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STABILITY OF COMBUSTION WAVES IN A SIMPLIFIED GAS-SOLID COMBUSTION MODEL IN POROUS MEDIA

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Abstract. We study the stability of the combustion waves that occur in a simplified model for injection of air into a porous medium that initially contains some solid fuel. We determine the essential spectrum of the linearized system at a traveling wave. For certain waves, we are able to use a weight function to stabilize the essential spectrum. We perform a numerical computation of the Evans function to show that some of these waves have no unstable discrete spectrum. The system is partly parabolic, so the linearized operator is not sectorial, and the weight function decays at one end. We use an extension of a recent result about partly parabolic systems that are stabilized by such weight functions to show nonlinear stability.

1. Introduction

This paper is devoted to the stability analysis of combustion waves that arise in a simplified, one-dimensional model of enhanced oil recovery using air injection. In this model, a combustion wave is just a continuous nonconstant traveling wave with constant end states. Understanding the stability of combustion waves helps to maximize oil recovery.

The system we consider models combustion when air is injected into a porous medium that initially contains some solid fuel. The model was proposed in [1] and studied in [5, 6, 7, 15]. It consists of three PDEs that give temperature, oxygen and fuel balance laws. It is a partly parabolic system that has diffusion in the temperature equation and no diffusion in the other equations; we ignore the diffusion of oxygen, and the solid fuel does not diffuse.

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Existence of combustion waves was proved in [7] for the case in which oxygen and heat are transported at the same velocity, and in [15] for the more important case in which oxygen is transported faster than temperature. In [15] six types of combustion waves that approach both end states exponentially and satisfy generic boundary conditions were found. Two are fast combustion waves that propagate faster than oxygen and temperature; two are slow combustion waves, called "reaction-trailing smolder waves" [2], that propagate more slowly than oxygen and temperature; and two are intermediate waves, called "reaction-leading smolder waves" [2, 14], that propagate more slowly than oxygen but faster than temperature.

In this work we study the stability of the combustion waves that were found in [15]. We begin by finding the spectrum of the operator obtained by linearizing the partial differential equation system about a traveling wave.

We first find the essential spectrum using the Fourier transform. It turns out that the essential spectrum is marginally stable (touches the imaginary axis) for all types of combustion waves. For the fast combustion waves we can find a weight function that stabilizes the essential spectrum (moves it to the left of the imaginary axis). We cannot find such a weight function for the other combustion waves. Therefore in the remainder of the paper we study stability of fast combustion waves only.

We continue the linear stability analysis for fast combustion waves by performing a numerical computation of the Evans function to find the discrete spectrum [12]. Some of the waves have no unstable discrete spectrum; others have an unstable eigenvalue because of a saddle-node bifurcation of traveling waves.

In proving nonlinear stability of the fast combustion waves with no unstable discrete spectrum, two issues remain: the system is only partly parabolic, so the linearized operator is not sectorial; and the weight function used to stabilize the essential spectrum decays at one end. With the assumption that there is no unstable discrete spectrum, we complete the proof of nonlinear stability using an extension of a result in [9]. The extension is achieved using [16].

The type of nonlinear stability that is shown is somehat unusual in that perturbations that are small in one norm are shown to decay in a different norm. However, this type of nonlinear stability is quite natural to this and other combustion problems; see the discussion after Theorem 5.1 and in [9].

The paper is organized as follows. We introduce the mathematical model and recall existence results for combustion waves in section 2, then linearize the system about the combustion waves and study the essential spectrum in section 3. For the fast combustion waves, numerical computation of the Evans function is performed to find the discrete spectrum in section 4. We study nonlinear stability of the fast combustion waves in section 5. A type of nonlinear stability follows from an extension of the main result of [9]. We explain this extension in appendix A.

The numerical computation of the Evans function in section 4 does not yield a rigorous proof of linear stability because there is no a priori bound on the location of possible eigenvalues. However, in section 6 we add small diffusion to the oxygen equation and show that for this modified system, a bound on the location of eigenvalues can be found. Our proof uses the technique of [11].

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2. Model and existence of combustion waves

The system we consider consists of three equations that give temperature (θ), fuel (ρ) and oxygen (Y) balance laws:

$$\partial_t\theta + a\partial_x\theta = \partial_{xx}\theta + \rho Y\Phi, \tag{2.1}$$

$$\partial_t\rho = -\rho Y\Phi, \tag{2.2}$$

$$\partial_t Y + b\partial_x Y = -\rho Y\Phi, \tag{2.3}$$

$$\Phi = \begin{cases} e^{-1/\theta}, & \theta > 0, \\ 0, & \theta \le 0, \end{cases}$$

where $a > 0$ and $b > 0$ are thermal and oxygen transport speeds, and Φ is unit reaction rate. Combustion is assumed to occur above a certain ignition temperature; we have normalized so that the ignition temperature is $\theta = 0$. The diffusion of oxygen is neglected. The equations have been nondimensionalized to reduce the number of parameters. For the derivation of the system see [7].

We assume $a < b$, which is correct in rock porous media since the thermal capacity of the gas is much less than the thermal capacity of the medium.

We use constant boundary conditions for (2.1)–(2.3) on $-\infty < x < \infty$, $t \ge 0$:

$$(\theta,\rho,Y)(-\infty,t) = (\theta^-,\rho^-,Y^-), \qquad (\theta,\rho,Y)(\infty,t) = (\theta^+,\rho^+,Y^+). \tag{2.4}$$

We assume the reaction cannot occur at the boundary. Thus at $x = \pm\infty$ we must have one of the following:

(1) low temperature $\theta \le 0$ (temperature control or TC);
(2) lack of fuel $\rho = 0$ (fuel control or FC);
(3) lack of oxygen $Y = 0$ (oxygen control or OC).

A traveling wave solution of (2.1)–(2.3) is a function $(\theta,\rho,Y)(\xi)$, $\xi = x - ct$, with $(\theta,\rho,Y)(-\infty) = (\theta^-,\rho^-,Y^-)$ and $(\theta,\rho,Y)(\infty) = (\theta^+,\rho^+,Y^+)$. We will sometimes denote a wave of velocity c that goes, for example, from a left state of type TC to a right state of type OC by $TC \xrightarrow{c} OC$.

We only consider generic boundary conditions, meaning that exactly one of the conditions $\theta^- \le 0$, $\rho^- = 0$, $Y^- = 0$ holds, and exactly one of the conditions $\theta^+ \le 0$, $\rho^+ = 0$, $Y^+ = 0$ holds. The other two values are positive at both left and right.

We limit our attention to waves that approach their end states exponentially [9]. Within the class of waves that satisfy generic boundary conditions, this limitation just means that we do not consider waves with $\theta^- = 0$ that approach the left state more slowly than exponentially. Such waves are generally considered nonphysical in that they only occur in solutions of initial value problems if the initial conditions are carefully prepared. Only traveling waves with velocity $c > 0$ are considered.

Theorem 2.1. *There exist six types of nonconstant traveling wave solutions of* (2.1)–(2.3), (2.4) *with positive velocity that satisfy generic boundary conditions and approach their end states exponentially, two fast ($c_f > b$), two slow ($c_s < a$), and two intermediate ($a < c_m < b$):*

$$(1)\ FC \xrightarrow{c_f} TC \qquad (3)\ TC \xrightarrow{c_s} OC \qquad (5)\ FC \xrightarrow{c_m} OC$$

$$(2)\, OC \xrightarrow{c_f} TC \qquad (4)\, FC \xrightarrow{c_s} OC \qquad (6)\, FC \xrightarrow{c_m} TC$$

The existence of these combustion waves was proved in [15].

3. Spectrum and exponential weight functions

In this section, we linearize the system about a traveling wave and begin to study the spectrum of the linearized operator $\mathcal{L}$. The spectrum of $\mathcal{L}$, which we denote $\mathrm{Sp}(\mathcal{L})$, consists of the discrete spectrum $\mathrm{Sp}_{\mathrm{d}}(\mathcal{L})$ and the essential spectrum $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L})$. The discrete spectrum is the set of all eigenvalues of $\mathcal{L}$ with finite multiplicity that are isolated in the spectrum, and the essential spectrum is the rest of the spectrum. We will study $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L})$ in this section.

Replacing the spatial coordinate x by the moving coordinate $\xi = x - ct$ in (2.1)–(2.3), we obtain

$$\partial_t\theta = \partial_{\xi\xi}\theta + (c-a)\partial_\xi\theta + F, \tag{3.1}$$
$$\partial_t\rho = c\partial_\xi\rho - F, \tag{3.2}$$
$$\partial_t Y = (c-b)\partial_\xi Y - F, \tag{3.3}$$

where $F = \rho Y\Phi$. A traveling wave $T^*(\xi) = (\theta^*(\xi), \rho^*(\xi), Y^*(\xi))$ with velocity c is a stationary solution of (3.1)–(3.3) with

$$\lim_{\xi\to-\infty} T^*(\xi) = T^- = (\theta^-, \rho^-, Y^-), \qquad \lim_{\xi\to+\infty} T^*(\xi) = T^+ = (\theta^+, \rho^+, Y^+).$$

We assume that $T^*(\xi)$ approaches $T^\pm$ at an exponential rate.

We linearize (3.1)–(3.3) at $T^*(\xi)$ and obtain

$$\partial_t\tilde\theta = \partial_{\xi\xi}\tilde\theta + (c-a)\partial_\xi\tilde\theta + F_\theta(T^*(\xi))\tilde\theta + F_\rho(T^*(\xi))\tilde\rho + F_Y(T^*(\xi))\tilde Y, \tag{3.4}$$
$$\partial_t\tilde\rho = c\partial_\xi\tilde\rho - F_\theta(T^*(\xi))\tilde\theta - F_\rho(T^*(\xi))\tilde\rho - F_Y(T^*(\xi))\tilde Y, \tag{3.5}$$
$$\partial_t\tilde Y = (c-b)\partial_\xi\tilde Y - F_\theta(T^*(\xi))\tilde\theta - F_\rho(T^*(\xi))\tilde\rho - F_Y(T^*(\xi))\tilde Y. \tag{3.6}$$

We write (3.4)–(3.6) as $X_t = \mathcal{L}X$, where

$$\mathcal{L} = \begin{pmatrix} \partial_{\xi\xi} + (c-a)\partial_\xi + F_\theta(T^*(\xi)) & F_\rho(T^*(\xi)) & F_Y(T^*(\xi)) \\ -F_\theta(T^*(\xi)) & c\partial_\xi - F_\rho(T^*(\xi)) & -F_Y(T^*(\xi)) \\ -F_\theta(T^*(\xi)) & -F_\rho(T^*(\xi)) & (c-b)\partial_\xi - F_Y(T^*(\xi)) \end{pmatrix}. \tag{3.7}$$

Definition 3.1. The traveling wave $T^*(\xi)$ is *spectrally stable* in a space $\mathcal{X}$ if

(1) 0 is an isolated simple eigenvalue of $\mathcal{L}$ on $\mathcal{X}$, with eigenfunction $T^{*\prime}(\xi)$, and
(2) there exists $\nu > 0$ such that the rest of the spectrum of $\mathcal{L}$ on $\mathcal{X}$ lies in $\operatorname{Re}\lambda < -\nu$.

In any space that contains $T^{*\prime}(\xi)$, $\mathcal{L}$ has an eigenvalue 0 with eigenfunction $T^{*\prime}(\xi)$.

Definition 3.2. The traveling wave $T^*(\xi)$ is *linearly stable* in a space $\mathcal{X}$ if the following hold.

(1) 1 is an isolated simple eigenvalue of the semigroup $e^{t\mathcal{L}}$ on $\mathcal{X}$, with eigenfunction $T^{*\prime}(\xi)$, and
(2) let $\mathcal{P}_s$ denote the Riesz spectral projection associated with $\mathrm{Sp}(\mathcal{L}) \setminus \{0\}$. Then there exist $\nu > 0$ and $K > 0$ such that $\|e^{t\mathcal{L}\mathcal{P}_s}\| < Ke^{-\nu t}$ for $t \geq 0$.

Linearized stability implies that every solution of (3.4)–(3.6) in $\mathcal{X}$ decays exponentially to a multiple of $T^{*\prime}(\xi)$.

There are two related constant-coefficient linear partial differential equations $X_t = \mathcal{L}^{\pm} X$, obtained by linearizing (3.1)–(3.3) at $T^{\pm}$. The spectrum of $\mathcal{L}^{\pm}$ in L^2 (or H^1, another space in which we shall be interested) can be computed using the Fourier transform

$$\hat{\mathcal{L}}^{\pm} = \begin{pmatrix} -\mu^2 + i\mu(c-a) + F_\theta(T^{\pm}) & F_\rho(T^{\pm}) & F_Y(T^{\pm}) \\ -F_\theta(T^{\pm}) & i\mu c - F_\rho(T^{\pm}) & -F_Y(T^{\pm}) \\ -F_\theta(T^{\pm}) & -F_\rho(T^{\pm}) & i\mu(c-b) - F_Y(T^{\pm}) \end{pmatrix}.$$

The right-hand boundary of the essential spectrum of $\mathcal{L}$ in L^2 or H^1 is the union of the right-hand boundaries of $\mathrm{Sp}(\mathcal{L}^-)$ and $\mathrm{Sp}(\mathcal{L}^+)$.

We shall treat fast combustion waves in detail, and then briefly discuss the other combustion waves.

3.1. **Spectrum of fast combustion waves.** There are two types of fast combustion waves, $FC \xrightarrow{c_f} TC$ and $OC \xrightarrow{c_f} TC$. Since the right state has type TC for both, we first compute the spectrum of $\hat{\mathcal{L}}^+$ at (θ^+, ρ^+, Y^+), where $\theta^+ \leq 0$, $\rho^+ > 0$ and $Y^+ > 0$. We obtain

$$\hat{\mathcal{L}}^{+} = \begin{pmatrix} -\mu^2 + i\mu(c_f - a) & 0 & 0 \\ 0 & i\mu c_f & 0 \\ 0 & 0 & i\mu(c_f - b) \end{pmatrix}. \tag{3.8}$$

The spectrum of $\hat{\mathcal{L}}^+$ in L^2 or H^1 is the set of λ that are eigenvalues of (3.8) for some μ in $\mathbb{R}$. Thus the eigenvalues are parameterized as

$$\lambda(\mu) = -\mu^2 + i\mu(c_f - a), \quad \lambda(\mu) = i\mu c_f, \quad \lambda(\mu) = i\mu(c_f - b).$$

We conclude that at (θ^+, ρ^+, Y^+), the spectrum of the linearization is a parabola in the left half-plane that touches the origin together with the imaginary axis.

Next we compute the spectrum at the left state.

(1) FC left state. We determine the spectrum of $\hat{\mathcal{L}}^-$ at a point (θ^-, ρ^-, Y^-) where $\theta^- > 0$, $\rho^- = 0$ and $Y^- > 0$. We obtain

$$\hat{\mathcal{L}}^{-} = \begin{pmatrix} -\mu^2 + i\mu(c_f - a) & Y^-\Phi(\theta^-) & 0 \\ 0 & i\mu c_f - Y^-\Phi(\theta^-) & 0 \\ 0 & Y^-\Phi(\theta^-) & i\mu(c_f - b) \end{pmatrix}. \tag{3.9}$$

The spectrum of $\hat{\mathcal{L}}^-$ in L^2 or H^1 is the set of λ that are eigenvalues of (3.9) for some μ in $\mathbb{R}$:

$$\lambda(\mu) = -\mu^2 + i\mu(c_f - a), \quad \lambda(\mu) = i\mu c_f - Y^-\Phi(\theta^-), \quad \lambda(\mu) = i\mu(c_f - b).$$

Thus the spectrum of the linearization consists of a parabola in the left half-plane that touches the origin, a vertical line in the open left half-plane, and the imaginary axis.

(2) OC left state. We determine the spectrum of $\hat{\mathcal{L}}^-$ at a point (θ^-, ρ^-, Y^-) where $\theta^- > 0$, $\rho^- > 0$ and $Y^- = 0$. We obtain

$$\hat{\mathcal{L}}^{-} = \begin{pmatrix} -\mu^2 + i\mu(c_f - a) & 0 & \rho^-\Phi(\theta^-) \\ 0 & i\mu c_f & -\rho^-\Phi(\theta^-) \\ 0 & 0 & i\mu(c_f - b) - \rho^-\Phi(\theta^-) \end{pmatrix}. \tag{3.10}$$

The spectrum of $\hat{\mathcal{L}}^-$ in L^2 or H^1 is the set of λ that are eigenvalues of (3.10) for some μ in $\mathbb{R}$:

$$\lambda(\mu) = -\mu^2 + i\mu(c_f - a), \quad \lambda(\mu) = i\mu(c_f - b) - \rho^-\Phi(\theta^-), \quad \lambda(\mu) = i\mu c_f.$$

Thus, as in the case of an FC left state, the spectrum consists of a parabola in the left half-plane that touches the origin, a vertical line in the open left half-plane, and the imaginary axis.

We don't have spectral stability in L^2 or H^1 for any fast combustion wave since both $\mathrm{Sp}(\mathcal{L}^+)$ and $\mathrm{Sp}(\mathcal{L}^-)$ touch the imaginary axis. Spectral stability can be obtained if these spectra can be moved to the left of the imaginary axis by working in a space with weighted norm.

3.2. **Weight function for fast combustion waves.** For $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$, let $\gamma_\alpha : \mathbb{R} \to \mathbb{R}$ be a fixed weight function of class α, i.e., γ_α is C^∞, $\gamma_\alpha(\xi) > 0$ for all ξ, $\gamma_\alpha(\xi) = e^{\alpha_- \xi}$ for large negative ξ, and $\gamma_\alpha(\xi) = e^{\alpha_+ \xi}$ for large positive ξ.

Let $\mathcal{X}_0$ denote one of the standard Banach spaces $L^2(\mathbb{R}, \mathbb{R}^3)$ or $H^1(\mathbb{R}, \mathbb{R}^3)$, and denote the norm by $\| \ \|_0$. Let $\mathcal{X}_\alpha$ denote the corresponding weighted space with weight function $\gamma_\alpha(\xi)$. More precisely, $x(\xi) \in \mathcal{X}_\alpha$ provided $\gamma_\alpha(\xi)x(\xi) \in \mathcal{X}_0$, and $\|x(\xi)\|_\alpha = \|\gamma_\alpha(\xi)x(\xi)\|_0$.

To study the spectrum of $\mathcal{L}$ as an operator on $\mathcal{X}_\alpha$, let $X = (\tilde{\theta}(\xi), \tilde{\rho}(\xi), \tilde{Y}(\xi)) \in \mathcal{X}_\alpha$, and let $W = \gamma_\alpha(\xi)X = (u(\xi), v(\xi), z(\xi)) \in \mathcal{X}_0$. Then the equation $X_t = \mathcal{L}X$ yields $\gamma_\alpha^{-1}W_t = \mathcal{L}\gamma_\alpha^{-1}W$. Multiplying both sides by γ_α, we obtain $W_t = \gamma_\alpha \mathcal{L} \gamma_\alpha^{-1} W$, where $\gamma_\alpha \mathcal{L} \gamma_\alpha^{-1}$ is a linear operator on $\mathcal{X}_0$. To find the spectrum of $\mathcal{L}$ on $\mathcal{X}_\alpha$, we instead find the spectrum of the isomorphic operator $\mathcal{L}_\alpha = \gamma_\alpha \mathcal{L} \gamma_\alpha^{-1}$ on $\mathcal{X}_0$. Let $\eta_\alpha = \gamma_\alpha \partial_\xi \gamma_\alpha^{-1}$ and let $\zeta_\alpha = \gamma_\alpha \partial_{\xi\xi} \gamma_\alpha^{-1}$. Then

$$\mathcal{L}_\alpha = \begin{pmatrix} \partial_{\xi\xi} + (c_f - a - 2\eta_\alpha)\partial_\xi + \zeta_\alpha + (c-a)\eta_\alpha + F_\theta(T^*) & F_\rho(T^*) & F_Y(T^*) \\ -F_\theta(T^*) & c_f\partial_\xi - c_f\eta_\alpha - F_\rho(T^*) & -F_Y(T^*) \\ -F_\theta(T^*) & -F_\rho(T^*) & (c_f - b)(\partial_\xi + \eta_\alpha) - F_Y(T^*) \end{pmatrix}. \tag{3.11}$$

In the equation

$$W_t = \mathcal{L}_\alpha W,$$

we let $\xi \to \pm\infty$, which yields the constant-coefficient linear differential equations

$$W_t = \mathcal{L}_\alpha^\pm W, \tag{3.12}$$

where

$$\mathcal{L}_\alpha^\pm = \begin{pmatrix} \partial_{\xi\xi} + (c_f - a - 2\alpha_\pm)\partial_\xi + \alpha_\pm^2 + a\alpha_\pm - c_f\alpha_\pm + F_\theta(T^\pm) & F_\rho(T^\pm) & F_Y(T^\pm) \\ -F_\theta(T^\pm) & c_f\partial_\xi - c_f\alpha_\pm - F_\rho(T^\pm) & -F_Y(T^\pm) \\ -F_\theta(T^\pm) & -F_\rho(T^\pm) & (c_f - b)(\partial_\xi - \alpha_\pm) - F_Y(T^\pm) \end{pmatrix}.$$

The right-hand boundary of the essential spectrum of $\mathcal{L}_\alpha$ is the union of the right-hand boundaries of $\mathrm{Sp}(\mathcal{L}_\alpha^-)$ and $\mathrm{Sp}(\mathcal{L}_\alpha^+)$. These spectra are the same in L^2 or H^1, so we compute them in L^2 using Fourier transform.

Since the right state is of type TC for all fast combustion waves, we first compute the spectrum of $\mathcal{L}_\alpha^+$ at the right end state (θ^+, ρ^+, Y^+) where $\theta^+ \le 0$, $\rho^+ > 0$ and $Y^+ > 0$. We obtain

$$\mathcal{L}_\alpha^+ = \begin{pmatrix} -\mu^2 + (c_f - a - 2\alpha_+)i\mu + \alpha_+^2 + (a - c_f)\alpha_+ & 0 & 0 \\ 0 & i\mu c_f - c_f\alpha_+ & 0 \\ 0 & 0 & i\mu(c_f - b) - (c_f - b)\alpha_+ \end{pmatrix}. \tag{3.13}$$

The spectrum of $\mathcal{L}_\alpha^+$ is the set of λ that are eigenvalues of (3.13) for some μ in $\mathbb{R}$:

$$\lambda(\mu) = -\mu^2 + (c_f - a - 2\alpha_+)i\mu + \alpha_+^2 + (a - c_f)\alpha_+,$$

$$\lambda(\mu) = i\mu c_f - c_f\alpha_+,$$

$$\lambda(\mu) = i\mu(c_f - b) - (c_f - b)\alpha_+.$$

To move the spectrum to the open left half-plane, we require that the real part of all eigenvalues be negative. This happens if and only if $0 < \alpha_+ < c_f - a$.

Next we compute the spectrum at the left state.

(1) FC left state. By a similar computation, we determine the spectrum of $\mathcal{L}_\alpha^-$ at a point (θ^-, ρ^-, Y^-) where $\theta^- > 0$, $\rho^- = 0$ and $Y^- > 0$. We again find that the spectrum moves to the open left half-plane if and only if $0 < \alpha_- < c_f - a$.

(2) OC left state. Similarly we determine the spectrum of $\mathcal{L}_\alpha^-$ at a point (θ^-, ρ^-, Y^-) where $\theta^- > 0$, $\rho^- > 0$ and $Y^- = 0$. Again we find that the spectrum moves to the open left half-plane if and only if $0 < \alpha_- < c_f - a$.

3.3. **Slow and intermediate waves.** The slow and intermediate combustion waves cannot be stabilized by weight functions of any class α. For slow combustion waves, which all have right state of type OC, to move the spectrum of $\mathcal{L}_\alpha^+$ to the open left half-plane would require a negative α_+ for the temperature equation and a positive α_+ for the fuel and oxygen equations. Therefore, there is no α_+ that moves the spectrum to the open left half-plane. Similarly, for intermediate combustion waves, which all have left state of type FC, to move the spectrum of $\mathcal{L}_\alpha^-$ to the open left half-plane would require a positive α_- for the temperature equation and a negative α_- for the fuel and oxygen equations.

4. Evans function for fast combustion waves

Given the temperature $\theta^+ \leq 0$ and the fuel concentration $\rho^+ > 0$ of a temperature-controlled right state, one can plot a curve in the (Y^+, c)-plane of values such that there exists a fast combustion wave with right state (θ^+, ρ^+, Y^+) and velocity c. Figure 4.1, reproduced from [15], shows such a curve, plotted with AUTO [8].

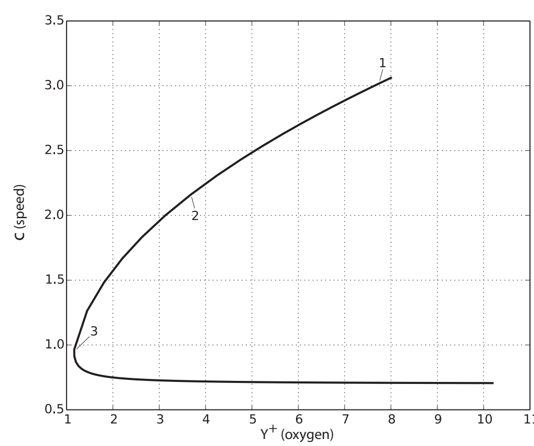

FIGURE 4.1. Traveling wave bifurcation diagram with $a = 0.5$, $b = 0.7$, $\theta^+ = -0.1$, $\rho^+ = 2$. The solutions between labels 1 and 2 are FC to TC waves; after that, solutions are OC to TC waves. The curve turns when Y^+ reaches a minimum value Y^+_{**} (label 3).

In this section we study numerically the discrete spectrum of fast combustion waves using the Evans function [3, 12, 13]. More precisely, we study the discrete spectrum of the operator $\mathcal{L}_\alpha$ defined in the previous section, where $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$ has been chosen to stabilize the essential spectrum; thus $0 < \alpha_\pm < c_f - a$. The Evans function is an analytic function $D(\lambda)$, defined to the right of the essential spectrum of $\mathcal{L}_\alpha$, that equals 0 at eigenvaues of $\mathcal{L}_\alpha$. By plotting $D(\lambda)$ on a closed curve C in the complex plane, one obtains a closed curve $D(C)$ whose winding number about 0 equals the number of eigenvalues of $\mathcal{L}_\alpha$ inside C, counting multiplicity. A well-chosen curve should yield all the eigenvalues of $\mathcal{L}_\alpha$, if any, in the right half-plane. Typically numerical evidence indicates that increasing the size of C past a certain point does not yield additional eigenvalues. In some problems, one can obtain an a priori bound on the discrete spectrum of $\mathcal{L}_\alpha$ in the right half-plane, thus proving that there are no eigenvalues outside a correctly chosen C. Unfortunately we do not have such a bound for the problem under study, so we just choose the curve C large enough that we do not observe additional eigenvalues in the right half plane when we further increase its size. In section 6 we will show that such a bound can be obtained if we add a small diffusion term to the oxygen equation.

The eigenvalue problem of (3.12) reads

$$\begin{aligned}
\lambda u &= u_{\xi\xi} + (c - a - 2\alpha_\pm)u_\xi + (\alpha_\pm^2 + a\alpha_\pm - c\alpha_\pm)u + F_\theta(T^*)u + F_\rho(T^*)v + F_Y(T^*)z, \\
\lambda v &= cv_\xi - c\alpha_\pm v - F_\theta(T^*)u - F_\rho(T^*)v - F_Y(T^*)z, \\
\lambda z &= (c - b)z_\xi - (c - b)\alpha_\pm z - F_\theta(T^*)u - F_\rho(T^*)v - F_Y(T^*)z.
\end{aligned} \tag{4.1}$$

We rewrite (4.1) as a first order system with parameter λ by letting $w = u_\xi$:

$$\begin{aligned}
u_\xi &= w, \\
w_\xi &= \lambda u - (c - a - 2\alpha_\pm)w - (\alpha_\pm^2 + a\alpha_\pm - c\alpha_\pm)u - F_\theta(T^*)u - F_\rho(T^*)v - F_Y(T^*)z, \\
v_\xi &= \frac{1}{c}(\lambda v + c\alpha_\pm v + F_\theta(T^*)u + F_\rho(T^*)v + F_Y(T^*)z), \\
z_\xi &= \frac{1}{c - b}(\lambda z + (c - b)\alpha_\pm z + F_\theta(T^*)u + F_\rho(T^*)v + F_Y(T^*)z).
\end{aligned} \tag{4.2}$$

System (4.2) is in the form

$$Z_\xi = A(\xi, \lambda)Z, \tag{4.3}$$

with A analytic in λ for each ξ.

We define the limit matrices $A_\pm(\lambda) = \lim_{\xi\to\pm\infty} A(\xi, \lambda)$; $A_\pm$ are analytic in λ. To the right of the essential spectrum of $\mathcal{L}_\alpha$, the dimension of the unstable subspace $U_-(\lambda)$ of $A_-(\lambda)$ is three, and that of the stable subspace $S_+(\lambda)$ of $A_+(\lambda)$ is one, which sum to four, the dimension of the phase space. To define the Evans function, we define linearly independent solutions $Z_1^-(\xi, \lambda)$, $Z_2^-(\xi, \lambda)$, $Z_3^-(\xi, \lambda)$ of (4.3), analytic in λ, that decay exponentially as $\xi \to -\infty$, and a nontrivial solution $Z_4^+(\xi, \lambda)$ of (4.3), analytic in λ, that decays exponentially as $\xi \to \infty$. We evaluate the solutions at $\xi = 0$, obtaining four vectors $Z_1^-, Z_2^-, Z_3^-, Z_4^+$, and define the Evans function

$$D(\lambda) = \det(Z_1^- Z_2^- Z_3^- Z_4^+).$$

Thus $D(\lambda) = 0$ if and only if (4.3) has a nontrivial solution that decays as $\xi \to \pm\infty$, i.e., if and only if λ is an eigenvalue of $\mathcal{L}_\alpha$. The order of the root equals the algebraic multiplicity of the eigenvalue [13].

We use STABLAB [4] to compute the Evans function with the values of a, b, θ^+ and ρ^+ given in Figure 4.1. (Actually, for this problem STABLAB uses the adjoint formulation of the Evans function, in terms of one vector and one covector.) The traveling wave system (3.1)–(3.3), with given right state (θ^+, ρ^+, Y^+), reduces to

$$\dot{\theta} = (a - c)(\theta - \theta^+) - c(\rho - \rho^+), \tag{4.4}$$

$$\dot{\rho} = \left(\frac{\rho - \rho^+}{c - b} + \frac{Y^+}{c}\right)\rho\Phi(\theta). \tag{4.5}$$

We begin by setting $Y^+ = 8$. We look for a value of c for which there is a traveling wave with left state having $\rho^- = 0$, i.e., we look for a $FC \xrightarrow{c_f} TC$ wave. For $c = 3.061$ we find the traveling wave shown in the first panel of the Figure 4.2. The point (Y^+, c) is near label 1 in Figure 4.1. The second panel of Figure 4.2 shows the Evans function $D(C)$ where C is the semicircle $(x + 10^{-4})^2 + y^2 = 250^2$, $x \geq -10^{-4}$, together with the vertical diameter. The curve has winding number one about 0; this can be seen from the third panel of Figure 4.2, which zooms in on the second panel near $\lambda = 0$. The winding number indicates that there is a simple eigenvalue at 0 and no other eigenvalues inside C. A similar result is obtained for other traveling waves in Figure 4.2 between labels 1 and 2, all of which are FC to TC waves. Increasing the size of C does not change the result.

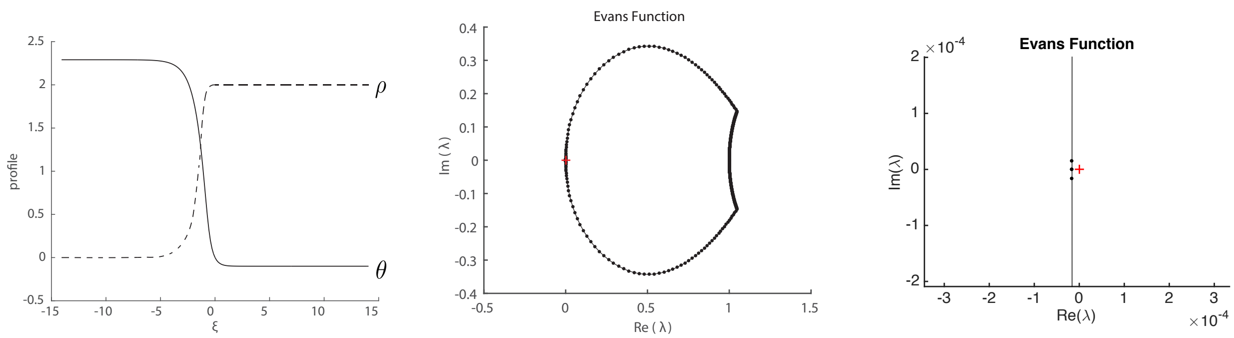

FIGURE 4.2. Left: profile for the system (4.4)–(4.5) with a, b, θ^+ and ρ^+ given in Figure 4.1, $Y^+ = 8$, and $c = 3.061$. Center: Evans function output for the curve C described in the text. We use 150 points on the circle part, 100 points along the vertical diameter, and take 256 Kato steps [4] between contour points. Right: zoom in near $\lambda = 0$, showing that 0 is inside the curve.

Next, with the same values of a, b, θ^+ and ρ^+, we take $Y^+ = 1.5$ and look for a value of c for which there is a traveling with left state having $Y^- = 0$, i.e., we look for a $OC \xrightarrow{c_f} TC$ wave. For $c = 1.2632$ we find the traveling wave shown in the first panel of the Figure 4.3. The point (Y^+, c) is between labels 2 and 3 in Figure 4.1. The second panel of Figure 4.3 shows the Evans function $D(C)$ where C is again the semicircle $(x + 10^{-4})^2 + y^2 = 250^2$, $x \geq -10^{-4}$, together with the vertical diameter. The curve again has winding number one about 0; this can be seen from the third panel of Figure 4.2, which zooms in on the second panel near $\lambda = 0$. The winding number indicates that there is a simple eigenvalue at 0 and no other eigenvalues inside C. A similar result is obtained for other traveling waves in Figure 4.2 between labels 2 and 3, all of which are OC to TC waves. Increasing the size of C does not change the result.

Thus the numerical evidence indicates that for traveling waves between labels 1 and 3 in Figure 4.1, the simple eigenvalue 0 is the only element of $\mathrm{Sp_d}(\mathcal{L}_\alpha)$ in $\{\lambda : \mathrm{Re}\,\lambda \geq 0\}$.

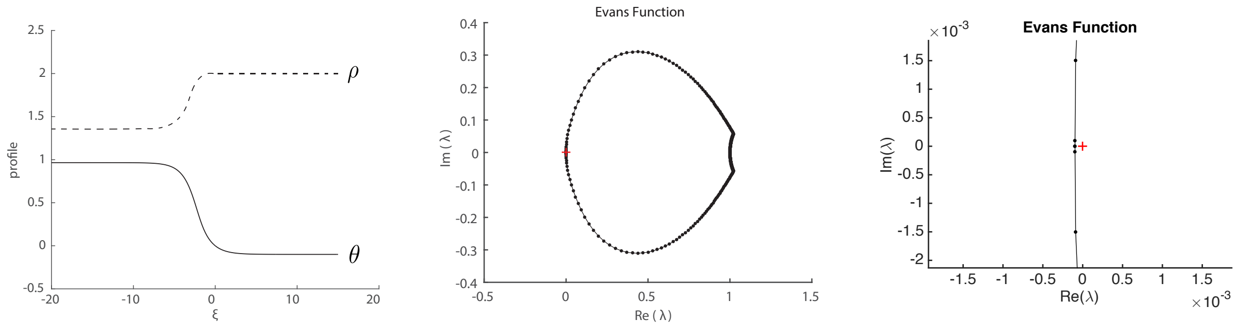

FIGURE 4.3. Left: profile for the system (4.4)–(4.5) with a, b, θ^+ and ρ^+ given in Figure 4.1, $Y^+ = 1.5$, and $c = 1.2632$. Center: Evans function output for the curve C described in the text. We use 150 points on the circle part, 100 points along the vertical diameter, and take 256 Kato steps [4] between contour points. Right: zoom in near $\lambda = 0$, showing that 0 is inside the curve.

As Y^+ reaches its minimum value at Y^+_{**} (label 3), the curve of Figure 4.1 turns. Solutions after label 3 still correspond to $OC \xrightarrow{c_f} TC$ waves. We set $Y^+ = 1.2$ and find that for $c = 0.7647$ we have the traveling wave shown in the first panel of the Figure 4.4. This point (Y^+, c) is on the lower branch of the curve in Figure 4.1. The second panel of Figure 4.4 shows the Evans function $D(C)$ where C is the semicircle $(x + 10^{-4})^2 + y^2 = 4$, $x \geq -10^{-4}$, together with the vertical diameter. The curve has winding number two about 0; this can be seen from the third panel of Figure 4.4, which zooms in on the second panel near $\lambda = 0$. There is a simple eigenvalue at 0 and a positive real eigenvalue in the right half-plane. (We checked that the second eigenvalue is in the right half-plane by shifting the semi-circle a little to the right of the imaginary axis; the winding number becomes one.) A similar result is obtained for other traveling waves on the lower branch of the curve in Figure 4.1. Therefore these traveling waves are not spectrally stable in $\mathcal{X}_\alpha$.

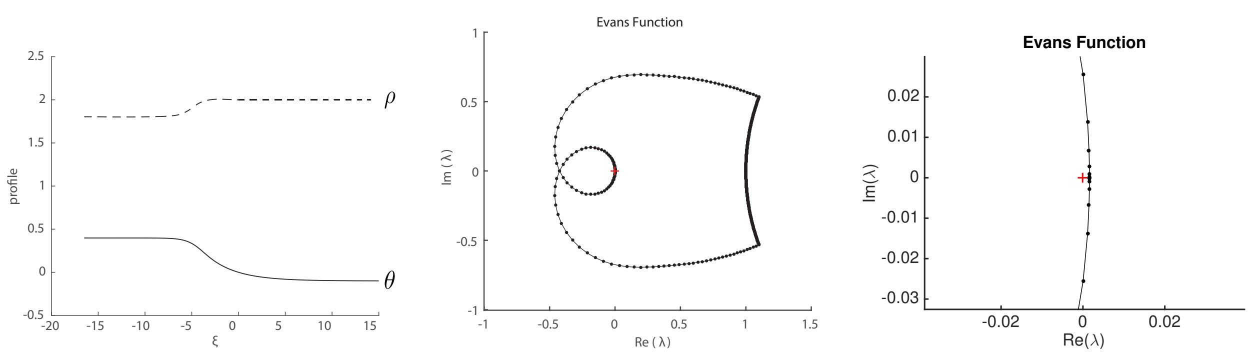

FIGURE 4.4. Left: profile for the system (4.4)–(4.5) with a, b, θ^+ and ρ^+ given in Figure 4.1, $Y^+ = 1.2$, and $c = 0.7647$. Center: Evans function output for the semicircular contour of radius 2 described in the text. We use 150 points on the circle part, 100 points along the vertical diameter, and take 256 Kato steps [4] between contour points. Right: zoom in near $\lambda = 0$, showing that the curve winds twice around 0.

5. Linear and nonlinear stability of fast combustion waves

In this section, we study the linear and nonlinear stability of fast combustion waves. In section 3, we saw that the essential spectrum of such a wave can be moved to the left of the imaginary axis by using a weight function $\gamma_\alpha(\xi)$, $\alpha = (\alpha_-, \alpha_+)$, with $0 < \alpha_\pm < c_f - a$. For some of the fast combustion waves, we showed numerically in section 4 that the linear operator $\mathcal{L}_\alpha$ has no eigenvalues in the half-plane $\operatorname{Re}\lambda \geq 0$ other than a simple eigenvalue zero. In this section we consider a fast combustion wave for which we assume that this is the case, i.e., a fast combustion wave that is spectrally stable in the weighted space $\mathcal{X}_\alpha$.

Linearized stability of the traveling wave in $\mathcal{X}_\alpha$ does not follow from spectral stability using standard results. Since the system (2.1)–(2.3) is partly parabolic, the linearized operator has vertical lines in its spectrum, so it is not a sectorial operator. Therefore the linearized system generates a C_0-semigroup, not an analytic semigroup. This difficulty is typical for systems with no diffusion in some equations.

However, linearized stability in $\mathcal{X}_\alpha$ does follow from spectral stability by a recent result of Yurov [16], for $\mathcal{X}_0$ equal to either $L^2(\mathbb{R}, \mathbb{R}^3)$ or $H^1(\mathbb{R}, \mathbb{R}^3)$. We postpone a discussion of this fact to Appendix A.

Unfortunately, nonlinear stability of the traveling wave to perturbations in $\mathcal{X}_\alpha$ does not follow from linearized stability using standard results. The essential difficulty is that the weight function $\gamma_\alpha(\xi)$ decays exponentially at the left, so $\mathcal{X}_\alpha$ includes functions that grow exponentially at the left. The square of such a function grows twice as fast at the left, so it need not be in $\mathcal{X}_\alpha$. This makes it diffucult to study nonlinear problems in this space.

Our goal in the rest of this section is to use Theorem 3.14 in [9] to obtain a type of nonlinear stability for fast combustion waves that are spectrally stable in $\mathcal{X}_\alpha$. Theorem 3.14 in [9] as stated does not apply to systems with transport terms ($a\partial_x\theta$ in (2.1) and $b\partial_x Y$ in (2.3)), so a generalization is needed. The necessary generalizaation again relies on Yurov's theorem. We postpone a discussion of this matter to appendix A.

Let $\beta = (0, \alpha_+)$, and let $\gamma_\beta(\xi)$ be a fixed weight function of class β, i.e., γ_β is C^∞, $\gamma_\beta(\xi) > 0$ for all ξ, $\gamma_\beta(\xi) = 1$ for large negative ξ, and $\gamma_\beta(\xi) = e^{\alpha_+\xi}$ for large positive ξ. Then $\mathcal{X}_\beta$ denotes the weighted space based on $\mathcal{X}_0$ with weight function γ_β.

We shall show:

Theorem 5.1. *Consider the system (3.1)–(3.3) with constants $c > b > a > 0$, with c chosen so that there is a stationary solution $T^*(\xi)$ of type FC to TC. Let $\mathcal{X}_0 = H^1(\mathbb{R}, \mathbb{R}^3)$. Let $\alpha = (\alpha_-, \alpha_+)$ with $0 < \alpha_- < \min(c - a, \frac{1}{c}Y^-\Phi(\theta^-))$ and $0 < \alpha_+ < c - a$. Assume the Evans function for the traveling wave $T^*(\xi)$ in the space $\mathcal{X}_\alpha$ has no zeros in the half-plane $\operatorname{Re}\lambda \geq 0$ other than a simple zero at the origin. Choose $\nu > 0$ such that the operator $\mathcal{L}_\alpha$ defined in subsection 3.2 satisfies $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_\alpha) \,\text{and}\, \lambda \neq 0\} < -\nu$. Let $\beta = (0, \alpha_+)$. Then there is a constant $C > 0$ such that the following is true. Suppose $T^0 \in T^* + \mathcal{X}_\beta$ with $\|T^0 - T^*\|_\beta$ small, and let $T(t)$ be the solution of (3.1)–(3.3) with $T(0) = T^0$. Then:*

(1) $T(t)$ is defined for all $t \geq 0$.
(2) $T(t) = \tilde{T}(t) + T^*(\xi - q(t))$ with $\tilde{T}(t)$ in a fixed subspace of $\mathcal{X}_\alpha$ complementary to the span of $T^{*\prime}$.
(3) $\|\tilde{T}(t)\|_\beta + |q(t)|$ is small for all $t \geq 0$.
(4) $\|\tilde{T}(t)\|_\alpha \leq Ce^{-\nu t}\|\tilde{T}^0\|_\alpha$.
(5) There exists q^* such that $|q(t) - q^*| \leq Ce^{-\nu t}\|\tilde{T}^0\|_\alpha$.

Let $\tilde{U} = (\tilde{M}, \tilde{N})$ with $\tilde{M} = (\tilde{u}_1, \tilde{u}_3)$ and $\tilde{N} = \tilde{u}_2$.

(6) $\|(\tilde{M}(t)\|_0 \leq C\|\tilde{T}^0\|_\beta$.
(7) $\|\tilde{N}(t)\|_0 \leq Ce^{-\nu t}\|\tilde{T}^0\|_\beta$.

For a fast traveling wave that has oxygen-controlled left state and temperature-controlled right state, the only changes in Theorem 5.1 are in the $(\tilde{M}, \tilde{N})$ decomposition: $\tilde{M} = (\tilde{u}_1, \tilde{u}_2)$ and $\tilde{N} = \tilde{u}_3$.

The results (6) and (7) have a physical interpretation. In the case of an FC left state, the combustion front moves to the right, leaving a high-temperature zone behind. Behind the combustion front the fuel is exhausted and oxygen is present. If we make a perturbation behind the front by adding

- fuel $(\tilde{u}_2)$, it immediately burns because of the high temperature and presence of oxygen;
- oxygen $(\tilde{u}_3)$, it does not react since there is no fuel;
- heat $(\tilde{u}_1)$, it diffuses.

On the other hand, in the case of an OC left state, behind the combustion front temperature is high, oxygen is exhausted and fuel is present. If we make a perturbation behind the front by adding

- fuel $(\tilde{u}_2)$, it does not react since there is no oxygen;
- oxygen $(\tilde{u}_3)$, it immediately reacts with the fuel until it is exhausted;
- heat $(\tilde{u}_1)$, it diffuses.

Theorem 5.1 follows from a generalization of Theorem 3.14 in [9], once the hypotheses are verified.

Since Theorem 3.14 in [9] is stated for traveling waves whose left state is the origin, we begin by rewriting (2.1)–(2.3) to achieve this. $T^*(\xi)$ is a traveling wave for (2.1)–(2.3) that is a fast combustion wave with fuel-controlled left state and temperature-controlled right state. Thus $T^- = (\theta^-, 0, Y^-)$ and $T^+ = (\theta^+, \rho^+, Y^+)$ with $\theta^+ \leq 0$ and θ^-, Y^-, ρ^+ and Y^+ all positive.

We make the change of variables $u_1 = \theta - \theta^-$, $u_2 = \rho$, and $u_3 = Y - Y^-$, which converts (3.1)–(3.3) to the system

$$\partial_t u_1 = \partial_{xx} u_1 - a\partial_x u_1 + u_2(u_3 + Y^-)\Phi(u_1 + \theta^-), \tag{5.1}$$

$$\partial_t u_2 = -u_2(u_3 + Y^-)\Phi(u_1 + \theta^-), \tag{5.2}$$

$$\partial_t u_3 = -b\partial_x u_3 - u_2(u_3 + Y^-)\Phi(u_1 + \theta^-). \tag{5.3}$$

Let $U^*(\xi) = (u_1^*(\xi), u_2^*(\xi), u_3^*(\xi))$ be the stationary solution of (5.1)–(5.3) that corresponds to $T^*(\xi)$. Then $U^- = (0, 0, 0)$ and $U^+ = (\theta^+ - \theta^-, \rho^+, Y^+ - Y^-)$.

The reaction terms in (5.1)–(5.3) comprise the function

$$R(U) = (u_2(u_3+Y^-)\Phi(u_1+\theta^-), -u_2(u_3+Y^-)\Phi(u_1+\theta^-), -u_2(u_3+Y^-)\Phi(u_1+\theta^-)). \tag{5.4}$$

Theorem 3.14 in [9] must be modified because it only applies to traveling waves for systems of the form $U_t = dU_{xx} + R(U)$, $d = \text{diag}(d_1, \ldots, d_n)$, with all $d_i \geq 0$. Thus transport terms such as $\partial_x u_1$ and $\partial_x u_3$ in (5.1)–(5.3) are not allowed. As mentioned above, we will address this point in appendix A. In the remainder of this section we will verify the remaining hypotheses of Theorem 3.14 in [9].

Hypothesis 1. The reaction terms in (5.1)–(5.3) are C^3.

In fact they are C^∞, so Hypothesis 1 is satisfied.

Hypothesis 2. The system (5.1)–(5.3) has a traveling wave solution $U^*(\xi)$, $\xi = x - ct$, with left state at the origin and right state U^+, for which there exist numbers $K > 0$ and $\omega_- < 0 < \omega_+$ such that for $\xi \le 0$, $\|U^*(\xi)\| \le Ke^{-\omega_-\xi}$, and for $\xi \ge 0$, $\|U^*(\xi) - U^+\| \le Ke^{-\omega_+\xi}$.

$U^*(\xi)$ is just $T^*(\xi)$ suitably translated. Since the linearization of (3.1)–(3.3) has only one positive eigenvalue at $(T^-, 0)$, namely $\frac{1}{c}Y^-\Phi(\theta^-)$, and only one negative eigenvalue at $(T^+, 0)$, namely $a - c$, we let

$$\omega_- = -\frac{1}{c}Y^-\Phi(\theta^-), \qquad \omega_+ = c - a.$$

The linearization of (5.1)–(5.3) at $U^*(\xi)$ is

$$\partial_t \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix} = \begin{pmatrix} \partial_{\xi\xi} + (c-a)\partial_\xi & 0 & 0 \\ 0 & c\partial_\xi & 0 \\ 0 & 0 & (c-b)\partial_\xi \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix} + DR(U^*(\xi)) \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix}, \tag{5.5}$$

where

$$DR(U^*(\xi)) = \begin{pmatrix} u_2^*(u_3^* + Y^-)\Phi'(u_1^* + \theta^-) & (u_3^* + Y^-)\Phi(u_1^* + \theta^-) & u_2^*\Phi(u_1^* + \theta^-) \\ -u_2^*(u_3^* + Y^-)\Phi'(u_1^* + \theta^-) & -(u_3^* + Y^-)\Phi(u_1^* + \theta^-) & -u_2^*\Phi(u_1^* + \theta^-) \\ -u_2^*(u_3^* + Y^-)\Phi'(u_1^* + \theta^-) & -(u_3^* + Y^-)\Phi(u_1^* + \theta^-) & -u_2^*\Phi(u_1^* + \theta^-) \end{pmatrix}.$$

Of course, (5.5) is just $\tilde{U}_t = \mathcal{L}\tilde{U}$, where $\mathcal{L}$ was defined in (3.7). (The translation does not affect the linearization.)

Hypothesis 3. There exists $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$ such that the following are true.

1. $0 < \alpha_- < -\omega_-$.
2. $0 \le \alpha_+ < \omega_+$.
3. For the system (5.5) and $\mathcal{X}_0 = L^2(\mathbb{R}, \mathbb{R}^3)$,
   (a) $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}_{\text{ess}}(\mathcal{L}_\alpha)\} < 0$, and
   (b) the only element of $\operatorname{Sp}(\mathcal{L}_\alpha)$ in $\{\lambda : \operatorname{Re}\lambda \ge 0\}$ is a simple eigenvalue 0.

Let $\alpha = (\alpha_-, \alpha_+)$ with $0 < \alpha_- < \min(c - a, \frac{1}{c}Y^-\Phi(\theta^-))$ and $0 < \alpha_+ < c - a$. From the verification of Hypothesis 2 and subsection 3.2, we see that α satisfies Hypothesis 3.

Hypothesis 4. There is a 2×2 matrix A such that $R(M, 0) = (AM, 0)$.

Decompose $\tilde{U}$-space such that $\tilde{U} = (\tilde{M}, \tilde{N})$ with $\tilde{M} = (\tilde{u}_1, \tilde{u}_3)$ and $\tilde{N} = \tilde{u}_2$. Since $R(u_1, 0, u_3) = (0, 0, 0)$ from (5.4), Hypothesis 4 is satisfied with $A = 0$.

The linearization of (5.1)–(5.3) at the end state $U^- = (0, 0, 0)$ is

$$\begin{pmatrix} \tilde{u}_{1t} \\ \tilde{u}_{2t} \\ \tilde{u}_{3t} \end{pmatrix} = \begin{pmatrix} \partial_{\xi\xi} + (c-a)\partial_\xi & Y^-\Phi(\theta^-) & 0 \\ 0 & c\partial_\xi - Y^-\Phi(\theta^-) & 0 \\ 0 & -Y^-\Phi(\theta^-) & (c-b)\partial_\xi \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix}, \tag{5.6}$$

or, equivalently, $= \tilde{U}_t = \mathcal{L}^-\tilde{U}$, , where $\mathcal{L}^-$ was defined in section 3.

From (5.6) we define $\mathcal{L}^{(1)}$, the restriction of $\mathcal{L}^-$ to u_1u_3-space, and $\mathcal{L}^{(2)}$, the restriction of $\mathcal{L}^-$ to u_2-space:

$$\mathcal{L}^{(1)} = \begin{pmatrix} \partial_{\xi\xi} + (c-a)\partial_\xi & 0 \\ 0 & (c-b)\partial_\xi \end{pmatrix}, \qquad \mathcal{L}^{(2)} = c\partial_\xi - Y^-\Phi(\theta^-). \tag{5.7}$$

Hypothesis 5.

1. For $\mathcal{X}_0 = L^2(\mathbb{R}, \mathbb{R}^3)$, the operator $\mathcal{L}^{(1)}$ on $\mathcal{X}_0^2$ generates a bounded semigroup.
2. For $\mathcal{X}_0 = L^2(\mathbb{R}, \mathbb{R}^3)$, the operator $\mathcal{L}^{(2)}$ on $\mathcal{X}_0$ satisfies $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}^{(2)})\} < 0$.

The operator $\mathcal{L}^{(1)}$ defined by (5.7) on $L^2(\mathbb{R},\mathbb{R}^3)$ is known to satisfy Hypothesis 5 (1), and the spectrum of the operator $c\partial_\xi - Y^-\Phi(\theta^-)$ on $L^2(\mathbb{R},\mathbb{R}^3)$ is contained in $\operatorname{Re}\lambda \le -Y^-\Phi(\theta^-) < 0$, so Hypothesis 5 (2) is satisfied.

6. Adding small diffusion to the model

In this section we add a small diffusion term to the oxygen equation in the system (2.1)–(2.3):

$$\partial_t\theta + a\partial_x\theta = \partial_{xx}\theta + \rho Y\Phi, \tag{6.1}$$

$$\partial_t\rho = -\rho Y\Phi, \tag{6.2}$$

$$\partial_t Y + b\partial_x Y = \epsilon\partial_{xx}Y - \rho Y\Phi. \tag{6.3}$$

It was shown in [15] that the new traveling waves are small perturbation of the old ones.

Replacing the spatial coordinate x by the moving coordinate $\xi = x - ct$ in (6.1)–(6.3), we obtain

$$\partial_t\theta = \partial_{\xi\xi}\theta + (c-a)\partial_\xi\theta + F, \tag{6.4}$$

$$\partial_t\rho = c\partial_\xi\rho - F, \tag{6.5}$$

$$\partial_t Y = \epsilon\partial_{xx}Y + (c-b)\partial_\xi Y - F, \tag{6.6}$$

where $F = \rho Y\Phi$.

If we linearize (6.4)–(6.6) at an endpoint of a traveling wave and compare to (3.4)–(3.6) evaluated at an endpoint, we find that a vertical line in the spectrum has changed to a parabola. We can find a weight function that moves the spectrum to the left of the imaginary axis as in section 3. Weight function for fast combustion waves:

(1) TC right state: if $0 < \alpha_+ < \min\{c-a, \frac{c-b}{\epsilon}\}$, then the spectrum lies in the open left half-plane.
(2) FC left state: if $0 < \alpha_- < \min\{c-a, \frac{c-b}{\epsilon}\}$, then the spectrum lies in the open left half-plane.
(3) OC left state: if $0 < \alpha_- < \min\{c-a, \frac{c-b+\sqrt{(b-c)^2+4\epsilon\rho_-\Phi(\theta^-)}}{2\epsilon}\}$, then the spectrum lies in the open left half-plane.

Slow and intermediate waves still cannot be stabilized by weight functions of any class α.

Using spectral energy estimates, we shall find a priori bounds on the unstable eigenvalues for the system (6.1)–(6.3) in an appropriate weighted space. Linearizing (6.1)–(6.3) at the combustion front $(\hat\theta, \hat\rho, \hat Y)$, we obtain

$$\partial_t\theta = \partial_{\xi\xi}\theta + (c-a)\partial_\xi\theta + h_1\theta + h_2 Y + h_3\rho, \tag{6.7}$$

$$\partial_t\rho = c\partial_\xi\rho - h_1\theta - h_2 Y - h_3\rho, \tag{6.8}$$

$$\partial_t Y = \epsilon\partial_{\xi\xi}Y + (c-b)\partial_\xi Y - h_1\theta - h_2 Y - h_3\rho, \tag{6.9}$$

where

$$h_1(\xi) = \frac{\hat\rho(\xi)\hat Y(\xi)}{\hat\theta(\xi)^2}\exp(-\frac{1}{\hat\theta(\xi)}), \quad h_2(\xi) = \hat\rho(\xi)\exp(-\frac{1}{\hat\theta(\xi)}), \quad h_3(\xi) = \hat Y(\xi)\exp(-\frac{1}{\hat\theta(\xi)}).$$

We now introduce a weight function of the form $e^{\alpha\xi}$ that moves the spectrum to the open left half-plane. This can be done only for fast combustion waves; provided ϵ is small, we can use any α with $0 < \alpha < c-a$.

If $(\theta(\xi), \rho(\xi), Y(\xi))$ is in a weighted space $\mathcal{X}_\alpha$ with weight function $e^{\alpha\xi}$, then $(\theta(\xi), \rho(\xi), Y(\xi)) = e^{-\alpha\xi}(u(\xi), v(\xi), z(\xi))$ with $(u(\xi), v(\xi), z(\xi))$ in $\mathcal{X}_0$. Substituting into (6.7)–(6.9) and multiplying by $e^{\alpha\xi}$, we obtain

$$\begin{aligned}
\partial_t u &= \partial_{\xi\xi} u + (c-a-2\alpha)\partial_\xi u + (h_1+\alpha^2+a\alpha-c\alpha)u + h_2 z + h_3 v,\\
\partial_t v &= c\partial_\xi v - h_1 u - h_2 z - (h_3+c\alpha)v,\\
\partial_t z &= \epsilon\partial_{\xi\xi} z + (c-b-2\epsilon\alpha)\partial_\xi z - h_1 u + (\epsilon\alpha^2+b\alpha-c\alpha-h_2)z - h_3 v.
\end{aligned}$$

The eigenvalue problem reads

$$\lambda u = \partial_{\xi\xi} u + (c-a-2\alpha)\partial_\xi u + (h_1+\alpha^2+a\alpha-c\alpha)u + h_2 z + h_3 v, \tag{6.10}$$

$$\lambda v = c\partial_\xi v - h_1 u - h_2 z - (h_3+c\alpha)v, \tag{6.11}$$

$$\lambda z = \epsilon\partial_{\xi\xi} z + (c-b-2\epsilon\alpha)\partial_\xi z - h_1 u + (\epsilon\alpha^2+b\alpha-c\alpha-h_2)z - h_3 v. \tag{6.12}$$

Lemma 6.1. *If (u, v, z) satisfies (6.10)–(6.12) for some nonzero λ, then the following two inequalities hold for all $\epsilon_1 > 0$ and $\epsilon_2 > 0$:*

$$\mathrm{Re}(\lambda)\int |u|^2 \le \int (h_1+\alpha^2+a\alpha-c\alpha)|u|^2 + \epsilon_1\int h_2|u|^2 + \frac{1}{4\epsilon_1}\int h_2|z|^2 + \epsilon_2\int h_3|u|^2 + \frac{1}{4\epsilon_2}\int h_3|v|^2 \tag{6.13}$$

and

$$\begin{aligned}
(\mathrm{Re}(\lambda) + |\,\mathrm{Im}(\lambda)|)\int |u|^2 \;\le\; & \int (h_1+\alpha^2+a\alpha-c\alpha)|u|^2 + \frac{(c-a-2\alpha)^2}{4}\int |u|^2 + \epsilon_1\int h_2|u|^2\\
&+\frac{1}{2\epsilon_1}\int h_2|z|^2 + \epsilon_2\int h_3|u|^2 + \frac{1}{2\epsilon_2}\int h_3|v|^2.
\end{aligned} \tag{6.14}$$

Proof. We multiply (6.10) by the conjugate $\bar{u}$ and integrate from $-\infty$ to ∞. We obtain

$$\lambda\int |u|^2 = (c-a-2\alpha)\int u'\bar{u} + \int (h_1+\alpha^2+a\alpha-c\alpha)|u|^2 + \int h_2 z\bar{u} + \int h_3 v\bar{u} - \int |u'|^2. \tag{6.15}$$

Since $\mathrm{Re}\int_{-\infty}^{\infty} u'\bar{u}d\xi = \int_{-\infty}^{\infty}(u'\bar{u}+\bar{u}'u)d\xi/2 = \int_{-\infty}^{\infty}(u\bar{u})'d\xi/2 = 0$, taking the real and imaginary parts of (6.15), we have

$$\mathrm{Re}(\lambda)\int |u|^2 = \int (h_1+\alpha^2+a\alpha-c\alpha)|u|^2 + \mathrm{Re}\int h_2 z\bar{u} + \mathrm{Re}\int h_3 v\bar{u} - \int |u'|^2, \tag{6.16}$$

$$|\,\mathrm{Im}(\lambda)|\int |u|^2 \le (c-a-2\alpha)\int |u'||\bar{u}| + |\,\mathrm{Im}\int h_2 z\bar{u}| + |\,\mathrm{Im}\int h_3 v\bar{u}|. \tag{6.17}$$

The inequality (6.13) follows by using Young's inequality on (6.16); we use Young's inequality in the form $ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2$ where a, b are any real numbers and $\epsilon > 0$. In Lemma 10.1, ϵ_1 and ϵ_2 come from this inequality.

The inequality (6.14) follows by adding (6.16) and (6.17) and using the fact that $|\,\mathrm{Re}(x\bar{y})| + |\,\mathrm{Im}(x\bar{y})| \le \sqrt{2}|x||y|$, where x, y are complex numbers, and using Young's inequality to get

$(c-a-2\alpha)|u'||u| \leq \frac{(c-a-2\alpha)^2|u|^2}{4} + |u'^2|$:

$$
\begin{aligned}
(\mathrm{Re}(\lambda) + |\operatorname{Im}(\lambda)|) \int |u|^2 \leq \\
\int (h_1 + \alpha^2 + a\alpha - c\alpha)|u|^2 + \frac{(c-a-2\alpha)^2}{4} \int |u|^2 + \sqrt{2} \int h_2|z||u| + \sqrt{2} \int h_3|v||u| \leq \\
\int (h_1 + \alpha^2 + a\alpha - c\alpha)|u|^2 + \frac{(c-a-2\alpha)^2}{4} \int |u|^2 \\
+ \epsilon_1 \int h_2|u|^2 + \frac{1}{2\epsilon_1} \int h_2|z|^2 + \epsilon_2 \int h_3|u|^2 + \frac{1}{2\epsilon_2} \int h_3|v|^2.
\end{aligned}
$$

□

Lemma 6.2. *If (u, v, z) satisfies (6.10)–(6.12) for some nonzero λ, then the following inequality holds for all $\epsilon_3 > 0$ and $\epsilon_4 > 0$:*

$$
\mathrm{Re}(\lambda) \int |v|^2 \leq \epsilon_3 \int h_1|v|^2 + \frac{1}{4\epsilon_3} \int h_1|u|^2 + \epsilon_4 \int h_2|v|^2 + \frac{1}{4\epsilon_4} \int h_2|z|^2 - \int (h_3 + c\alpha)|v|^2. \tag{6.18}
$$

Proof. We multiply (6.11) by the conjugate $\bar{v}$ and integrate from $-\infty$ to ∞. We obtain

$$
\lambda \int |v|^2 = c \int v'\bar{v} - \int h_1 u\bar{v} - \int h_2 z\bar{v} - \int (h_3 + c\alpha)|v|^2. \tag{6.19}
$$

Taking the real part of (6.19), we have

$$
\mathrm{Re}(\lambda) \int |v|^2 = -\mathrm{Re} \int h_1 u\bar{v} - \mathrm{Re} \int h_2 z\bar{v} - \int (h_3 + c\alpha)|v|^2. \tag{6.20}
$$

The inequality (6.18) follows by using Young's inequality on (6.20). □

Lemma 6.3. *If (u, v, z) satisfies (6.10)–(6.12) for some nonzero λ, then the following two inequalities hold for all $\epsilon_5 > 0$ and $\epsilon_6 > 0$:*

$$
\mathrm{Re}(\lambda) \int |z|^2 \leq \epsilon_5 \int h_1|u|^2 + \frac{1}{4\epsilon_5} \int h_1|z|^2 + \epsilon_6 \int h_3|v|^2 + \frac{1}{4\epsilon_6} \int h_3|z|^2 + \int (\epsilon\alpha^2 + b\alpha - c\alpha - h_2)|z|^2 \tag{6.21}
$$

and

$$
\begin{aligned}
(\mathrm{Re}(\lambda) + |\operatorname{Im}(\lambda)|) \int |z|^2 \leq \frac{(c-b-2\epsilon\alpha)^2}{4\epsilon} \int |z|^2 + \epsilon_5 \int h_1|u|^2 + \frac{1}{2\epsilon_5} \int h_1|z|^2 \\
+ \epsilon_6 \int h_3|v|^2 + \frac{1}{2\epsilon_6} \int h_3|z|^2 + \int (\epsilon\alpha^2 + b\alpha - c\alpha - h_2)|z|^2.
\end{aligned} \tag{6.22}
$$

Proof. We multiply (6.12) by the conjugate $\bar{z}$ and integrate from $-\infty$ to ∞. We obtain

$$
\lambda \int |z|^2 = (c-b-2\epsilon\alpha) \int z'\bar{z} - \int h_1 u\bar{z} - \int h_3 \bar{z} v + \int (\epsilon\alpha^2 + b\alpha - c\alpha - h_2)|z|^2 - \epsilon \int |z'|^2. \tag{6.23}
$$

Taking the real and imaginary parts of (6.23), we have

$$
\mathrm{Re}(\lambda) \int |z|^2 = -\mathrm{Re} \int h_1 u\bar{z} - \mathrm{Re} \int h_3 \bar{z} v + \int (\epsilon\alpha^2 + b\alpha - c\alpha - h_2)|z|^2 - \epsilon \int |z'|^2, \tag{6.24}
$$

$$
|\operatorname{Im}(\lambda)| \int |z|^2 \leq (c-b-2\epsilon\alpha) \int |z'||\bar{z}| + |\operatorname{Im} \int h_1 u\bar{z}| + |\operatorname{Im} \int h_3 v\bar{z}|. \tag{6.25}
$$

The inequality (6.21) follows by using Young's inequality on (6.24). The inequality (6.22) follows by adding (6.24) and (6.25) together and using the fact that $|\operatorname{Re}(x\bar{y})| + |\operatorname{Im}(x\bar{y})| \leq \sqrt{2}|x||y|$, where x, y are complex numbers, and using Young's inequality to get $(c-b-2\epsilon\alpha)|z'||z| \leq \frac{(c-b-2\epsilon\alpha)^2|z|^2}{4} + |z'^2|$:

$$\begin{aligned}(\operatorname{Re}(\lambda) + |\operatorname{Im}(\lambda)|) \int |z|^2 \leq \\ \frac{(c-b-2\epsilon\alpha)^2}{4\epsilon}\int |z|^2 + \int (\epsilon\alpha^2 + b\alpha - c\alpha - h_2)|z|^2 + \sqrt{2}\int h_1|z||u| + \sqrt{2}\int h_3|v||z| \leq \\ \frac{(c-b-2\epsilon\alpha)^2}{4\epsilon}\int |z|^2 + \int (\epsilon\alpha^2 + b\alpha - c\alpha - h_2)|z|^2 \\ + \epsilon_5 \int h_1|u|^2 + \frac{1}{2\epsilon_5}\int h_1|z|^2 + \epsilon_6\int h_3|v|^2 + \frac{1}{2\epsilon_6}\int h_3|z|^2.\end{aligned}$$

□

Theorem 6.4. *If (u, v, z) satisfies (6.10)–(6.12) for some nonzero λ, then the following inequality holds for all $0 < \delta < 1$:*

$$\operatorname{Re}(\lambda) \leq \frac{1}{1-\delta}\sup_{\xi} h_1 + \frac{(1-\delta)^2 + 2\delta}{8\delta}\sup_{\xi}\{h_2 + h_3\} + \max\{\alpha^2 + a\alpha, \epsilon\alpha^2 + b\alpha\}. \tag{6.26}$$

Proof. First we multiply (6.13) by $k > 0$ and add to (6.18) and (6.21). We obtain

$$\begin{aligned}\operatorname{Re}(\lambda)\int (k|u|^2 + |v|^2 + |z|^2) \leq \\ (k + \epsilon_5 + \frac{1}{4\epsilon_3})\int h_1u^2 + \epsilon_3\int h_1v^2 + \frac{1}{4\epsilon_5}\int h_1z^2 + k\epsilon_1\int h_2u^2 \\ + \epsilon_4\int h_2v^2 + (\frac{k}{4\epsilon_1} + \frac{1}{4\epsilon_4} - 1)\int h_2z^2 + k\epsilon_2\int h_3u^2 + (\frac{k}{4\epsilon_2} + \epsilon_6 - 1)\int h_3v^2 \\ + \frac{1}{4\epsilon_6}\int h_3z^2 + \max\{\alpha^2 + a\alpha, \epsilon\alpha^2 + b\alpha\}\int (ku^2 + z^2) - c\alpha\int (ku^2 + v^2 + z^2).\end{aligned}$$

Set $\frac{k}{4\epsilon_1} + \frac{1}{4\epsilon_4} = 1$, $\frac{k}{4\epsilon_2} + \epsilon_6 = 1$, and take $\epsilon_4 = \epsilon_1$ and $\epsilon_6 = \frac{1}{4\epsilon_2}$. Then $\epsilon_1 = \epsilon_2 = \epsilon_4 = \frac{k+1}{4}$ and $\epsilon_6 = \frac{1}{k+1}$. Also set $\epsilon_3 = \frac{1}{1-\delta}$, $\epsilon_5 = \frac{1-\delta}{4}$, and $k = \frac{(1-\delta)^2}{2\delta}$. Then we have

$$\begin{aligned}\operatorname{Re}(\lambda)\int (k|u|^2 + |v|^2 + |z|^2) \leq \frac{1}{1-\delta}\int h_1(k|u|^2 + |v|^2 + |z|^2) \\ + \frac{(1-\delta)^2 + 2\delta}{8\delta}\int h_2(k|u|^2 + |v|^2) + \frac{(1-\delta)^2 + 2\delta}{8\delta}\int h_3(k|u|^2 + |z|^2) \\ + \max\{\alpha^2 + a\alpha, \epsilon\alpha^2 + b\alpha\}\int (ku^2 + z^2) - c\alpha\int (ku^2 + v^2 + z^2).\end{aligned}$$

Therefore,

$$\operatorname{Re}(\lambda) \leq \frac{1}{1-\delta}\sup_{\xi} h_1 + \frac{(1-\delta)^2 + 2\delta}{8\delta}\sup_{\xi}\{h_2 + h_3\} + \max\{\alpha^2 + a\alpha, \epsilon\alpha^2 + b\alpha\}.$$

□

Theorem 6.5. *If (u, v, z) satisfies (6.10)–(6.12) for some nonzero λ, then the following inequality holds for all $0 < \delta < 1$:*

$$\mathrm{Re}(\lambda) + |\mathrm{Im}(\lambda)| \leq \max_{\xi} \Big\{ \alpha^2 + a\alpha - c\alpha + \frac{(c-a-2\alpha)^2}{4} + (1-\delta)h_2 + \frac{h_3}{1-\delta} + \frac{(2-\delta)}{4\delta(1-\delta)} \frac{\hat{v}^2}{\hat{u}^4} h_3 + \frac{5h_1}{4} + \epsilon\alpha^2 + b\alpha - c\alpha + \frac{(c-b-2\epsilon\alpha)^2}{4\epsilon} + \frac{h_1}{1-\delta} + \frac{h_3}{2(1-\delta)} + \frac{(2-\delta)}{2\delta} \frac{\hat{v}^2}{\hat{z}^2} h_3 \Big\}. \tag{6.27}$$

Proof. To show (6.27) we need to revise Lemma 6.2. First we replace $(\hat{\theta}, \hat{\rho}, \hat{Y})$ in $h_1(\xi), h_2(\xi)$ and $h_3(\xi)$ with $(\hat{u}, \hat{v}, \hat{z})$. Note that we can write h_1 and h_2 in terms of h_3: $h_1 = \frac{\hat{v}}{\hat{u}^2} h_3$ and $h_2 = \frac{\hat{v}}{\hat{z}} h_3$. In (6.20) we replace $h_1 u\bar{v}$ and $h_2 z\bar{v}$ with $\frac{\hat{v}}{\hat{u}^2} h_3 u\bar{v}$ and $\frac{\hat{v}}{\hat{z}} h_3 z\bar{v}$ and apply Young's inequality. We obtain

$$h_1 u\bar{v} = \frac{\hat{v}}{\hat{u}^2} h_3 u\bar{v} \leq \epsilon_3 h_3 |v|^2 + \frac{\hat{v}^2}{4\epsilon_3 \hat{u}^4} h_3 |u|^2$$

and

$$h_2 z\bar{v} = \frac{\hat{v}}{\hat{z}} h_3 z\bar{v} \leq \epsilon_4 h_3 |v|^2 + \frac{\hat{v}^2}{4\epsilon_4 \hat{z}^2} h_3 |z|^2.$$

Substituting these expressions into (6.20), we obtain

$$\mathrm{Re}(\lambda) \int |v|^2 \leq \epsilon_3 \int h_3 |v|^2 + \frac{1}{4\epsilon_3} \int \frac{\hat{v}^2}{\hat{u}^4} h_3 |u|^2 + \epsilon_4 \int h_3 |v|^2 + \frac{1}{4\epsilon_4} \int \frac{\hat{v}^2}{\hat{z}^2} h_3 |z|^2 - \int (h_3 + c\alpha)|v|^2. \tag{6.28}$$

We multiply (6.14) and (6.22) by k_1 and k_2 respectively and then add to (6.28), which yields

$$\begin{aligned}
&(\mathrm{Re}(\lambda) + |\mathrm{Im}(\lambda)|) \int (k_1 u^2 + k_2 z^2) + \mathrm{Re}(\lambda) \int v^2 \leq \\
&\int \left(h_1 + \alpha^2 + a\alpha - c\alpha + \epsilon_1 h_2 + \epsilon_2 h_3 + \frac{(c-a-2\alpha)^2}{4} + \frac{h_3}{4\epsilon_3 k_1} \frac{\hat{v}^2}{\hat{u}^4} + \frac{\epsilon_5 k_2 h_1}{k_1} \right) k_1 |u|^2 \\
&\quad + \int \left(\epsilon\alpha^2 + b\alpha - c\alpha + \frac{(c-b-2\epsilon\alpha)^2}{4\epsilon} + \frac{h_1}{2\epsilon_5} + \frac{h_3}{2\epsilon_6} + \frac{h_3}{4\epsilon_4 k_2} \frac{\hat{v}^2}{\hat{z}^2} \right) k_2 |z|^2 \\
&\quad + \left(\epsilon_3 + \epsilon_4 + \frac{k_1}{2\epsilon_2} + k_2 \epsilon_6 - 1\right) \int h_3 |v|^2 + \left(\frac{k_1}{2\epsilon_1} - k_2\right) \int h_2 |z|^2.
\end{aligned}$$

Take $\epsilon_1 = \epsilon_6 = 1 - \delta$, $\epsilon_3 = \epsilon_4 = \epsilon_5 = \frac{1-\delta}{2}$, $\epsilon_2 = \frac{1}{1-\delta}$ and $k_1 = \frac{2\delta}{2-\delta}$. Then $\epsilon_3 + \epsilon_4 + \frac{k_1}{2\epsilon_2} + k_2\epsilon_6 = 1$ and $\frac{k_1}{2k_2\epsilon_1} = 1$. Thus we get

$$\begin{aligned}
&(\mathrm{Re}(\lambda) + |\mathrm{Im}(\lambda)|) \int (k_1 u^2 + k_2 z^2) + \mathrm{Re}(\lambda) \int v^2 \leq \\
&\int \left(\alpha^2 + a\alpha - c\alpha + \frac{(c-a-2\alpha)^2}{4} + (1-\delta)h_2 + \frac{h_3}{1-\delta} + \frac{(2-\delta)}{4\delta(1-\delta)} \frac{\hat{v}^2}{\hat{u}^4} h_3 + \frac{5h_1}{4} \right) k_1 |u|^2 \\
&\quad + \int \left(\epsilon\alpha^2 + b\alpha - c\alpha + \frac{(c-b-2\epsilon\alpha)^2}{4\epsilon} + \frac{h_1}{1-\delta} + \frac{h_3}{2(1-\delta)} + \frac{(2-\delta)}{2\delta} \frac{\hat{v}^2}{\hat{z}^2} h_3 \right) k_2 |z|^2.
\end{aligned}$$

We have a contradiction when

$$\begin{aligned}&\mathrm{Re}(\lambda) + |\,\mathrm{Im}(\lambda)| \geq \\ &\quad \max_{\xi} \Big\{ \alpha^2 + a\alpha - c\alpha + \frac{(c-a-2\alpha)^2}{4} + (1-\delta)h_2 + \frac{h_3}{1-\delta} + \frac{(2-\delta)}{4\delta(1-\delta)}\frac{\hat{v}^2}{\hat{u}^4}h_3 + \frac{5h_1}{4}, \\ &\qquad \epsilon\alpha^2 + b\alpha - c\alpha + \frac{(c-b-2\epsilon\alpha)^2}{4\epsilon} + \frac{h_1}{1-\delta} + \frac{h_3}{2(1-\delta)} + \frac{(2-\delta)}{2\delta}\frac{\hat{v}^2}{\hat{z}^2}h_3 \Big\}.\end{aligned}$$

□

The inequalities (6.26) and (6.27) define a trapezoidal region of possible unstable spectrum. We could use this region together with an Evans function calculation to rigorously rule out eigenvalues with $\mathrm{Re}\,\lambda \geq 0$.

Appendix A. Linear and nonlinear stability theorems

Consider a system of the form

$$\begin{aligned}\partial_t U &= d\partial_{xx}U + \tilde{a}\partial_x U + R_1(U,V), \\ \partial_t V &= \tilde{b}\partial_x V + R_2(U,V),\end{aligned} \tag{A.1}$$

with

$$U = U(x,t) \in \mathbb{R}^{N_1}, \quad V = V(x,t) \in \mathbb{R}^{N_2}, \quad x \in \mathbb{R}, \quad t \geq 0,$$

$$d = \mathrm{diag}(d_1,\ldots,d_{N_1}) \text{ with all } d_k > 0, \quad \tilde{a} = (\tilde{a}_{kl}) \text{ of size } N_1 \times N_1, \quad \tilde{b} = \mathrm{diag}(b_1,\ldots,b_{N_2}).$$

The matrices d and $\tilde{b}$ are constant, and the maps R_j are C^2. For the moment we refrain from giving assumptions on $\tilde{a}$.

After replacing x by $\xi = x - ct$, (A.1) becomes

$$\begin{aligned}\partial_t U &= d\partial_{\xi\xi}U + a\partial_\xi U + R_1(U,V), \\ \partial_t V &= b\partial_\xi V + R_2(U,V),\end{aligned} \tag{A.2}$$

with $a = \tilde{a} + \mathrm{diag}(c,\ldots,c)$ and $b = \tilde{b} + \mathrm{diag}(c,\ldots,c)$. Thus b is a constant diagonal matrix.

We denote the differentials of the maps R_j by $R_{j1} = \partial_U R_j$, $R_{j2} = \partial_V R_j$. Let $T^*(\xi) = (U^*(\xi), V^*(\xi))$ be a traveling wave solution of (A.1) with velocity c that approaches its end states exponentially. Then the linearization of (A.2) at $T^*(\xi)$ is

$$\begin{pmatrix} U_t \\ V_t \end{pmatrix} = \mathcal{L}\begin{pmatrix} U \\ V \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} d\partial_{\xi\xi} + a\partial_\xi + R_{11} & R_{12} \\ R_{21} & b\partial_\xi + R_{21} \end{pmatrix}, \tag{A.3}$$

where $R_{jk} = R_{jk}(U^*(\xi), V^*(\xi))$.

To study $\mathcal{L}$ on a weighted Banach space $\mathcal{X}_\alpha$, with weight function $\gamma_\alpha(\xi)$, we instead study the isomorphic operator $\mathcal{L}_\alpha = \gamma_\alpha \mathcal{L}\gamma_\alpha^{-1}$ on $\mathcal{X}_0$. $\mathcal{L}_\alpha$ has the form

$$\mathcal{L}_\alpha = \begin{pmatrix} d\partial_{\xi\xi} + \hat{a}\partial_\xi + S_{11} & S_{12} \\ S_{21} & b\partial_\xi + S_{21} \end{pmatrix}, \tag{A.4}$$

in which the $S_{jk}(\xi)$ are continuously differentiable and $S'_{jk}(\xi) \to 0$ exponentially as $\xi \to \pm\infty$. Note that d and b are unchanged from (A.3). Compare (3.11).

In the course of proving Theorem 3.14 in [9], one must show that if a traveling wave $T^*(\xi)$ is spectrally stable in $\mathcal{X}_\alpha$, then it is linearly stable in $\mathcal{X}_\alpha$. (Theorem 3.14 in [9] is stated for traveling waves with left state at the origin, but the translation required to achieve this does not affect the linearization of the system at the traveling wave.)

The proof in [9] that spectral stability in $\mathcal{X}_\alpha$ implies linear stability in $\mathcal{X}_\alpha$ appeals to Theorem 3.1 in [10], which implies that for a linear operator in the form (A.4) on $\mathcal{X}_0 = L^2(\mathbb{R}, \mathbb{R}^{N_1+N_2})$ or $\mathrm{BUC}(\mathbb{R}, \mathbb{R}^{N_1+N_2})$, in which $d = \mathrm{diag}(d_1, \ldots, d_{N_1})$ with all d_k positive constants, $\hat{a}$ is a constant matrix, b is a constant multiple of the identity matrix, and the $S_{jk}(\xi)$ are continuous and approach constant limits exponentially as $\xi \to \pm\infty$, spectral stability implies linear stability. The same result holds in $H^1(\mathbb{R}, \mathbb{R}^{N_1+N_2})$ provided the S_{jk} are continuously differentiable and $S'_{jk}(\xi) \to 0$ exponentially as $\xi \to \pm\infty$. This result on BUC or H^1 is used to prove Theorem 3.14 in [9] for perturbations of the traveling wave in BUC or H^1.

Because of the requirement in Theorem 3.1 of [10] that b be a constant multiple of the identity matrix, Theorem 3.14 in [9] was stated for reaction-diffusion systems (no transport terms, i.e., $\tilde{a} = \tilde{b} = 0$ in (A.1)), in which case $b = cI$. We remark that Theorem 3.1 as stated in [10] is not quite sufficient to prove that spectral stability in $\mathcal{X}_\alpha$ implies linear stability $\mathcal{X}_\alpha$, since the matrix $\hat{a}$ in (A.4) need not be constant even when $\tilde{a} = 0$ in (A.1). This can be seen from (3.11); in the term $-2\eta_a\partial_\xi$ in (3.11), $\eta_a(\xi)$ is not constant unless the weight function is just an exponential function $e^{\alpha\xi}$. The simplest fix is to note that Theorem 3.1 in [10], for L^2 or BUC, could have allowed a continuous $\hat{a}(\xi)$ that approaches end states exponentially with no change in the proof. Then the theorem could have been extended to H^1 provided $\hat{a}(\xi)$ as well as the $S_{jk}(\xi)$ are continuously differentiable and their derivatives go to 0 exponentially as $\xi \to \pm\infty$.

Note that the restriction to $\tilde{a} = 0$ in [9] was not necessary. With the slight generalization of Theorem 3.1 in [10] just mentioned, one could have allowed $\tilde{a}$ to be a continuously differentiable function whose derivative goes to 0 exponentially as $\xi \to \pm\infty$.

In order to use Theorem 3.14 of [9] in Section 5 of this paper, it must be generalized to allow systems (A.1) in which $\tilde{b}$ is an arbitrary diagonal matrix. (Note that for the system (2.1)–(2.3), $\tilde{b} = \mathrm{diag}(0, b)$.) In fact we can generalize Theorem 3.14 in [9] to allow $\tilde{a}$ to be a continuously differentiable function whose derivative goes to 0 exponentially as $\xi \to \pm\infty$, and $\tilde{b}$ to be an arbitrary diagonal matrix, with the other hypotheses unchanged. The key step in the proof of the generalization is to show that if the traveling wave $T^*(\xi)$ is spectrally stable in $\mathcal{X}_\alpha$, then it is linearly stable in $\mathcal{X}_\alpha$. To do this one can appeal to Yurov's recent result, Theorem 1.1 in [16], which implies that for a linear operator in the form (A.4) on L^2, in which $d = \mathrm{diag}(d_1, \ldots, d_{N_1})$ with all d_k positive constants, $\hat{a}(\xi)$ and the $S_{jk}(\xi)$ are continuous and approach end states exponentially, and b is a constant diagonal matrix, spectral stability implies linear stability. By an argument in Section 3 of [10], the same result holds on H^1, provided $\hat{a}(\xi)$ and the $S_{jk}(\xi)$ are continuously differentiable and their derivatives go to 0 exponentially as $\xi \to \pm\infty$. However, Yurov's result does not imply the same result on BUC. Thus the generalization of Theorem 3.14 in [9] that is needed in Section 5 of this paper allows perturbations in H^1 but not in BUC. That is why Theorem 5.1 of this paper only allows perturbations in H^1.

References

[1] I.Y. Akkutlu and Y.C. Yortsos, The dynamics of in-situ combustion fronts in porous media, *J. of Combustion and Flame*, **134** (2003) 229–247.

[2] A.P. Aldushin, I.E. Rumanov and B.J Matkowsky, Maximal energy accumulation in a superadiabatic filtration combustion wave, *J. of Combustion and Flame*, **118** (1999) 76–90.

[3] J. Alexander, R. Gardner, and C. Jones, A topological invariant arising in the stability analysis of travelling waves, *J. Reine Angew. Math.*, **410** (1990) 167–212.

[4] B. Barker, J. Humpherys and K. Zumbrun, STABLAB: A MATLAB-Based Numerical Library for Evans Function Computation, *http://www.impact.byu.edu/stablab/*, 2009.

[5] G. Chapiro, *Gas-Solid Combustion in Insulated Porous Media*, Doctoral thesis, IMPA, 2009. http://www.preprint.impa.br.

[6] G. Chapiro, A. A. Mailybaev, A.J. de Souza, D. Marchesin and J. Bruining, Asymptotic approximation of long-time solution for low-temperature filtration combustion, *Comput. Geosciences*, **16** (2012) 799–808.

[7] G. Chapiro, D. Marchesin and S. Schecter, Combustion waves and Riemann solutions in light porous foam, *Journal of Hyperbolic Differential Equations*, **11** (2014) 295–328.

[8] E.J. Doedel, A.R. Champneys, T.F. Fairgrieve, Y.A. Kuznetsov, B. Oldeman, R. Paffenroth, B. Sandstede, X. Wang and C. Zhang, AUTO-07P: Continuation and Bifurcation Software for Ordinary Differential Equations, Concordia University, Montreal, Canada (2007)

[9] A. Ghazaryan, Y. Latushkin and S. Schecter, Stability of traveling waves for a class of reaction diffusion systems that arise in chemical reaction models, *SIAM J.Math. Anal.*, **42** (2010) 2434–2472.

[10] A. Ghazaryan, Y. Latushkin and S. Schecter, Stability of traveling waves for degenerate systems of reaction diffusion equations, *Indiana Univ. Math. J.*, **60** (2011) 443–472.

[11] A. Ghazaryan, J. Humpherys and J. Lytle, Spectral behavior of combustion fronts with high exothermicity, *SIAM J. on Appl. Math.* **73** (2013), 422-437.

[12] V. Gubernov, G.N. Mercer, H.S. Sidhu and R.O. Weber, Evans function stability of combustion waves, *SIAM J.Appl. Math.*, **63(4)** (2003) 1259–1275.

[13] B. Sandstede, Stability of traveling waves, *in Handbook of Dynamical Systems*, **2** (2002) 983–1055.

[14] D.A. Schult, B.J. Matkowsky, V.A. Volpert, and A.C. Fernandez-Pello, Forced forward smolder combustion, *Combustion and Flame*, **104(1-2)** (1996) 1–26.

[15] F. Ozbag, S. Schecter and G. Chapiro, Traveling waves in a simplified gas-solid combustion model in porous pedia, *Advances in differential equations.*

[16] V. Yurov, *Stability estimates for semigroups and partly parabolic reaction diffusion equation*, Doctoral thesis, University of Missouri, 2013.