U_+) = \tilde{r}_2(\psi_2(s, U_+)), \tag{4.5}$$

$$\psi_2(\lambda_2(U_+), U_+) = U_+. \tag{4.6}$$

Then there is a rarefaction wave of type R_2 for $U_t + F(U)_x = 0$ from U_- to U_+ with speed s if and only if

$$U_- - \psi_2(s, U_+) = 0 \qquad \text{with} \quad s = \lambda_2(U_-) < \lambda_2(U_+). \tag{4.7}$$

Equations (4.4) and (4.7) are defining equations for rarefaction waves of types R_1 and R_2, respectively. The nondegeneracy conditions for rarefaction waves of type R_i, which are implicit in our definition of rarefaction, are the speed inequality and the genuine nonlinearity condition (2.1). It is easy to see that properties (D1)–(D3) hold.

Next we consider shock waves. If there is to be a shock wave solution of $U_t + F(U)_x = 0$ from U_- to U_+ with speed s, we must have that:

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0; \tag{E0}$$

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \text{ has an orbit from } U_- \text{ to } U_+. \tag{C0}$$

The two-component equation (E0) is a defining equation. Condition (C0) is an open condition, and therefore is regarded as a nondegeneracy condition, for all but transitional shock waves.

In Tables 4.1–4.4 we list additional defining equations and nondegeneracy conditions for shock waves of various types. The additional defining equations are either equality of the shock speed with a characteristic speed or, for transitional shock waves, a separation equation that implies condition (C0). The wave nondegeneracy conditions are open conditions. The tables omit several types of nondegeneracy conditions, which we assume implicitly: (a) $U_- \neq U_+$; (b) inequality conditions on the eigenvalues that are implied by the shock type (e.g., for an $R \cdot S$ shock, $\lambda_1(U_-) < \lambda_2(U_-) < s$ and $\lambda_1(U_+) < s < \lambda_2(U_+)$); and (c) condition (C0) when it is an open condition (given the defining equations and the listed nondegeneracy conditions).

The additional defining equations and nondegeneracy conditions for classical and overcompressive shock waves are given in Tables 4.1–4.3; the reader should also refer to Figs. 2.1–2.3. In these tables, conditions (C1)–(C5) are that the connection Γ is not *distinguished*; this means the following. For $RS \cdot S$ and $RS \cdot RS$ shock waves, the connection Γ should not lie in the unstable manifold of U_- (i.e., the unique invariant curve tangent to an eigenvector with positive eigenvalue). For $S \cdot SA$ and $SA \cdot SA$ shock waves, the connection Γ should not lie in the stable manifold of U_+. For $RS \cdot SA$ shock waves, the connection Γ should not lie in either the unstable manifold of U_- or the stable manifold of U_+.

To treat the transitional shock waves (refer to Fig. 2.4), suppose that $w^*\colon U_-^* \xrightarrow{s^*} U_+^*$ is a shock wave for $U_t + F^*(U)_x = 0$ of type $S \cdot S$, $S \cdot RS$, or $SA \cdot RS$. (Shock waves of type $SA \cdot S$ are related to those of type $S \cdot RS$ by the correspondence (2.18).) Thus we suppose that, for the differential equation

$$\dot{U} = F^*(U) - F^*(U_-^*) - s^*(U - U_-^*), \tag{4.8}$$

TABLE 4.1

Additional Defining Equations and Nondegeneracy Conditions for Slow Shock Waves

Type of Shock	Additional defining equations		Nondegeneracy conditions	
$R \cdot S$	none		none	
$R \cdot RS$	$\lambda_1(U_+) - s = 0$	(E1)	$D\lambda_1(U_+)\, r_1(U_+) \neq 0$	(G1)
			$l_1(U_+)(U_+ - U_-) \neq 0$	(B1)
$RS \cdot S$	$\lambda_1(U_-) - s = 0$	(E2)	$D\lambda_1(U_-)\, r_1(U_-) \neq 0$	(G2)
			not distinguished connection	(C1)
$RS \cdot RS$	$\lambda_1(U_-) - s = 0$	(E3)	$D\lambda_1(U_-)\, r_1(U_-) \neq 0$	(G3)
	$\lambda_1(U_+) - s = 0$	(E4)	$D\lambda_1(U_+)\, r_1(U_+) \neq 0$	(G4)
			$l_1(U_+)(U_+ - U_-) \neq 0$	(B2)
			not distinguished connection	(C2)

TABLE 4.2

Additional Defining Equations and Nondegeneracy Conditions for Fast Shock Waves

Type of shock	Additional defining equations		Nondegeneracy conditions	
$S \cdot A$	none		none	
$SA \cdot A$	$\lambda_2(U_-) - s = 0$	(E5)	$D\lambda_2(U_-)\, r_2(U_-) \neq 0$	(G5)
			$l_2(U_-)(U_+ - U_-) \neq 0$	(B3)
$S \cdot SA$	$\lambda_2(U_+) - s = 0$	(E6)	$D\lambda_2(U_+)\, r_2(U_+) \neq 0$	(G6)
			not distinguished connection	(C3)
$SA \cdot SA$	$\lambda_2(U_-) - s = 0$	(E7)	$D\lambda_2(U_-)\, r_2(U_-) \neq 0$	(G7)
	$\lambda_2(U_+) - s = 0$	(E8)	$D\lambda_2(U_+)\, r_2(U_+) \neq 0$	(G8)
			$l_2(U_-)(U_+ - U_-) \neq 0$	(B4)
			not distinguished connection	(C4)

TABLE 4.3

Additional Defining Equations and Nondegeneracy Conditions for Overcompressive Shock Waves

Type of shock	Additional defining equations		Nondegeneracy conditions	
$R \cdot A$	none		none	
$R \cdot SA$	$\lambda_2(U_+) - s = 0$	(E9)	$D\lambda_2(U_+)\, r_2(U_+) \neq 0$	(G9)
$RS \cdot A$	$\lambda_1(U_-) - s = 0$	(E10)	$D\lambda_1(U_-)\, r_1(U_-) \neq 0$	(G10)
$RS \cdot SA$	$\lambda_1(U_-) - s = 0$	(E11)	$D\lambda_1(U_-)\, r_1(U_-) \neq 0$	(G11)
	$\lambda_2(U_+) - s = 0$	(E12)	$D\lambda_2(U_+)\, r_2(U_+) \neq 0$	(G12)
			not distinguished connection	(C5)

U_-^* is an equilibrium of saddle or saddle-attractor type, U_+^* is an equilibrium of saddle or repeller-saddle type, and there is a solution $\tilde{U}: \mathbb{R} \to \mathbb{R}^2$ such that $\lim_{\xi \to \pm\infty} \tilde{U}(\xi) = U_\pm^*$ and $\tilde{U}(\xi) \in \Gamma^*$ for all $\xi \in \mathbb{R}$.

If U_-^* is a saddle of Eq. (4.8), let $W_-(U_-^*, s^*)$ denote its unstable manifold; if U_+^* is a saddle of Eq. (4.8), let $W_\pm(U_-^*, s^*)$ denote its stable manifold. Similarly, if $U_\pm^*$ is a repeller-saddle or saddle-attractor, let $W_\pm(U_-^*, s^*)$ denote one of its center manifolds. The manifolds $W_\pm(U_-^*, s^*)$ both perturb smoothly to invariant manifolds of Eq. (2.2), denoted $W_\pm(U_-, s)$. When U_-^* is a saddle, $W_-(U_-, s)$ is just the unstable manifold of the saddle U_- of Eq. (2.2); when U_+^* is a saddle, $W_+(U_-, s)$ is the stable manifold of the saddle of Eq. (2.2) near U_+^*.

Let Σ be a line segment through $\tilde{U}(0)$ transverse to $\dot{\tilde{U}}(0)$ in the direction V. See Fig. 4.1. Then $W_\pm(U_-, s)$ meet Σ in points $\bar{U}_\pm(U_-, s)$, and

$$\bar{U}_-(U_-, s) - \bar{U}_+(U_-, s) = S(U_-, s)\, V. \tag{4.9}$$

The function S is called the *separation function*; it is defined on a neighborhood of (U_-^*, s^*), and, of course, $S(U_-^*, s^*) = 0$. The partial derivatives of S are given as follows [22]. The linear differential equation

$$\dot{\phi} + \phi[DF(\tilde{U}(\xi)) - s^* I] = 0 \tag{4.10}$$

has, up to constant multiple, a unique bounded solution. For the correct choice of this constant,

$$\frac{\partial S}{\partial s}(U_-^*, s^*) = -\int_{-\infty}^{\infty} \phi(\xi)(\tilde{U}(\xi) - U_-^*)\, d\xi, \tag{4.11}$$

$$D_{U_-} S(U_-^*, s^*) = -\left(\int_{-\infty}^{\infty} \phi(\xi)\, d\xi\right) \{DF(U_-^*) - s^* I\}. \tag{4.12}$$

Since we want to treat the fast $SA \cdot S$ shock waves analogously to the slow $S \cdot RS$ shock waves, using the correspondence (2.18), we shall also consider the family of differential equations

$$\dot{U} = F^*(U) - F^*(U_+^*) - s^*(U - U_+^*) \tag{4.13}$$

with solution $\tilde{U}$ from U_-^* to U_+^*, as well as families of invariant manifolds $\tilde{W}_\pm$ and separation function $\tilde{S}$, defined for F near F^*, U near U_-^*, s near s^*, and U_+ near U_+^*. Then

$$\frac{\partial \tilde{S}}{\partial s}(s^*, U_+^*) = -\int_{-\infty}^{\infty} \phi(\xi)(\tilde{U}(\xi) - U_+^*)\, d\xi, \tag{4.14}$$

$$D_{U_+} \tilde{S}(s^*, U_+^*) = -\left(\int_{-\infty}^{\infty} \phi(\xi)\, d\xi\right) \{DF(U_+^*) - s^* I\}. \tag{4.15}$$

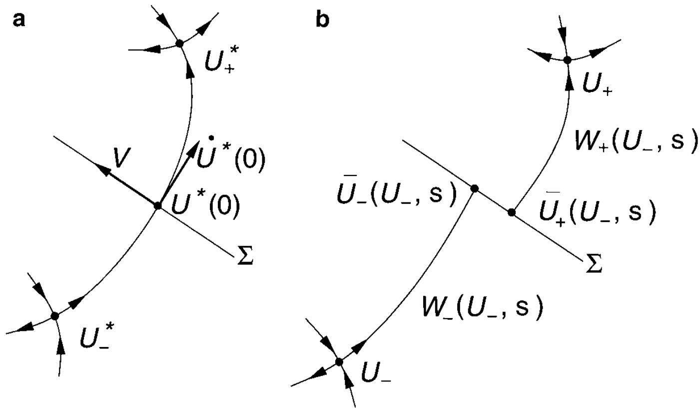

FIG. 4.1. Geometry of the separation function. The diagram (a) corresponds to the differential equation $\dot{U} = F(U) - F(U_-^*) - s^*(U - U_-^*)$; diagram (b) corresponds to $\dot{U} = F(U) - F(U_-) - s(U - U_-)$.

Additional local defining equations and nondegeneracy conditions for transitional shock waves are given in Table 4.4. In this table, conditions (T2)–(T4) are transversality conditions. Condition (T2) is that there is a vector W such that

$$\begin{pmatrix} l_1(U_+) \\ \int_{-\infty}^{\infty} \phi(\xi)\, d\xi \end{pmatrix} W$$

$$\text{and} \quad \begin{pmatrix} l_1(U_+)(U_+ - U_-) \\ \int_{-\infty}^{\infty} \phi(\xi)(U(\xi) - U_-)\, d\xi \end{pmatrix} \quad \text{are linearly independent.} \tag{T2}$$

Condition (T3) is that there is a vector W such that

$$\begin{pmatrix} l_2(U_-) \\ \int_{-\infty}^{\infty} \phi(\xi)\, d\xi \end{pmatrix} W$$

$$\text{and} \quad \begin{pmatrix} l_2(U_-)(U_- - U_+) \\ \int_{-\infty}^{\infty} \phi(\xi)(U(\xi) - U_+)\, d\xi \end{pmatrix} \quad \text{are linearly independent.} \tag{T3}$$

Condition (T4) is that

$$\begin{pmatrix} l_1(U_+)\, r_1(U_-) & l_1(U_+)(U_+ - U_-) \\ (\int_{-\infty}^{\infty} \phi(\xi)\, d\xi)\, r_1(U_-) & \int_{-\infty}^{\infty} \phi(\xi)(U(\xi) - U_-)\, d\xi \end{pmatrix} \quad \text{is invertible.} \tag{T4}$$

It is easy to see that if $w^*\colon U_-^* \xrightarrow{s^*} U_+^*$ is an elementary wave of some type T for $U_t + F^*(U)_x = 0$, then there are neighborhoods $\mathscr{U}_\pm$ of $U_\pm^*$, $\mathscr{I}$ of

TABLE 4.4

Additional Defining Equations and Nondegeneracy Conditions for Transitional Shock Waves

Type of shock	Additional defining equations		Nondegeneracy conditions	
$S \cdot S$	$S(U_-, s) = 0$	(S1)	$DS(U_-, s) \neq 0$	(T1)
$S \cdot RS$	$\lambda_1(U_+) - s = 0$	(E13)	$D\lambda_1(U_+)\, r_1(U_+) \neq 0$	(G13)
	$S(U_-, s) = 0$	(S2)	transversality	(T2)
$SA \cdot S$	$\lambda_2(U_-) - s = 0$	(E14)	$D\lambda_2(U_-)\, r_2(U_-) \neq 0$	(G14)
	$\tilde{S}(s, U_+) - s = 0$	(S3)	transversality	(T3)
$SA \cdot RS$	$\lambda_2(U_-) - s = 0$	(E15)	$D\lambda_2(U_-)\, r_2(U_-) \neq 0$	(G15)
	$\lambda_1(U_+) - s = 0$	(E16)	$D\lambda_1(U_+)\, r_1(U_+) \neq 0$	(G16)
	$S(U_-, s) = 0$	(S4)	transversality	(T4)

s^*, and $\mathscr{F}$ of F^* such that the left-hand sides of the equations for waves of type T that we have given constitute a C^1 map $G_T\colon \mathscr{U}_- \times \mathscr{I} \times \mathscr{U}_+ \times \mathscr{F} \to \mathbb{R}^e$, where $e = 2, 3, 4$, or 5. (Recall that $\mathscr{F} \subseteq \mathscr{B}$, $\mathscr{B}$ being a the Banach space of C^2 maps). By checking the number e for each wave type, the reader can verify the assignment of Riemann numbers in Table 2.2.

THEOREM 4.1. *Let $w^*\colon U_-^* \xrightarrow{s^*} U_+^*$ be an elementary wave of type T for $U_t + F^*(U)_x = 0$, and assume that the appropriate wave nondegeneracy conditions from Tables* 4.1–4.4 *are satisfied at* (U_-^*, s^*, U_+^*, F^*). *Then there are neighborhoods $\mathscr{U}_\pm$ of $U_\pm^*$, $\mathscr{I}$ of s^*, $\mathscr{F}$ of F^*, and $\mathscr{N}$ of w^* such that G_T is defined on $\mathscr{U}_- \times \mathscr{I} \times \mathscr{U}_+ \times \mathscr{F}$ and satisfies properties* (D1)–(D3).

Proof. As stated above, the result holds for rarefaction waves. For shock waves, the proof of properties (D1) and (D3) uses only the nondegeneracy conditions of classes G and C:

Case 1. $w^*\colon U_-^* \xrightarrow{s^*} U_+^*$ has type $R \cdot S$. Then for (U_-, s, U_+, F) near (U_-^*, s^*, U_+^*, F^*), condition (E0) holds if and only if, for the differential equation (2.2), U_- is a repeller equilibrium and U_+ is a saddle equilibrium. Moreover, the connection from U_- to U_+ is stable to perturbation.

Case 2. $w^*\colon U_-^* \xrightarrow{s^*} U_+^*$ has type $R \cdot RS$. Then for (U_-, s, U_+, F) near (U_-^*, s^*, U_+^*, F^*), conditions (E0) and (E1) hold if and only if for the differential equation (2.2), U_- is a repeller equilibrium and U_+ is a repeller-saddle equilibrium. The local center manifold of U_+ is near that of U_+^*, and the nondegeneracy condition (G1) ensures that U_+ and U_+^* have the same quadratic behavior on their center manifolds. Thus the connection of Eq. (4.8) from U_-^* to U_+^* perturbs to a connection of Eq. (2.2) from U_- to U_+.

Case 3. $w^*: U_-^* \xrightarrow{s^*} U_+^*$ has type $RS \cdot S$. The argument is similar to case 2, but notice that to prove persistence of the connection, we use condition (C1) as well as (G2).

Case 4. $w^*: U_-^* \xrightarrow{s^*} U_+^*$ has type $RS \cdot RS$. Then for (U_-, s, U_+, F) near (U_-^*, s^*, U_+^*, F^*), conditions (E0), (E3), and (E4) hold if and only if U_- and U_+ are both repeller-saddle equilibria for the differential equation (2.2). To prove persistence of the connection, we invoke conditions (G3), (G4), and (C2).

Case 5. The proofs for shock waves of fast type follow from the preceding cases and the correspondence (2.18).

Case 6. The proofs for the overcompressive shock waves are left to the reader.

Case 7. We will not give detailed arguments for the transition shock waves; we simply remark that the only nondegeneracy conditions that are used are those of class G. Also, we note that zeroes of the separation function correspond to connections that lie near Γ^*. There may, of course, be connections from U_- to U_+ that do not lie near Γ^* (see Fig. 4.2 for an example).

Property (D2) is verified for shock waves of types $R \cdot S$, $R \cdot RS$, $RS \cdot S$, $RS \cdot RS$, $S \cdot S$, $SA \cdot RS$, and $S \cdot RS$ in Secs. 5–7. (See part 1 of Proposition 5.1, Lemma 5.3, the proof of part 2 of Proposition 5.2, Lemma 5.4, Proposition 6.3, and Lemmas 7.1 and 6.2.) The overcompressive cases $R \cdot A$, $R \cdot SA$, $RS \cdot A$, and $RS \cdot SA$ can be proved in a similar manner and

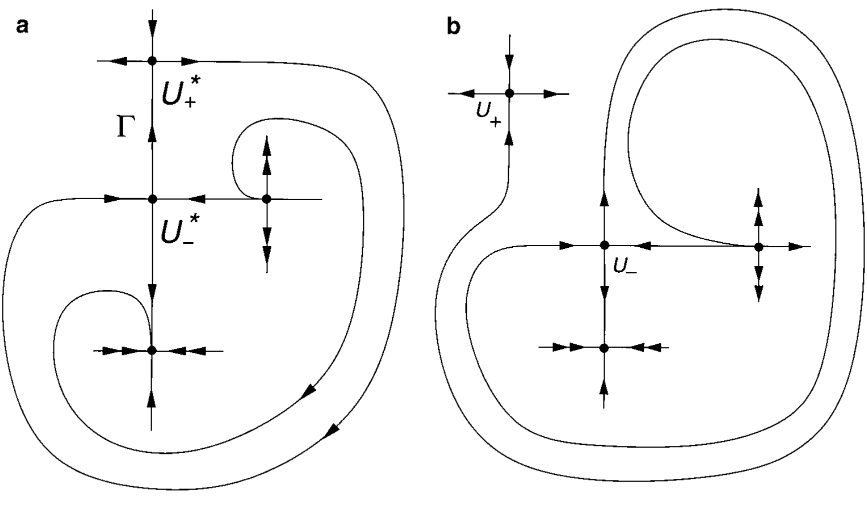

FIG. 4.2. The connection Γ from U_-^* to U_+^* in diagram (a) perturbs to a connection from U_- to U_+ in diagram (b) that does not lie in a small neighborhood of Γ.

are left to the reader. The remaining cases follow from the transformation (2.18). Note that the nondegeneracy conditions of class C are not used in these proofs. ∎

5. 1-Wave and 2-Wave Groups

To simplify the notation in most of the remainder of the paper, we will not show the dependence of the local defining maps G_T and G on the flux function F, and we will denote the flux function under consideration by F rather than F^*. Also, we will frequently denote an elementary wave simply by $U_- \xrightarrow{s} U_+$ and an allowed sequence of elementary waves by $U_0 \xrightarrow{s_1} \cdots \xrightarrow{s_n} U_n$.

In this section we prove the Structural Stability Theorem in the absence of transitional wave groups and $SA \cdot RS$ waves. We first analyze the 1- and 2-wave groups separately, and then prove our result.

Proposition 5.1. 1. *Let $U_-^* \xrightarrow{s^*} U_+^*$ be an $R \cdot S$ shock wave for $U_t + F(U)_x = 0$, so that the left-hand side of the defining equation $(E0)$ is a map G_T from (U_-, s, U_+)-space to $\mathbb{R}^2$. Then the linear map*

$$DG_T(U_-^*, s^*, U_+^*) \mid \{(\dot{U}_-, \dot{s}, \dot{U}_+) : \dot{U}_- = 0\} \tag{5.1}$$

is surjective. Moreover, there is a nonzero vector $\partial U_+/\partial s$ such that $DG_T(U_-^*, s^*, U_+^*) \cdot (0, \dot{s}, \dot{U}_+) = 0$ if and only if $\dot{U}_+ = (\partial U_+/\partial s)\dot{s}$.

2. *Let*

$$U_0^* \xrightarrow{s_1^*} \cdots \xrightarrow{s_k^*} U_k^* \tag{5.2}$$

be a composite 1-wave group ($k \geqslant 2$) or a 1-rarefaction wave ($k = 1$) for $U_t + F(U)_x = 0$, with local defining map G. Assume that each wave satisfies its nondegeneracy conditions. Then

$$DG(U_0^*, s_1^*, ..., s_k^*, U_k^*) \mid \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_k, \dot{U}_k) : \dot{U}_0 = 0\} \tag{5.3}$$

is surjective. Moreover, if $\tilde{k}$ is the index of the last rarefaction (so that $\tilde{k} = k-1$ or k), then there are nonzero vectors $\partial U_i/\partial s_k$, $\tilde{k} \leqslant i \leqslant k$, such that

$$DG(U_0^*, s_1^*, ..., s_k^*, U_k^*) \cdot (0, \dot{s}_1, ..., \dot{s}_k, \dot{U}_k) = 0 \tag{5.4}$$

if and only if: $\dot{s}_i = \dot{s}_k$ and $\dot{U}_i = (\partial U_i/\partial s_k)\, \dot{s}_k$ for $\tilde{k} \leqslant i \leqslant k$; $\dot{s}_i = 0$ and $\dot{U}_i = 0$ for $0 < i < \tilde{k}$.

Remark. The sum of the Riemann numbers of the elementary waves (2.14) in a 1-wave group is 1. Therefore, for a 1-wave group, G is a map from $\mathbb{R}^{3k+2}$ into $\mathbb{R}^{3k-1}$.

Proposition 5.1 has the following familiar interpretation in terms of wave curves.

In the situation of part 1 of Proposition 5.1, if we fix $U_- = U_-^*$ in the defining equation $G_T(U_-, s, U_+) = 0$, then by the implicit function theorem we can solve for U_+ as a function of s near $s = s^*$, $U_+ = U_+^*$. The curve $U_+(s)$ is part of the 1-shock curve based at U_-^*; its tangent vector at $s = s^*$ is $\partial U_+/\partial s$.

In the situation of part 2 of Proposition 5.1, if we fix $U_0 = U_0^*$ in $G(U_0, s_1, ..., s_k, U_k) = 0$, we can solve for the remaining U_i and s_i in terms of s_k near $(s_1^*, ..., s_k^*, U_k^*)$. For $i < \tilde{k}$ these are actually independent of s_k. The curve $U_k(s_k)$ is part of the composite 1-wave curve based at U_0^*, or part of the 1-rarefaction curve based at U_0^* if $k = 1$; its tangent vector at $s = s^*$ is $\partial U_k/\partial s_k$.

Once Proposition 5.1 is proved, the correspondence (2.18) immediately yields.

Proposition 5.2. 1. *Let* $U_-^* \xrightarrow{s^*} U_+^*$ *be an* $S \cdot A$ *shock wave for* $U_t + F(U)_x = 0$, *so that the left-hand side of the defining equation* $(E0)$ *is a map* G_T *from* (U_-, s, U_+)*-space to* $\mathbb{R}^2$. *Then*

$$DG_T(U_-^*, s^*, U_+^*) \mid \{(\dot{U}_-, \dot{s}, U_+) : U_+ = 0\} \tag{5.5}$$

is surjective. Moreover, there is a nonzero vector $\partial U_-/\partial s$ *such that* $DG(U_-^*, s^*, U_+^*) \cdot (\dot{U}_-, \dot{s}, 0) = 0$ *if and only if* $\dot{U}_- = (\partial U_-/\partial s)\dot{s}$.

2. *Let*

$$U_l^* \xrightarrow{s_{l+1}^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{5.6}$$

be a composite 2*-wave group* $(n - l \geqslant 2)$ *or a* 2*-rarefaction* $(n - l = 1)$ *for* $U_t + F(U)_x = 0$, *with local defining map* G. *Assume each wave satisfies its nondegeneracy conditions. Then*

$$DG(U_l^*, s_{l+1}^*, ..., s_n^*, U_n^*) \mid \{(\dot{U}_l, \dot{s}_{l+1}, ..., \dot{s}_n, \dot{U}_n) : \dot{U}_n = 0\} \tag{5.7}$$

is surjective. Moreover, if $\tilde{l} + 1$ *is the index of the first rarefaction (so that* $\tilde{l} = l$ *or* $l + 1$*), then there are nonzero vectors* $\partial U_i/\partial s_{l+1}$, $l \leqslant i \leqslant \tilde{l}$, *such that*

$$DG(U_l^*, s_{l+1}^*, ..., s_n^*, U_n^*) \cdot (\dot{U}_l, \dot{s}_{l+1}, ..., \dot{s}_n, 0) = 0 \tag{5.8}$$

if and only if: $\dot{s}_{i+1} = \dot{s}_{l+1}$ *and* $\dot{U}_i = (\partial U_i/\partial s_{l+1})\, \dot{s}_{l+1}$ *for* $l \leqslant i \leqslant \tilde{l}$; $\dot{s}_{i+1} = 0$ *and* $\dot{U}_i = 0$ *for* $\tilde{l} < i < n$.

The geometric interpretation of Proposition 5.2 is in terms of backwards wave curves. In the situation of part 1 of Proposition 5.2, if we fix

$U_+ = U_+^*$ in $G_T(U_-, s, U_+) = 0$, we can solve for U_- as a function of s near $U_- = U_-^*$, $s = s^*$. The curve $U_-(s)$ is part of the backwards 2-shock curve based at U_+^*; its tangent vector at $s = s^*$ is $\partial U_-/\partial s$. Similarly, in the situation of part 2 of Proposition 5.2, if we fix $U_n = U_n^*$ in $G(U_l, s_{l+1}, ..., s_n, U_n) = 0$, we can solve for the remaining U_i and s_i in terms of s_{l+1}, near $(U_l^*, s_{l+1}^*, ..., s_n^*)$. The curve $U_l(s_{l+1})$ is part of the backwards composite 2-wave curve based at U_n^*, or part of the backwards 2-rarefaction curve based at U_n^* if $l = n-1$; this tangent vector at $s = s^*$ is $\partial U_l/\partial s_{l+1}$.

Proof of Proposition 5.1. 1. We have

$$DG_T(U_-^*, s^*, U_+^*) \cdot (0, \dot{s}, \dot{U}_+) = (DF(U_+^*) - s^*I)\, \dot{U}_+ - \dot{s}(U_+^* - U_-^*) = 0 \tag{5.9}$$

if and only if

$$\dot{U}_+ = (DF(U_+^*) - s^*I)^{-1}\, (U_+^* - U_-^*)\dot{s}. \tag{5.10}$$

Thus the kernel of the linear map (5.1) from $\mathbb{R}^3$ to $\mathbb{R}^2$ is one-dimensional, so that it is surjective. Moreover, Eq. (5.10) implies that

$$\frac{\partial U_+}{\partial s} = (DF(U_+^*) - s^*I)^{-1}\, (U_+^* - U_-^*) \neq 0. \tag{5.11}$$

2. The proof of this part will be given after two lemmas.

LEMMA 5.3. *Let* $U_-^* \xrightarrow{s^*} U_+^*$ *be an* $R \cdot RS$ *shock wave for* $U_t + F(U)_x = 0$, *so that the left-hand sides of the defining equations* $(E0)$ *and* $(E1)$ *form a map* G_T *from* (U_-, s, U_+)*-space to* $\mathbb{R}^2$. *Then*

$$DG_T(U_-^*, s^*, U_+^*) \mid \{(\dot{U}_-, \dot{s}, \dot{U}_+) : \dot{U}_- = 0\} \tag{5.12}$$

is invertible if and only if the wave nondegeneracy conditions are verified.

Proof. Referring to Table 4.1, we linearize Eqs. (E0) and (E1) at (U_-^*, s^*, U_+^*) and apply to the vector $(0, \dot{s}, \dot{U}_+)$, obtaining

$$DF(U_+^*) - s^*I)\, \dot{U}_+ - \dot{s}(U_+^* - U_-^*) = 0 \tag{5.13}$$

$$D\lambda_1(U_+^*)\, \dot{U}_+ - \dot{s} = 0. \tag{5.14}$$

To show that this system of three equations in three unknowns has no nontrivial solutions when (G1) and (B1) hold, let l_i and r_i denote $l_i(U_+^*)$ and

$r_i(U_+^*)$, respectively, write $\dot{U}_+ = ar_1 + br_2$, and multiply the first equation by l_1 and l_2. We obtain:

$$l_1\{(DF(U_+^*) - s^*I)(ar_1 + br_2) - \dot{s}(U_+^* - U_-^*)\} = -\dot{s}l_1(U_+^* - U_-^*) = 0, \tag{5.15}$$

$$l_2\{(DF(U_+^*) - s^*I)(ar_1 + br_2) - \dot{s}(U_+^* - U_-^*)\}$$
$$= b(\lambda_2(U_+^*) - s^*) - \dot{s}l_2(U_+^* - U_-^*) = 0, \tag{5.16}$$

$$D\lambda_1(U_+^*)\, r_1 + bD\lambda_1(U_+^*)\, r_2 - \dot{s} = 0. \tag{5.17}$$

Equation (5.15) and assumption (B1) imply $\dot{s} = 0$. Then Eq. (5.16) and $s^* = \lambda_1(U_+^*) \neq \lambda_2(U_+^*)$ imply $b = 0$. Finally Eq. (5.17) and assumption (G1) imply that $a = 0$. Conversely, if (G1) or (B1) fails, one easily finds nontrivial solutions. ∎

LEMMA 5.4. *Let* $U_-^* \xrightarrow{s^*} U_+^*$ *be an* $RS \cdot RS$ *shock wave for* $U_t + F(U)_x = 0$, *so that the left-hand sides of the defining equations* (*E*0), (*E*3), *and* (*E*4) *form a map* G_T *from* (U_-, s, U_+)*-space to* $\mathbb{R}^4$. *If assumptions* (*G*3), (*G*4), *and* (*B*2) *are satisfied, then* $DG_T(U_-^*, s^*, U_+^*)$ *is surjective, and the one-dimensional kernel is spanned by a vector whose* U_-*-component is linearly independent from* $r_1(U_-^*)$.

Proof. Referring to Table 4.1, we linearize Eqs. (E0), (E3), and (E4) at (U_-^*, s^*, U_+^*) and apply to the vector $(\dot{U}_-, \dot{s}, \dot{U}_+)$, obtaining

$$(DF(U_+^*) - s^*I)\,\dot{U}_+ - (DF(U_-^*) - s^*I)\,\dot{U}_- - \dot{s}(U_+^* - U_-^*) = 0, \tag{5.18}$$

$$D\lambda_1(U_-^*)\,\dot{U}_- - \dot{s} = 0. \tag{5.19}$$

$$D\lambda_1(U_+^*)\,\dot{U}_+ - \dot{s} = 0, \tag{5.20}$$

It suffices to show that the only solution of Eqs. (5.18)–(5.20) with $\dot{U}_-$ being multiple of $r_1(U_-^*)$ is $\dot{U}_- = \dot{U}_+ = 0$, $\dot{s} = 0$.

Write $\dot{U}_+ = ar_1(U_+^*) + br_2(U_+^*)$, $\dot{U}_- = cr_1(U_-^*)$, and multiply Eq. (5.18) by $l_1(U_+^*)$ and $l_2(U_+^*)$. We obtain

$$-\dot{s}l_1(U_+^*)(U_+^* - U_-^*) = 0, \tag{5.21}$$

$$(\lambda_2(U_+^*) - s^*)\,b - \dot{s}l_2(U_+^*)(U_+^* - U_-^*) = 0, \tag{5.22}$$

$$D\lambda_1(U_+^*)(ar_1(U_+^*) + br_2(U_+^*)) - \dot{s} = 0, \tag{5.23}$$

$$D\lambda_1(U_-^*)\, cr_1(U_-^*) - \dot{s} = 0, \tag{5.24}$$

Equation (5.21) and assumption (B2) imply that $\dot{s}=0$. Then Eq. (5.22) and strict hyperbolicity imply that $b=0$, and Eq. (5.24) and assumption (G3) imply that $c=0$. Since $\dot{s}=b=0$, Eq. (5.23) and assumption (G4) imply that $a=0$. ∎

Remark. If any of the nondegeneracy conditions (G3), (G4), or (B2) fails to hold, then $DG_T(U_-^*, s^*, U_+^*)$, restricted to the four-dimensional space considered in the proof, fails to be invertible.

We are now ready to prove the second assertion of Proposition 5.1. For simplicity of exposition, we shall do this only for a representative case for $n=5$ with the following wave types:

$$U_0^* \xrightarrow{s_1^*} U_1^* \quad \text{of type} \quad R \cdot RS, \tag{5.25}$$

$$U_1^* \xrightarrow{s_2^*} U_2^* \quad \text{of type} \quad R_1, \tag{5.26}$$

$$U_2^* \xrightarrow{s_2^*} U_3^* \quad \text{of type} \quad RS \cdot RS, \tag{5.27}$$

$$U_3^* \xrightarrow{s_4^*} U_4^* \quad \text{of type} \quad R_1, \tag{5.28}$$

$$U_4^* \xrightarrow{s_5^*} U_5^* \quad \text{of type} \quad RS \cdot S. \tag{5.29}$$

Then the defining system of equations $G(U_0, s_1, ..., s_5, U_5)=0$ is as follows:

$$F(U_1)-F(U_0)-s_1(U_1-U_0)=0, \tag{5.30}$$

$$\lambda_1(U_1)-s_1=0, \tag{5.31}$$

$$U_2-\psi_1(U_1, s_2)=0, \tag{5.32}$$

$$F(U_3)-F(U_2)-s_3(U_3-U_2)=0, \tag{5.33}$$

$$\lambda_1(U_2)-s_3=0, \tag{5.34}$$

$$\lambda_1(U_3)-s_3=0, \tag{5.35}$$

$$U_4-\psi_1(U_3, s_4)=0, \tag{5.36}$$

$$F(U_5)-F(U_4)-s_5(U_5-U_4)=0, \tag{5.37}$$

$$\lambda_1(U_4)-s_5=0. \tag{5.38}$$

Differentiating at $(U_0^*, s_1^*, ..., s_5^*, U_5^*)$ yields:

$$(DF(U_1^*)-s_1^* I)\,\dot{U}_1-(DF(U_0^*)-s_1^* I)\,\dot{U}_0-\dot{s}_1(U_1^*-U_0^*)=0, \tag{5.39}$$

$$D\lambda_1(U_1^*)\,\dot{U}_1-\dot{s}_1=0, \tag{5.40}$$

$$\dot{U}_2-D\psi_1(U_1^*, s_2^*)(\dot{U}_1, \dot{s}_2)=0, \tag{5.41}$$

$$(DF(U_3^*) - s_3^* I)\,\dot U_3 - (DF(U_2^*) - s_3^* I)\,\dot U_2 - \dot s_3(U_3^* - U_2^*) = 0, \tag{5.42}$$

$$D\lambda_1(U_2^*)\,\dot U_2 - \dot s_3 = 0, \tag{5.43}$$

$$D\lambda_1(U_3^*)\,\dot U_3 - \dot s_3 = 0, \tag{5.44}$$

$$\dot U_4 - D\psi_1(U_3^*, s_4^*)(\dot U_3, \dot s_4) = 0, \tag{5.45}$$

$$(DF(U_5^*) - s_5^* I)\,\dot U_5 - (DF(U_4^*) - s_5^* I)\,\dot U_4 - \dot s_5(U_5^* - U_4^*) = 0, \tag{5.46}$$

$$D\lambda_1(U_4^*)\,\dot U_4 - \dot s_5 = 0. \tag{5.47}$$

Let $\dot U_0 = 0$. By Lemma 5.3, Eqs. (5.39)–(5.40) imply that $\dot s_1 = 0$, $\dot U_1 = 0$. Then Eq. (5.41) implies that $\dot U_2 = c r_1(U_2^*)$ for some c. Therefore Lemma 5.4 and Eqs. (5.42)–(5.44) imply that $\dot U_2 = \dot U_3 = 0$ and $\dot s_3 = 0$. Since $\dot U_2 = 0$, Eq. (5.41) implies that $\dot s_2 = 0$.

Because $\dot U_3 = 0$, Eq. (5.45) implies that $\dot U_4 = \dot s_4 \tilde r_1(U_4^*)$. (Recall that $\tilde r_1(U)$ has been normalized so that $D\lambda_1(U)\,\tilde r_1(U) \equiv 1$.) By Eq. (5.47), $\dot s_4 = \dot s_5$. Moreover, $(DF(U_4^*) - s_5^* I)\,\dot U_4 = 0$, so by Eq. (5.46), $\dot U_5 = (DF(U_5^*) - s_5^* I)^{-1}\,(U_5^* - U_4^*)\,\dot s_5$. Thus we are led to define

$$\frac{\partial U_4}{\partial s_5} = \tilde r_1(U_4^*), \tag{5.48}$$

$$\frac{\partial U_5}{\partial s_5} = (DF(U_5^*) - s_5^* I)^{-1}\,(U_5^* - U_4^*). \tag{5.49}$$

Therefore, in this representative case, the linear map (5.3) from $\mathbb{R}^{15}$ to $\mathbb{R}^{14}$ has one-dimensional kernel, so that it is surjective. ∎

Remark. If any of the nondegeneracy conditions (G1), (B1), (G3), (G4), or (B2) fails, then the linear map (5.3) is not surjective. If condition (G2) fails, then $\dot U_5 = 0$.

Finally, we prove the main result of this section. In this result and its proof, we return to denoting the flux function under consideration by F^*, and to showing the dependence of the local defining map G on F.

THEOREM 5.5 (Structural Stability for Classical Riemann Solutions). *Suppose that the allowed sequence of elementary waves* (2.7) *has* $\sum_{i=1}^{n} \rho(w_i^*) = 2$. *Assume that there is an integer* k, $0 < k < n$, *such that*

$$U_0^* \xrightarrow{s_1^*} \cdots \xrightarrow{s_k^*} U_k^* \tag{5.50}$$

is a 1-*wave group and*

$$U_k^* \xrightarrow{s_{k+1}^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{5.51}$$

is a 2-*wave group. Let* $\partial U_k/\partial s_k$ *be the tangent to the* 1-*wave curve defined in Proposition* 5.1; *let* $\partial U_k/\partial s_{k+1}$ *be the tangent to the* 2-*wave curve defined in Proposition* 5.2. *Assume hypotheses* (H1) *and* (H3) *of the Structural Stability Theorem, and assume that*

($\mathrm{H2}_1$) $\partial U_k/\partial s_k$ *and* $\partial U_k/\partial s_{k+1}$ *are linearly independent.*

Then the wave sequence (2.7) *is structurally stable.*

Assumption ($\mathrm{H2}_1$) says that the 1-wave curve based at U_0^* and the backwards 2-wave curve based at U_n^* meet transversally at U_k^*. This is the *wave group interaction condition* in the absence of transitional wave groups and $SA \cdot RS$ waves. A result similar to Theorem 5.5 follows from the work of Liu [15], in which global assumptions on the flux function assure transversality of wave curves. The work of Furtado [6] implies a similar result, in the context of the Lax admissibility criterion, if local transversality is assumed.

Proof.

Step 1. By Theorem 4.1, applied to each wave, there are neighborhoods $\mathscr{U}_i$ of U_i^*, $\mathscr{I}_i$ of s_i^*, and $\mathscr{F}$ of F^*, such that the local defining map $G=(G_1, ..., G_n)$ of the wave sequence (2.7), which maps $\mathscr{U}_0 \times \mathscr{I}_1 \times \cdots \times \mathscr{I}_n \times \mathscr{U}_n \times \mathscr{F}$ to $\mathbb{R}^{3n-2}$, has the property that $G(U_0, s_1, ..., s_n, U_n, F)=0$ implies the existence of waves w_i: $U_i \xrightarrow{s_i} U_{i+1}$ for $U_t+F(U)_x=0$ of the correct types, for which the maps $\bar{\Gamma}_i$ are continuous.

Step 2. Assume that $G(U_0, s_1, ..., s_n, U_n, F)=0$. We must show that $\sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_n$.

Step 2.1. First we show that the last wave of the 1-wave group has speed strictly less than the first wave of the 2-wave group, i.e., $\sigma_k < \sigma_{k+1}$. It suffices to show that $\sigma_k^* < \sigma_{k+1}^*$. Here σ_i^* is a speed interval for the wave sequence (2.7). Now $U_{k-1}^* \xrightarrow{s_k^*} U_k^*$ is a wave of type R_1, $R \cdot S$, or $RS \cdot S$, and $U_k^* \xrightarrow{s_{k+1}^*} U_{k+1}^*$ is a wave of type R_2, $S \cdot A$, or $S \cdot SA$.

Step 2.1.1. Suppose that $U_{k-1}^* \xrightarrow{s_k^*} U_k^*$ is of type R_1. Then $\sigma_k^* = [\lambda_1(U_{k-1}^*), \lambda_1(U_k^*)]$. If the next wave is of type R_2, then $\sigma_{k+1}^* = [\lambda_2(U_k^*), \lambda_2(U_{k+1}^*)]$. If it is of type $S \cdot *$, then $\sigma_{k+1}^* = [s_{k+1}^*, s_{k+1}^*]$ and $\lambda_1(U_k^*) < s_{k+1}^*$. In either case $\sigma_k^* < \sigma_{k+1}^*$.

Step 2.1.2. If $U_k^* \xrightarrow{s_{k+1}^*} U_{k+1}^*$ is of type R_2, the argument is similar.

Step 2.1.3. If $U_{k-1}^* \xrightarrow{s_k^*} U_k^*$ is of type $* \cdot S$ and the next wave is of type $S \cdot *$, then $\sigma_k^* = [s_k^*, s_k^*]$, $\sigma_{k+1}^* = [s_{k+1}^*, s_{k+1}^*]$, and $s_k^* < s_{k+1}^*$ by assumption (H3).

Step 2.2. Next we consider waves within the 1-wave group; we argue that $\sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_k$. We will give the argument only in the case $k=5$

with waves as in Eqs. (5.25)–(5.29). The inequalities hold because from Eqs. (5.30)–(5.38) we have that:

$$\sigma_1 = [s_1, s_1] \quad \text{with} \quad s_1 = \lambda_1(U_1), \tag{5.52}$$

$$\sigma_2 = [\lambda_1(U_1), \lambda_1(U_2)], \tag{5.53}$$

$$\sigma_3 = [s_3, s_3] \quad \text{with} \quad s_3 = \lambda_1(U_2) = \lambda_1(U_3), \tag{5.54}$$

$$\sigma_4 = [\lambda_1(U_3), \lambda_1(U_4)], \tag{5.55}$$

$$\sigma_5 = [s_5, s_5] \text{ with } s_5 = \lambda_1(U_4). \tag{5.56}$$

Step 2.3. The argument that $\sigma_{k+1} \leqslant \sigma_{k+2} \leqslant \cdots \leqslant \sigma_n$ (i.e., the argument for waves within the 2-wave group) is similar to step 2.2.

Step 3. Next, we show that

$$DG(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*) \mid \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = \dot{U}_n = 0, \dot{F} = 0\} \tag{5.57}$$

is invertible. To do this we show that the only solution of

$$DG(U_0^*, s_1^*, U_1^*, s_2^*, ..., s_n^*, U_n^*, F^*) \cdot (0, \dot{s}_1, \dot{U}_1, \dot{s}_2, ..., \dot{s}_n, 0, 0) = 0 \tag{5.58}$$

is the trivial one. If Eq. (5.58) holds, then by Proposition 5.1, for $\tilde{k} = k - 1$ or k, we have that

$$\dot{s}_i = \dot{s}_k \quad \text{and} \quad \dot{U}_i = \frac{\partial U_i}{\partial s_k} \dot{s}_k \quad \text{for} \quad \tilde{k} \leqslant i \leqslant k; \tag{5.59}$$

$$\dot{s}_i = 0 \quad \text{and} \quad \dot{U}_i = 0 \quad \text{for} \quad 0 < i < \tilde{k}. \tag{5.60}$$

By Proposition 5.2, for $\tilde{l} = k$ or $k + 1$, we have that:

$$\dot{s}_{i+1} = \dot{s}_{k+1} \quad \text{and} \quad \dot{U}_i = \frac{\partial U_i}{\partial s_{k+1}} \dot{s}_{k+1} \quad \text{for} \quad k \leqslant i \leqslant \tilde{l}; \tag{5.61}$$

$$\dot{s}_{i+1} = 0 \quad \text{and} \quad \dot{U}_i = 0 \quad \text{for} \quad \tilde{l} < i < n. \tag{5.62}$$

Since $\partial U_k / \partial s_k$ and $\partial U_k / \partial s_{k+1}$ are linearly independent by $(\mathrm{H2}_1)$, Eqs. (5.59) and (5.61) imply that $\dot{s}_k = 0 = \dot{s}_{k+1}$. It follows easily that the only solution of Eq. (5.58) is the trivial one.

Step 4. Finally, we note that under our hypotheses, continuity, in the Hausdorff topology, of the rarefaction waves follows from basic theorems on the perturbation of solutions to ordinary differential equations, while continuity of the connecting orbits for shock waves follows from stability to perturbation of the connections representing classical shock waves. ∎

Remark. If ($H2_1$) does not hold, then the linear map (5.57) is not invertible.

6. Transitional Wave Groups

In this section we extend the proof of Theorem 5.5 to the case in which transitional wave groups, but not $SA \cdot RS$ waves, are present. We also discuss the geometry of transitional wave groups.

Let

$$U_k^* \xrightarrow{s_{k+1}^*} \cdots \xrightarrow{s_l^*} U_l^* \tag{6.1}$$

be a sequence of one or more transitional wave groups for $U_t + F(U)_x = 0$. Then the local defining map G^t of the wave sequence (6.1) is a map into $\mathbb{R}^{3(l-k)}$, since the sum of the Riemann numbers for the $l-k$ waves is 0. Let V and W be vectors in $\mathbb{R}^2$. The wave sequence (6.1) is *good* with respect to (V, W) provided that $DG^t(U_k^*, s_{k+1}^*, ..., s_l^*, U_l^*)$, restricted to

$$\{(\dot{U}_k, \dot{s}_{k+1}, ..., \dot{s}_l, \dot{U}_l) : \dot{U}_k \text{ is a multiple of } V \text{ and } \dot{U}_l \text{ is a multiple of } W\}, \tag{6.2}$$

is invertible. (The space (6.2) has dimension $3(l-k)$.)

We now give the main result of this section, which is analogous to Theorem 5.5. Again, in this result and its proof, we return to denoting the flux function under consideration by F^*, and to showing the dependence of the local defining map on F.

Theorem 6.1 (Structural Stability with Transitional Wave Groups). *Suppose that the allowed sequence of elementary waves (2.7) has $\sum_{i=1}^n \rho(w_i^*) = 2$. Assume that there are integers k and l, $0 < k < l < n$, such that*

$$U_0^* \xrightarrow{s_1^*} \cdots \xrightarrow{s_k^*} U_k^* \text{ is a 1-wave group;} \tag{6.3}$$

$$U_k^* \xrightarrow{s_{k+1}^*} \cdots \xrightarrow{s_l^*} U_l^* \text{ is a sequence of transitional wave groups;} \tag{6.4}$$

$$U_l^* \xrightarrow{s_{l+1}^*} \cdots \xrightarrow{s_n^*} \text{ is a 2-wave group.} \tag{6.5}$$

Let G^t be the local defining map of $U_k^* \xrightarrow{s_{k+1}^*} \cdots \xrightarrow{s_l^*} U_l^*$; let $\partial U_k/\partial s_k$ be the tangent to the 1-*wave curve defined in Proposition* 5.1; *let $\partial U_l/\partial s_{l+1}$ be the tangent to the* 2-*wave curve defined in Proposition* 5.2. *Assume hypothesis* (H1) *and* (H3) *of the Structural Stability Theorem, and assume that*

(H2$_2$) *G^t is good with respect to $(\partial U_k/\partial s_k, \partial U_l/\partial s_{l+1})$.*

Then the wave sequence (2.7) *is structurally stable.*

Proof. We follow the steps in the proof of Theorem 5.5.

Step 1. By Theorem 4.1, applied to each wave, there are neighborhoods $\mathcal{U}_i$ of U_i^*, $\mathcal{I}_i$ of s_i^*, and $\mathcal{F}$ to F^* such that the local defining map $G=(G_1, ..., G_n)$ of the wave sequence (2.7), which maps $\mathcal{U}_0 \times \mathcal{I}_1 \times \cdots \times \mathcal{I}_n \times \mathcal{U}_n \times \mathcal{F}$ to $\mathbb{R}^{3n-2}$, has the property that $G(U_0, s_1, ..., s_n, U_n, F)=0$ implies the existence of waves w_i: $U_i \xrightarrow{s_i} U_{i+1}$ for $U_t + F(U)_x = 0$ of the correct types, for which the maps $\bar{\Gamma}_i$ are continuous.

Step 2. This step is essentially the same. First we show that the last wave of any wave group in the wave sequence (2.7) has speed strictly less than the first wave of the next wave group. Note that if the two wave groups are transitional, the two waves in question are $* \cdot S$ and $S \cdot *$ waves, so that assumption (H3) is needed. Then we treat waves within wave groups, taking Eqs. (6.27)–(6.31) below as our model for transitional wave groups.

Step 3. Suppose that Eq.(5.58) holds. By Proposition 5.1, for $\tilde{k}=k-1$ or k, we have that:

$$\dot{s}_i = \dot{s}_k \qquad \text{and} \qquad \dot{U}_i = \frac{\partial U_i}{\partial s_k} \dot{s}_k \qquad \text{for} \quad \tilde{k} \leqslant i \leqslant k; \tag{6.6}$$

$$\dot{s}_i = 0 \qquad \text{and} \qquad \dot{U}_i = 0 \qquad \text{for} \quad 0 < i < \tilde{k}. \tag{6.7}$$

By Proposition 5.2, for $\tilde{l}=l$ or $l+1$, we have that:

$$\dot{s}_{i+1} = \dot{s}_{l+1} \qquad \text{and} \qquad \dot{U}_i = \frac{\partial U_i}{\partial s_{l+1}} \dot{s}_{l+1} \qquad \text{for} \quad l \leqslant i \leqslant \tilde{l}; \tag{6.8}$$

$$\dot{s}_{i+1} = 0 \qquad \text{and} \qquad \dot{U}_i = 0 \qquad \text{for} \quad \tilde{l} < i < n. \tag{6.9}$$

Then from Eqs. (6.6), (6.8), and (H2$_2$) we conclude that

$$\dot{s}_k = 0 \qquad \text{and} \qquad \dot{s}_{l+1} = 0, \tag{6.10}$$

$$\dot{s}_i = 0 \quad \text{for} \quad k+1 \leqslant i \leqslant l \qquad \text{and} \qquad \dot{U}_i = 0 \quad \text{for} \quad k \leqslant i \leqslant l. \tag{6.11}$$

It follows easily that the only solution of Eq. (5.58) is the trivial one, so that the linear map (5.57) is invertible.

Step 4. In addition to the observations made in Step 4 of Theorem 5.5, we note that the connections defined by zeroes of the separation function vary continuously in the Hausdorff topology. ∎

Remark. If ($H2_2$) does not hold, then the linear map (5.57) is not invertible.

The remainder of this section is devoted to the question of when ($H2_2$) holds and to its geometric interpretation in terms of wave curves.

We first state a lemma on shock waves of type $S \cdot RS$.

LEMMA 6.2. *Let* $U^*_- \xrightarrow{s^*} U^*_+$ *be an* $S \cdot RS$ *shock wave for* $U_t + F(U)_x = 0$, *so that the left-hand sides of the defining equations* ($E0$), ($E13$), ($S2$) *form a map* G_T *from* (U_-, s, U_+)*-space to* $\mathbb{R}^4$. *Assume the nondegeneracy conditions are satisfied. Let* V *be a vector in* U_-*-space such that*

$$\begin{pmatrix} l_1(U^*_+) \\ \int_{-\infty}^{\infty} \phi(\xi)\, d\xi \end{pmatrix} (DF(U^*_-) - s^* I)\, V \qquad \text{and} \qquad \begin{pmatrix} l_1(U^*_+)(U^*_+ - U^*_-) \\ \int_{-\infty}^{\infty} \phi(\xi)(U(\xi) - U^*_-)\, d\xi \end{pmatrix} \tag{6.12}$$

are linearly independent. Then the linear map

$$DG_T(U^*_-, s^*, U^*_+) \mid \{(\dot U_-, \dot s, \dot U_+) : \dot U_- \text{ is a multiple of } V\} \tag{6.13}$$

is invertible.

Remark. The existence of vectors V satisfying condition (6.12) follows from condition (T2). If V does not satisfy condition (6.12), or if the nondegeneracy condition (G13) fails, then the linear map (6.13) is not invertible.

Proof. Referring to Table 4.4, we linearize Eqs. (E0), (E13), and (S2) at (U^*_-, s^*, U^*_+) and apply to the vector $(\dot U_-, \dot s, \dot U_+)$, obtaining

$$(DF(U^*_+) - s^* I)\, \dot U_+ - (DF(U^*_-) - s^* I)\, \dot U_- - \dot s (U^*_+ - U^*_-) = 0, \tag{6.14}$$

$$D\lambda_1(U^*_+)\, \dot U_+ - \dot s = 0, \tag{6.15}$$

$$\int_{-\infty}^{\infty} \phi(\xi) \{ -(DF(U^*_-) - s^* I)\, \dot U_- - \dot s (U(\xi) - U^*_-) \}\, d\xi = 0. \tag{6.16}$$

Let $r_i = r_i(U_+^*)$, $l_i = l_i(U_+^*)$, write $\dot{U}_+ = ar_1 + br_2$, and let $\dot{U}_- = cV$. We multiply Eq. (6.14) by l_1 and l_2 and obtain:

$$-l_1\{c(DF(U_-^*) - s^*I)\, V + \dot{s}(U_+^* - U_-^*)\} = 0, \tag{6.17}$$

$$(\lambda_2(U_+^*) - s^*)\, b - l_2\{c(DF(U_-^*) - s^*I)\, V + \dot{s}(U_+^* - U_-^*)\} = 0, \tag{6.18}$$

$$D\lambda_1(U_+^*)(ar_1 + br_2) - \dot{s} = 0, \tag{6.19}$$

$$-\left(\int_{-\infty}^{\infty} \phi(\xi)\, d\xi\right) \{c(DF(U_-^*) - s^*I)\, V\}$$

$$-\dot{s} \int_{-\infty}^{\infty} \phi(\xi)(U(\xi) - U_-^*)\, d\xi = 0. \tag{6.20}$$

Now Eqs. (6.12), (6.17), and (6.20) imply that $c = \dot{s} = 0$, so that Eq. (6.18) and $s^* = \lambda_1(U_+^*)$ imply that $b = 0$. Then condition (G13) and Eq. (6.19) imply that $a = 0$. ∎

The next proposition is the key to understanding transitional wave groups geometrically.

PROPOSITION 6.3. *Let*

$$U_k^* \xrightarrow{s_{k+1}^*} \cdots \xrightarrow{s_l^*} U_l^* \tag{6.21}$$

be a single transitional wave group for $U_t + F(U)_x = 0$, *with local defining map G. Assume each wave satisfies its nondegeneracy conditions. Then there exists a subspace* Δ *of* U_k*-space, of dimension* 0 *or* 1, *such that if* $V \notin \Delta$, *then*

$$DG(U_k^*, ..., U_l^*) \mid \{(\dot{U}_k, ..., \dot{U}_l) : \dot{U}_k \text{ is a multiple of } V\} \tag{6.22}$$

is surjective onto $\mathbb{R}^{3(l-k)}$, *and the projection of the one-dimensional kernel to* U_l*-space is one-dimensional. Conversely, if* $V \in \Delta$, *then one of these conclusions fails.*

Proof. *Step* 1. Suppose that the wave sequence (6.21) is a single $S \cdot S$ wave. We rewrite the linear map (6.22) in this case as

$$DG_T(U_-^*, s^*, U_+^*) \mid \{(\dot{U}_-, \dot{s}, \dot{U}_+) : \dot{U}_- \text{ is a multiple of } V\}. \tag{6.23}$$

By linearizing (E0) and (S1) at (U_-^*, s^*, U_+^*) and applying to $(\dot{U}_-, \dot{s}, \dot{U}_+)$, we rewrite the equation $DG_T(U_-^*, s^*, U_+^*)(\dot{U}_-, \dot{s}, \dot{U}_+) = 0$ as

$$(DF(U_+^*) - s^*I)\, \dot{U}_+ - (DF(U_-^*) - s^*I)\, \dot{U}_- - \dot{s}(U_+^* - U_-^*) = 0, \tag{6.24}$$

$$DS(U_-^*, s^*)(\dot{U}_-, \dot{s}) = 0. \tag{6.25}$$

Case (a). Suppose that $(\partial S/\partial s)(U_-^*, s^*) \neq 0$. Let $\dot{U}_- = cV$, $V \neq 0$ arbitrary. Then we can solve Eq. (6.25) for $\dot{s}$, and then solve Eq. (6.24) for $\dot{U}_+$:

$$\dot{U}_+ = (DF(U_+^*) - s^*I)^{-1} \left\{ (DF(U_-^*) - s^*I) - \frac{(U_+^* - U_-^*)\, D_{U_-} S(U_-^*, s^*)}{(\partial S/\partial s)(U_-^*, s^*)} \right\} cV. \tag{6.26}$$

Therefore the linear map (6.23) has a one-dimensional kernel, so that it is surjective.

Since the linear operator in braces in (6.26) is a rank one perturbation of an invertible operator, it is invertible or has rank 1. We must set Δ equal to the kernel of this operator; thus the dimension of Δ is 0 or 1. If $V \notin \Delta$, one sees from (6.26) that the projection of the kernel of the linear map (6.23) to U_+-space is one-dimensional.

Case (b). Suppose that $(\partial S/\partial s)(U_-^*, s^*) = 0$. Then by (T1), $U_+^* \neq U_-^*$ and $D_{U_-} S(U_-^*, s^*) \neq 0$. Let $\Delta = \operatorname{Ker} D_{U_-} S(U_-^*, s^*)$, a one-dimensional subspace of $\mathbb{R}^2$. If $V \notin \Delta$ and $\dot{U}_- = cV$, then Eqs. (6.24)–(6.25) are satisfied if and only if $c = 0$, $\dot{s}$ is arbitary, and $\dot{U}_+ = (DF(U_+^*) - s^*I)^{-1} (U_+^* - U_-^*)\dot{s}$.

Step 2. Suppose that the wave sequence (6.21) is a composite transitional wave group of the slow form (2.15). We shall do only the case $n = 5$ with the following wave types (we set $k = 0$ to simplify the notation):

$$U_0^* \xrightarrow{s_0^*} U_1^* \qquad \text{of type} \quad S \cdot RS, \tag{6.27}$$

$$U_1^* \xrightarrow{s_2^*} U_2^* \qquad \text{of type} \quad R_1, \tag{6.28}$$

$$U_2^* \xrightarrow{s_3^*} U_3^* \qquad \text{of type} \quad RS \cdot RS, \tag{6.29}$$

$$U_3^* \xrightarrow{s_4^*} U_4^* \qquad \text{of type} \quad R_1, \tag{6.30}$$

$$U_4^* \xrightarrow{s_5^*} U_5^* \qquad \text{of type} \quad RS \cdot S. \tag{6.31}$$

Then the equation $G(U_0, s_1, ..., s_5, U_5) = 0$ becomes:

$$F(U_1) - F(U_0) - s(U_1 - U_0) = 0, \tag{6.32}$$

$$\lambda_1(U_1) - s_1 = 0, \tag{6.33}$$

$$S(U_0, s_1) = 0, \tag{6.34}$$

followed by equations Eqs. (5.32)–(5.38). Notice G is a map into $\mathbb{R}^{15}$. Differentiating at $(U_0^*, ..., U_n^*)$, we obtain

$$(DF(U_1)-s_1 I)\,\dot{U}_1-(DF(U_0)-s_1 I)\,\dot{U}_0-\dot{s}_1(U_1-U_0)=0, \quad (6.35)$$

$$D\lambda_1(U_1)\,\dot{U}_1-\dot{s}_1=0, \quad (6.36)$$

$$\int \phi(\xi)\{-(DF(U_0^*)-s_1^* I)\,\dot{U}_0-\dot{s}_1(U(\xi)-U_0)\}\,d\xi=0, \quad (6.37)$$

followed by equations Eqs. (5.41)–(5.47).

Let Δ denote the space of vectors $\dot{U}_0$ such that

$$\begin{pmatrix} l_1(U_1^*) \\ \int_{-\infty}^{\infty} \phi(\xi)\,d\xi \end{pmatrix}(DF(U_0^*)-s^* I)\,\dot{U}_0 \qquad \text{and} \qquad \begin{pmatrix} l_1(U_1^*)(U_1^*-U_0^*) \\ \int_{-\infty}^{\infty} \phi(\xi)(U(\xi)-U_0^*)\,d\xi \end{pmatrix} \quad (6.38)$$

are linearly dependent. The space Δ is one-dimensional because of condition (T2). Suppose that $V \notin \Delta$. We shall identify the kernel of the linear map (6.22).

Let $(\dot{U}_0, \dot{s}_1, ..., \dot{s}_5, \dot{U}_5)$ satisfy Eqs. (6.35)–(6.37) and (5.41)–(5.47) with $\dot{U}_0 = cV$. By Lemma (6.2), $\dot{U}_0=\dot{U}_1=0$ and $\dot{s}_1=0$. Continuing as in the proof of the second assertion of Proposition (5.1), we find that the solution space is one-dimensional and that its projection to U_5-space is the one-dimensional space

$$\tilde{\Delta}=\{\dot{U}_5 : (DF(U_5^*)-s_5^* I)\,\dot{U}_5 \text{ is a multiple of } U_5^*-U_4^*\}. \quad (6.39)$$

Step 3. For the mapping $G(U_0, s_1, ..., s_5, U_5)$ studied in step 2, suppose that $W \notin \tilde{\Delta}$. We shall show that

$$DG(U_0^*, s_1^*, ..., s_5^*, U_5^*) \mid \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_5, \dot{U}_5) : \dot{U}_5 \text{ is a multiple of } W\} \quad (6.40)$$

is surjective and that the projection of the one-dimensional kernel to U_0-space is one-dimensional (in fact it is Δ). Using the correspondence (2.18), this implies that the proposition is true for a typical transitional wave group (6.21) of the fast form (2.16).

Suppose $V \notin \Delta$ and $W \notin \tilde{\Delta}$. Let K_1 denote the kernel of

$$DG(U_0^*, s_1^*, ..., s_5^*, U_5^*) \mid \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_5, \dot{U}_5) : \dot{U}_0 \text{ is a multiple of } V\}, \quad (6.41)$$

K_2 the kernel of the linear map (6.40), and K the kernel of $DG(U_0^*, s_1^*, ..., s_5^*, U_5^*)$. Then

$$K_1 = K \cap \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_5, \dot{U}_5) : \dot{U}_0 \text{ is a multiple of } V\}, \quad (6.42)$$

$$K_2 = K \cap \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_5, \dot{U}_5) : \dot{U}_5 \text{ is a multiple of } W\}. \quad (6.43)$$

We have that $\dim K_1 = 1$ (from step 2), that $\dim K_2 \geqslant 1$ (since the map (6.40) has domain of dimension 16 and range of dimension 15), and that $\dim K = 2$ (because the surjectivity of the linear map (6.41) implies that of $DG(U_0^*, s_1^*, ..., s_5^*, U_5^*)$). But $K_1 \cap K_2 = \{0\}$, since by step 2 the projection of K_1 to U_5-space is precisely $\tilde{\Delta}$. Therefore $\dim K_2 = 1$. Since $K_1 \cap K_2 = \{0\}$, the projection of K_2 to U_0-space is not contained in the span of V. Thus this projection is one-dimensional and transverse to V. Since this is true for any $V \notin \Delta$, in fact the projection is Δ. ∎

Remark. If any wave in the sequence (6.21) fails to satisfy a nondegeneracy condition of class G, B, or S, then it is impossible to find a vector V in U_0-space such that both conclusions of Proposition 6.3 hold.

The geometric significance of Proposition 6.3 is the following. Let $\mathscr{C}$ be a regular curve (one-dimensional submanifold) in U_k-space through U_k^* whose tangent line at U_k^* is Span V, $V \notin \Delta$. Then

$$\{(U_k, s_{k+1}, ..., s_l, U_l) : U_k \in \mathscr{C} \text{ and } G(U_k, s_{k+1}, ..., s_l, U_l) = 0\} \quad (6.44)$$

is itself a regular curve near $(U_k^*, s_{k+1}^*, ..., s_l^*, U_l^*)$, and the projection of this curve to U_l-space is a regular curve $\tilde{\mathscr{C}}$ through U_l^*. Thus the transitional wave group (6.21) transforms "most" regular curves in U_k-space through U_k^* into regular curves in U_l-space through U_l^*.

The details of how the transformation occurs, however, vary with the nature of the transitional wave group.

Case 1. Suppose that the wave sequence (6.21) is a single $S \cdot S$ wave that satisfies the nondegeneracy condition. Then the local defining map G_T goes from (U_-, s, U_+)-space to $\mathbb{R}^3$, and $G_T^{-1}(0)$ is a two-dimensional manifold whose tangent space at (U_-^*, s^*, U_+^*) is

$$K = \text{Ker } DG_T(U_-^*, s^*, U_+^*). \quad (6.45)$$

Let Π_- denote projection to U_--space and Π_+ projection to U_+-space. We distinguish three cases (see Fig. 6.1).

Case 1(a). Suppose that $\dim \Pi_- K = 2 = \dim \Pi_+ K$. Any regular curve $\mathscr{C}$ through U_-^* is transformed into a regular curve $\tilde{\mathscr{C}}$ through U_+^*, and the tangent line to $\tilde{\mathscr{C}}$ at U_+^* depends on that to $\mathscr{C}$ at U_-^*. This is the case $(\partial S/\partial s)(U_-^*, s^*) \neq 0$ and $\Delta = \{0\}$.

Case 1(b). Suppose that $\dim \Pi_- K = 2$ and $\dim \Pi_+ K = 1$. Generically, the projection of $G_T^{-1}(0)$ to Π_+-space has a fold. Any regular curve $\mathscr{C}$ through U_-^* that is transverse to $\Delta = \Pi_- \text{ Ker}(\Pi_+ \mid K)$ is transformed into a regular curve $\tilde{\mathscr{C}}$ through U_+^*; the tangent space to $\tilde{\mathscr{C}}$ is always $\Pi_+ K$. This is the case $(\partial S/\partial s)(U_-^*, s^*) \neq 0$ and $\dim \Delta = 1$.

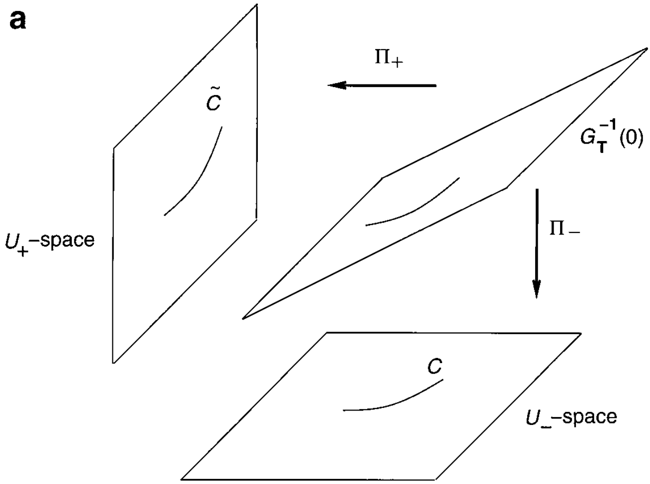

FIG. 6.1. The geometry of $G_T^{-1}(0)$ and the projections Π_- and Π_+ for a single $S \cdot S$ wave. The three cases (a), (b), and (c) are described in the text.

Case 1(c). Suppose that dim $\Pi_- K = 1$. Generically, the projection to U_--space has a fold. Consider a regular curve $\mathscr{C}$ through U_-^* that is transverse to $\Delta = \Pi_- K$. The portion of $\mathscr{C}$ lying to one side of Δ is transformed into a regular curve $\tilde{\mathscr{C}}$ through U_+^*, whose tangent space is always $\Pi_+ \operatorname{Ker}(\Pi_- \mid K)$. This is the case$(\partial S/\partial s)(U_-^*, s^*) = 0$.

Case 2. Suppose that the wave sequence (6.21) is a composite transitional wave curve of the slow form (2.15). From Lemma 6.2 it follows that there is a certain curve $\mathscr{D}$ consisting of states U_k that are the left states of $S \cdot RS$ shock waves. This curve goes through U_k^*; the corresponding right states and wave speeds are near U_{k+1}^* and s_{k+1}^*, respectively. The line tangent to $\mathscr{D}$ at U_k^* is Δ. Now

$$\{(U_k, s_{k+1}, ..., s_l, U_l) : U_k = U_k^* \text{ and } G(U_k, s_{k+1}, ..., s_l, U_l) = 0\} \tag{6.46}$$

is a curve with regular projection to U_l-space; the image is a curve $\tilde{\mathscr{D}}$ with tangent space $\tilde{\Delta}$. In fact, if the last wave of the sequence (6.21) is of type R_1, then the set (6.46) is defined by the equations

$$U_i = U_i^* \qquad \text{for} \quad k \leqslant i \leqslant l-1, \tag{6.47}$$

$$s_i = s_i^* \qquad \text{for} \quad k+1 \leqslant i \leqslant l-1, \tag{6.48}$$

$$U_l = \psi(U_{l-1}^*, s_l) \qquad \text{for} \quad s_l \text{ near } s_l^*, \tag{6.49}$$

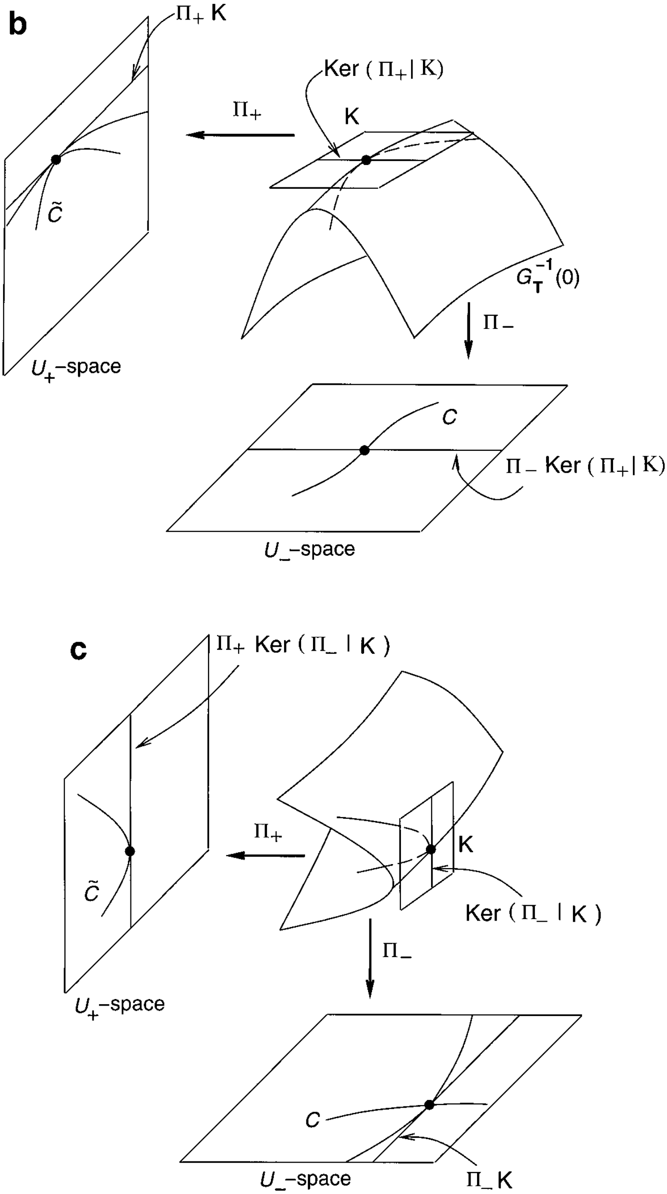

FIG. 6.1—*Continued*

so that $\tilde{\mathscr{D}}$ is the 1-rarefaction curve through U_l^*. If, instead, the last wave of the sequence (6.21) is of type $RS \cdot S$, then the set (6.46) is defined by the equations

$$U_i = U_i^* \qquad \text{for} \quad k \leqslant i \leqslant l-2, \tag{6.50}$$

$$s_i = s_i^* \qquad \text{for} \quad k+1 \leqslant i \leqslant l-2, \tag{6.51}$$

$$U_{l-1} = \psi(U_{l-1}^*, s_{l-1}) \qquad \text{for} \quad s_{l-1} \text{ near } s_l^*, \tag{6.52}$$

$$s_l = s_{l-1}, \tag{6.53}$$

$$F(U_l) - F(U_{l-1}) - s_l(U_l - U_{l-1}) = 0. \tag{6.54}$$

Thus $\tilde{\mathscr{D}}$ is the curve $U_l(s_l)$, with $U_l(s_l^*) = U_l^*$, given above. Any curve $\mathscr{C}$ through U_k^* that is transverse to $\mathscr{D}$ transfers to $\tilde{\mathscr{D}}$. Thus the image curve (not just its tangent line) is independent of $\mathscr{C}$.

Case 3. In the situation of the previous case, a curve in U_l-space transverse to $\tilde{\mathscr{D}}$ transfers to $\mathscr{D}$ in U_k-space. Applying the symmetry (2.18), one sees how the transfer works for fast composite wave curves. Again the image curve is independent of $\mathscr{C}$.

To complete our discussion of a single transitional wave group, we not that if in Theorem 5.1 there is a single transitional wave group, then hypothesis ($H2_2$) has the following interpretation: the one-wave curve is transverse to $\varDelta$ in U_k-space and the transformed one-wave curve in U_l-space (which exists by the previous discussion) is transverse to the 2-wave curve.

Next we discuss sequences of $r \geqslant 2$ transitional wave groups. Let $k = m_0 < m_1 < \cdots < m_r = l$. Let

$$U_{m_0}^* \to \cdots \to U_{m_1}^* \to \cdots \to U_{m_r}^* \tag{6.55}$$

be a sequence of transitional wave groups; for each $i = 1, ..., r$,

$$U_{m_{i-1}}^* \to \cdots \to U_{m_i}^* \tag{6.56}$$

is a transitional wave group. The local defining map G^t of the sequence (6.55), which maps from $(U_{m_0}, ..., U_{m_r})$-space to $\mathbb{R}^{3(m_r - m_0)}$, decomposes as $G^t = (G^1, ..., G^r)$, where $G^i = (G_{m_{i-1}}, ..., G_{m_i})$. Associated with each G^i is a map $\tilde{G}^i$ defined on $(U_{m_{i-1}}, ..., U_{m_i})$-space; $\tilde{G}^i$ is the local defining map of the sequence (6.56). Let Π^i denote the projection of $(U_{m_{i-1}}, ..., U_{m_i})$-space to U_{m_i}-space. Let $\Sigma^0 = \text{span } V \subset U_{m_0}$-space,

$$K^i = \operatorname{Ker} D\tilde{G}^i(U_{m_{i-1}}^*, ..., U_{m_i}^*) \mid \{(\dot{U}_{m_{i-1}}, ..., \dot{U}_{m_i}) : \dot{U}_{m_{i-1}} \in \Sigma^{i-1}\}, \tag{6.57}$$

$$\Sigma^i = \Pi^i K^i \subset U_{m_i}\text{-space}. \tag{6.58}$$

Using these definitions, for a given $V \in U_{m_0}$-space, we can define inductively $\Sigma^0, K^1, \Sigma^1, K^2, \Sigma^2, ..., K^r, \Sigma^r$. We also note that if each wave in the ith transitional wave group satisfies its nondegeneracy conditions, then a subspace Δ^{i-1} of $U_{m_{i-1}}$-space can be defined by Proposition 5.3.

THEOREM 6.4. *In the above situation, assume each wave in the sequence* (6.55) *satisfies its nondegeneracy conditions. Then* G^t *is good with respect to* (V, W) *if and only if*:

(1) *for* $i = 0, ..., r-1$, Σ^i *and* Δ^i *are transverse*;

(2) Σ^r *and* W *are transverse.*

In view of Theorem 6.4 and the previous discussion, hypothesis $(\mathrm{H2}_2)$ of Theorem 6.1 has the following geometric interpretation when there are r transitional wave groups: the first transitional wave group transforms the 1-wave curve to a regular curve $\mathscr{C}^1$, the second transforms $\mathscr{C}^1$ to a regular curve $\mathscr{C}^2$, ..., the rth transforms $\mathscr{C}^{r-1}$ to a regular curve $\mathscr{C}^r$, and $\mathscr{C}^r$ is transverse to the backwards 2-wave curve.

Proof. Assume G^t is good with respect to (V, W). We shall show that for $i = 1, ..., r$:

(a) $D\tilde{G}^i(U^*_{m_{i-1}}, ..., U^*_{m_i}) \mid \{\dot{U}_{m_{i-1}}, ..., \dot{U}_{m_{i-1}}) : \dot{U}_{m_{i-1}} \in \Sigma^{i-1}\}$ is surjective;

(b) Σ^i is one-dimensional.

Then by Proposition 6.3, Σ^i and Δ^i are transverse for $i = 0, ..., r-1$.

Since G^t is good with respect to (V, W), clearly (1) holds for $i = 1$. Therefore $\dim K^1 = 1$. Suppose that $\dim \Sigma^1 = 0$. Then there is a nonzero vector $(\dot{U}_{m_0}, ..., \dot{U}_{m_r})$ with $\dot{U}_i = 0$, $m_1 \leqslant i \leqslant m_r$; $\dot{s}_i = 0$, $m_1 < i \leqslant m_r$; $\dot{U}_0$ a multiple of V; and $DG^t(U^*_{m_0}, ..., U^*_{m_r})(\dot{U}_{m_0}, ..., \dot{U}_{m_r}) = 0$. This contradicts the assumption that G^t is good with respect to (V, W). Therefore $\dim \Sigma^1 = 1$.

Proceeding inductively, suppose that $2 \leqslant j \leqslant r$ and for $i = 1, ..., j-1$, (a) and (b) hold. Since the linear map $DG(U^*_{m_0}, ..., U^*_{m_r})$, restricted to

$$\{(\dot{U}_{m_0}, ..., \dot{U}_{m_r}) : \dot{U}_{m_0} \text{ is a multiple of } V \text{ and } \dot{U}_{m_r} \text{ is a multiple of } W\}, \tag{6.59}$$

is invertible, for any $Z \in \mathbb{R}$ the system

$$D\tilde{G}^1(U^*_{m_0}, ..., U^*_{m_1})(\dot{U}_{m_0}, ..., \dot{U}_{m_1}) = 0 \tag{6.60}$$

$$\vdots$$

$$D\tilde{G}^{j-1}(U^*_{m_{j-2}}, ..., U^*_{m_{j-1}})(\dot{U}_{m_{j-2}}, ..., \dot{U}_{m_{j-1}}) = 0 \tag{6.61}$$

$$D\tilde{G}^j(U^*_{m_{j-1}}, ..., U^*_{m_j})(\dot{U}_{m_{j-1}}, ..., \dot{U}_{m_j}) = Z \tag{6.62}$$

$$D\tilde{G}^{j+1}(U^*_{m_j}, ..., U^*_{m_{j+1}})(\dot{U}_{m_j}, ..., \dot{U}_{m_{j+1}}) = 0 \tag{6.63}$$

$$\vdots$$

$$D\tilde{G}^r(U^*_{m_{r-1}}, ..., U^*_{m_r})(\dot{U}_{m_{r-1}}, ..., \dot{U}_{m_r}) = 0 \tag{6.64}$$

has a solution with $\dot{U}_{m_0}$ a multiple of V and $\dot{U}_{m_r}$ a multiple of W. By the induction hypothesis and Eq. (6.61), we must have $\dot{U}_{m_{j-1}} \in \Sigma^{j-1}$. Then using Eq. (6.63) we see that (a) holds for $i=j$. Therefore $\dim K^j = 1$. If $\dim \Sigma^j = 0$, then we can construct a nonzero vector $(\dot{U}_{m_0}, ..., \dot{U}_{m_r})$ with $\dot{U}_i = 0$ for $m_j \leqslant i \leqslant m_r$, $\dot{s}_i = 0$ for $m_j < i \leqslant m_r$, $\dot{U}_0$ a multiple of V, and

$$DG^t(U^*_{m_0}, ..., U^*_{m_r})(\dot{U}_{m_0}, ..., \dot{U}_{m_r}) = 0. \tag{6.65}$$

This is impossible, so that $\dim \Sigma^j = 1$. This completes the proof by induction of statement (1).

To prove statement (2) given statement (1), we simply note that if Σ^r contains W, we can easily construct a nonzero vector in the kernel of the linear map (6.59).

Now assume that statements (1) and (2) hold. If $(\dot{U}_{m_0}, ..., \dot{U}_{m_r})$ is in the kernel of the linear map (6.59), then

$$(\dot{U}_{m_0}, ..., \dot{U}_{m_1}) \in K^1, ..., (\dot{U}_{m_{r-1}}, ..., \dot{U}_{m_r}) \in K^r. \tag{6.66}$$

But statement (2) implies that $\dot{U}_{m_r} = 0$. Therefore the vectors $(\dot{U}_{m_{r-1}}, ..., \dot{U}_{m_r}) = 0, ..., (\dot{U}_{m_0}, ..., \dot{U}_{m_1}) = 0$. ∎

7. Doubly Sonic Transitional Waves

In this section we extend the proof of Theorem 6.1 to the case in which $SA \cdot RS$ waves are present.

LEMMA 7.1. *Let* $U^*_- \xrightarrow{s^*} U^*_+$ *be an* $SA \cdot RS$ *shock wave for* $U_t + F(U)_x = 0$, *so that the left-hand sides of the defining equations* ($E0$), ($E15$), ($E16$), ($S4$) *form a map* G_T *from* (U_-, s, U_+)*-space to* $\mathbb{R}^5$. *Assume that the nondegeneracy conditions are satisfied. Then* $DG_T(U^*_-, s^*, U^*_+)$ *is invertible.*

Proof. Referring to Table 4.4, we linearize Eqs. (E0), (E15), (E16), and (S4) at (U^*_-, s^*, U^*_+) and apply to the vector $(\dot{U}_-, \dot{s}, \dot{U}_+)$, obtaining

$$(DF(U_+^*)-s^*I)\,\dot{U}_+-(DF(U_-^*)-s^*I)\,\dot{U}_--\dot{s}(U_+^*-U_-^*)=0, \tag{7.1}$$

$$D\lambda_2(U_-^*)\,\dot{U}_--\dot{s}=0, \tag{7.2}$$

$$D\lambda_1(U_+^*)\,\dot{U}_+-\dot{s}=0, \tag{7.3}$$

$$\int_{-\infty}^{\infty}\phi(\xi)\{-(DF(U_-^*)-s^*I)\,\dot{U}_--\dot{s}(U(\xi)-U_-^*)\}\,d\xi=0. \tag{7.4}$$

Write $\dot{U}_-=ar_1(U_-^*)+br_2(U_-^*)$ and $\dot{U}_+=cr_1(U_+^*)+dr_2(U_+^*)$ and multiply Eqs. (7.1)–(7.4) by $l_1(U_+^*)$ and $l_2(U_+^*)$. We obtain:

$$-l_1(U_+^*)\{(\lambda_1(U_-^*)-s^*)\,ar_1(U_-^*)+\dot{s}(U_+^*-U_-^*)\}=0, \tag{7.5}$$

$$(\lambda_2(U_+^*)-s^*)\,d-l_2(U_+^*)\{(\lambda_1(U_-^*)-s^*)\,ar_1(\dot{U}_-^*)+\dot{s}(U_+^*-U_-^*)\}=0, \tag{7.6}$$

$$D\lambda_2(U_-^*)(ar_1(U_-^*)+br_2(U_-^*))-\dot{s}=0, \tag{7.7}$$

$$D\lambda_1(U_+^*)(cr_1(U_+^*)+dr_2(U_+^*))-\dot{s}=0, \tag{7.8}$$

$$-\int_{-\infty}^{\infty}\phi(\xi)\,d\xi(\lambda_1(U_-^*)-s^*)\,ar_1(U_+^*)-\dot{s}\int_{-\infty}^{\infty}\phi(\xi)(U(\xi)-U_-^*)\,d\xi=0. \tag{7.9}$$

By condition (T4), Eqs. (7.5) and (7.9) imply that $a=\dot{s}=0$. Then Eqs. (7.7) and (G15) imply that $b=0$, Eq. (7.6) implies $d=0$, and Eqs. (7.8) and (G16) imply that $c=0$. ∎

THEOREM 7.2 (Structural Stability with Doubly Sonic Transitional Waves). *Suppose that the allowed sequence of elementary waves* (2.7) *has* $\sum_{i=1}^{n}\rho(w_i^*)=2$. *Assume that this sequence has* $m\geqslant 1$ *waves of type* $SA\cdot RS$ *separating wave sequences* $g_0, \ldots, g_m$. *Assume hypotheses* ($H1$) *and* ($H3$) *of the Structural Stability Theorem, and assume that each sequence* g_i, $i=0, \ldots, m$, *satisfies the appropriate hypothesis* ($H2_1$) *of Theorem* 5.5 *or hypothesis* ($H2_2$) *of Theorem* 6.1. *Then the wave sequence* (2.7) *is structurally stable.*

Proof. We follow the steps of Theorems 5.5 and 6.1. Step 1 is essentially the same as in Theorem 6.1. For Step 2, we first show strict inequality between the last speed of one wave group and the first speed of the next wave group within the same g_i. However, for $i=0, \ldots, m-1$, the 2-wave group of g_i, the following $SA \cdot RS$ wave, and the 1-wave group of g_{i+1} should be amalgamated into a wave sequence

$$(w_{l(i)}, ..., w_{p(i+1)}) \tag{7.10}$$

for which it is only true that

$$\sigma_{l(i)} \leqslant \cdots \leqslant \sigma_{p(i+1)}. \tag{7.11}$$

For Step 3, let $w_{k_1}, ..., w_{k_m}$ be the $SA \cdot RS$ waves, so that the g_i are

$$g_0\colon U_0^* \to \cdots \to U_{k_1-1}^* \tag{7.12}$$

$$g_1\colon U_{k_1}^* \to \cdots \to U_{k_2-2}^* \tag{7.13}$$

$$\vdots$$

$$g_m\colon U_{k_m}^* \to \cdots \to U_n^*. \tag{7.14}$$

Let G be the local defining map of the wave sequence (2.7), and suppose that

$$DG(U_0^*, s_1^*, ..., s_n^*, U_n^*) \cdot (0, \dot{s}_1, ..., \dot{s}_n, 0) = 0. \tag{7.15}$$

From Lemma 7.1 we have that

$$\dot{U}_{k_1-1} = \dot{U}_{k_1} = 0, \dot{s}_{k_1} = 0, \tag{7.16}$$

$$\dot{U}_{k_2-1} = \dot{U}_{k_2} = 0, \dot{s}_{k_2} = 0, \tag{7.17}$$

$$\vdots$$

$$\dot{U}_{k_m-1} = \dot{U}_{k_m} = 0, \dot{s}_{k_m} = 0. \tag{7.18}$$

From this and $\dot{U}_0 = \dot{U}_n = 0$, Step 3 in the proofs of Theorem 5.5 and 6.1 tells us that within each g_j, all other $\dot{U}_i = 0$ and $\dot{s}_i = 0$. Finally, Step 4 is the same as in Theorem 6.1. ∎

To obtain a geometric interpretation of Theorem 7.2, consider a Riemann problem

$$U_t + F^*(U)_x = 0, \tag{7.19}$$

$$U(x, 0) = \begin{cases} U_L^* & \text{for} \quad x < 0, \\ U_R^* & \text{for} \quad x > 0. \end{cases} \tag{7.20}$$

with solution (2.7) satisfying $U_0^* = U_L^*$, $U_n^* = U_R^*$, and the assumptions of Theorem 7.2. As in the proof of Theorem 7.2, let the m waves of type

$SA \cdot RS$ in the sequence (2.7) be $U^*_{k_i-1} \xrightarrow{s^*_{k_i}} U^*_{k_i}$, $i=1, ..., m$. Suppose that we vary the Riemann problem data slightly to

$$U(x, 0) = \begin{cases} U_L & \text{for} \quad x<0, \\ U_R & \text{for} \quad x>0. \end{cases} \tag{7.21}$$

The new Riemann solutions is as follows:

1. The middle portion of the new solution,

$$U_{k_1-1} \rightarrow \cdots \rightarrow U_{k_m} \tag{7.22}$$

is exactly the same as the middle portion of the old situation. In particular, $U_{k_1-1} = U^*_{k_1-1}$ and $U_{k_m} = U^*_{k_m}$.

2. The initial portion of the new solution,

$$U_0 \rightarrow \cdots U_{k_1-1}, \tag{7.23}$$

is a sequence of the same types of waves as the first portion of the old solution. Of course, $U_0 = U_L$.

3. The last portion of the new solution,

$$U_{k_m} \rightarrow \cdots \rightarrow U_n, \tag{7.24}$$

is a sequences of the same types of waves as the last portion of the old solution. Of course, $U_n = U_R$.

At present it is not known whether $SA \cdot RS$ waves occur in any physically meaningful systems of conservation laws.

8. Discussion

In this paper we have presented a large class of Riemann solutions, whose component shock waves have viscous profiles, that are structurally stable with respect to perturbation of the left state, the right state, and the flux function.

It should not be hard to show that this class contains every Riemann solution whose structural stability is exhibited by its local defining map, subject to the restrictions that (1) rarefaction waves lie in the strictly hyperbolic region $\mathscr{U}_F$ and (2) shock waves have their left and right states in $\mathscr{U}_F$. Most of the proof of this is contained in various remarks made in the course of this paper. These remarks cover the necessity of assumptions (H1) and (H2) of the Structural Stability Theorem; some additional work will be necessary to verify the necessity of (H3).

We conjecture that if a Riemann solution is structurally stable, then its stability is exhibited by its local defining map. If this were true, then the class of structurally stable solutions presented here would be complete.

It also should not be hard to remove restrictions (1) and (2) above. The only new waves that appear to arise in structurally stable Riemann solutions are the following. (a) A new type of transitional wave group, consisting of a 2-rarefaction wave to a special point in the boundary of $\mathscr{U}_F$, followed by a 1-rarefaction wave from that point, can occur. These transitional rarefaction waves are discussed in Ref. [9]. (b) Shock waves of with repeller and attractor equilibria can now have complex eigenvalues. However, it should be noted that the physical significance of mixed-type models with data in the elliptic region is often unclear.

Of course, it would be interesting to extend the results of this paper to systems of $N > 2$ conservation laws. It would also be interesting to study the effect of removing the restriction $D(U) \equiv I$. In physical applications, the viscosity matrix is not the identity, and not even constant. There is work in progress on this effect for a class of models with quadratic flux functions [7]. For all of these extensions of the current work, the concept of wave manifold [8] should be helpful.

Acknowledgments

Dan Marchesin and Bradley Plohr thank Eli Isaacson (University of Wyoming) and Frederico Furtado (Universidade de Campinas) for allowing them to draw freely from unpublished joint work on structural stability of wave curves and of Riemann solutions in the context of the Lax admissibility condition. Some of this work is contained in Ref. [6].

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