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Existence of Dafermos profiles for singular shocks

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Abstract

For a model system of two conservation laws, we show that singular shocks have Defermos profiles.

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1. Introduction

Keyfitz and Kranzer [10,13] showed that the Riemann problem for the strictly hyperbolic, genuinely nonlinear system of conservation laws

$$u_{1t} + (u_1^2 - u_2)_x = 0, \tag{1.1}$$

$$u_{2t} + (\tfrac{1}{3}u_1^3 - u_1)_x = 0 \tag{1.2}$$

does not always have a solution consisting of combinations of rarefactions and shock waves. They could, however, always produce a unique solution to the Riemann problem for (1.1)–(1.2) if they allowed *singular shocks*. Singular shocks satisfy only a modified form of the Rankine–Hugoniot condition; thus they do not have viscous profiles.

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Roughly speaking, a shock wave is a Heaviside function, whereas a singular shock is a Heaviside function plus a δ-function concentrated at the discontinuity [11,22].

Keyfitz and Kranzer proposed an approach to singular shocks via the Dafermos regularization of (1.1)–(1.2), which is the artificial system

$$u_{1t} + (u_1^2 - u_2)_x = \varepsilon t u_{1xx}, \tag{1.3}$$

$$u_{2t} + (\tfrac{1}{3}u_1^3 - u_1)_x = \varepsilon t u_{2xx}. \tag{1.4}$$

They conjectured that the singular shocks they wanted to use could be approximated, for small $\varepsilon > 0$, by self-similar solutions $(u^\varepsilon, v^\varepsilon)(\frac{x}{t})$ of (1.3)–(1.4) that grow arbitrarily large near the discontinuity as $\varepsilon \to 0$. On the assumption that such *Dafermos profiles* exist, Keyfitz and Kranzer constructed their asymptotic approximations to lowest order in ε.

The result of this paper is that the conjectured self-similar solutions of (1.3)–(1.4) exist. The proof avoids the problem of matching difficult asymptotic expansions by using geometric singular perturbation theory [6,7]. More precisely, we use the blowing-up approach to geometric singular perturbation problems that lack normal hyperbolicity [4,5,15]. The idea of using this method to study self-similar solutions of the Dafermos regularization is due to Szmolyan [25]; see also [19–21,16].

A generalization of the Keyfitz–Kranzer system ($\frac{1}{3}$ replace by $\frac{\gamma}{3}$ with $0 < \gamma \leq 1$) is discussed in [17]. The results of the present paper hold for this generalization. Sever [22] identifies a class of problems for which the lowest-order asymptotic approximations to Dafermos profiles can be constructed. Another example of a system that admits singular shocks is treated in [12]. We have not checked that our result holds for these problems.

In order to provide a context for the idea of Keyfitz and Kranzer, let us review some background about systems of conservation laws.

A *system of conservation laws* in one-space dimension is a partial differential equation of the form

$$u_t + f(u)_x = 0, \tag{1.5}$$

with $t \geq 0$, $x \in \mathbb{R}$, $u(x,t) \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ a smooth map. A *shock wave* for (1.5) is given by

$$u(x,t) = \begin{cases} u_- & \text{for } x < st, \\ u_+ & \text{for } x > st. \end{cases} \tag{1.6}$$

The triple (u_-, s, u_+) is required to satisfy the *Rankine–Hugoniot condition*

$$f(u_+) - f(u_-) - s(u_+ - u_-) = 0. \tag{1.7}$$

This condition follows from the requirement that (1.6) be a weak solution of (1.5) [23].

Too many shock waves satisfy the Rankine–Hugoniot condition; an additional criterion is needed to select the physically realistic ones. A *viscous regularization* of (1.5) is a partial differential equation of the form

$$u_t + f(u)_x = (B(u)u_x)_x, \tag{1.8}$$

where $B(u)$ is an $n \times n$ matrix whose eigenvalues all have positive real part. The shock wave (1.6) satisfies the *viscous profile criterion* for $B(u)$ if (1.8) has a traveling wave solution $u(x - st)$ that satisfies the boundary conditions

$$u(-\infty) = u_-, \quad u(+\infty) = u_+. \tag{1.9}$$

A traveling wave solution of (1.8) satisfying the boundary conditions (1.9) exists if and only if the *traveling wave ODE*

$$\dot{u} = B(u)^{-1}(f(u) - f(u_-) - s(u - u_-)) \tag{1.10}$$

has an equilibrium at u_+ (it automatically has one at u_-) and a connecting orbit from u_- to u_+. The condition that (1.10) have an equilibrium at u_+ is just the Rankine–Hugoniot condition (1.7).

A *Riemann problem* for (1.5) is (1.5) together with the initial condition

$$u(x,0) = \begin{cases} u_L & \text{for } x < 0, \\ u_R & \text{for } x > 0. \end{cases} \tag{1.11}$$

One seeks piecewise continuous weak solutions of Riemann problems in the scale-invariant form $u(x,t) = \hat{u}(\xi)$, $\xi = \frac{x}{t}$. Usually, one requires that the solution consist of a finite number of constant parts, continuously changing parts (rarefaction waves), and jump discontinuities (shock waves). Shock waves occur when

$$\lim_{\xi \to s-} \hat{u}(\xi) = u_- \neq u_+ = \lim_{\xi \to s+} \hat{u}(\xi).$$

One way to decide which shock waves to allow is to have in mind a fixed regularization (1.8). For a Riemann solution associated with the viscosity $B(u)$, the triple (u_-, s, u_+) is required to satisfy the viscous profile criterion for $B(u)$.

An alternative approach to Riemann problems uses the *Dafermos regularization* of a system of conservation laws [2]. The Dafermos regularization of (1.5) associated with the viscosity matrix $B(u)$ is

$$u_t + f(u)_x = \varepsilon t(B(u)u_x)_x. \tag{1.12}$$

Like the Riemann problem, but unlike (1.8), (1.12) has many scale-invariant solutions $u(x,t)=\hat{u}(\xi)$, $\xi=\frac{x}{t}$. They satisfy the nonautonomous second-order ODE

$$(Df(u)-\xi I)\frac{du}{d\xi}=\varepsilon\frac{d}{d\xi}\left(B(u)\frac{du}{d\xi}\right), \tag{1.13}$$

where we have written u instead of $\hat{u}$. Corresponding to the initial condition (1.11), we use the boundary conditions

$$u(-\infty)=u_L, \quad u(+\infty)=u_R. \tag{1.14}$$

For u_R close to u_L, Tzavaras [24] has shown that Riemann solutions associated with $B(u)\equiv I$ can be approximated by solutions of the boundary-value problem (1.13)–(1.14) with $B(u)\equiv I$ and $\varepsilon>0$ small.

A *structurally stable* Riemann solution is one that is stable to perturbation of u_L, u_R and f, in the sense that nearby Riemann problems have solutions with the same number of waves, of the same types [18]. It appears to be the case that the structurally stable Riemann solutions associated with a given $B(u)$ have, for small $\varepsilon>0$, solutions of (1.13)–(1.14) nearby. For results in this direction, see [25,19,21]; for some non-structurally stable Riemann solutions, see [16]. In these papers, a Riemann solution $\hat{u}(\frac{x}{t})$ of (1.5), (1.11) that is associated with a given $B(u)$ is viewed as a singular solution of (1.13)–(1.14) with $\varepsilon=0$. This singular solution includes lines of normally hyperbolic equilibria (corresponding to constant states in the Riemann solution), curves of equilibria that are not normally hyperbolic (corresponding to rarefactions), and orbits connecting equilibria (shock waves; the orbits correspond to the solutions of (1.10) associated with the shock waves). The proofs that for small $\varepsilon>0$ there are nearby solutions of the boundary-value problem (1.13)–(1.14) use geometric singular perturbation theory.

These results suggest that in looking for solutions of the Riemann problem (1.5), (1.11) that are associated with the viscosity $B(u)$, one should accept any function $\hat{u}(\xi)$ that arises as the limit as $\varepsilon\to 0$ of solutions of the Dafermos boundary-value problem (1.13)–(1.14). This is essentially the idea of Keyfitz and Kranzer, with $B(u)\equiv I$, that leads to singular shocks. The solutions of (1.13)–(1.14) that they use become unbounded as $\varepsilon\to 0$. Nevertheless, they converge pointwise to a function that is discontinuous at a single point, and in measure to this function plus a δ-function concentrated at the discontinuity.

The rest of the paper is organized as follows. The geometry of the Dafermos regularization is reviewed in Section 2. In Section 3 we specialize to the Keyfitz–Kranzer system. Blow-up is performed in Section 4. A useful lemma on flow past a "corner equilibrium" is proved in Section 5. Manifolds of corner equilibria arise in blown-up geometric singular perturbation problems precisely where inner and outer solutions must be matched. When such equilibria are normally hyperbolic, this lemma plays the same role in tracking the flow past them that the Exchange Lemma [9,8] plays at certain other manifolds of equilibria. Finally, the result on existence of Dafermos profiles for singular shocks is stated precisely and proved in Section 6.

2. Dafermos regularization

We consider the nonautonomous second-order ODE (1.13) with $B(u) \equiv I$. Following [25], we convert it into an autonomous first-order ODE by letting $v = \varepsilon \frac{du}{d\xi}$ and treating ξ as a state variable:

$$\varepsilon u' = v, \tag{2.1}$$

$$\varepsilon v' = (Df(u) - \xi I)v, \tag{2.2}$$

$$\xi' = 1. \tag{2.3}$$

As an autonomous ODE, the system (2.1)–(2.3) is a singular perturbation problem written in the slow time θ, with $\frac{d\xi}{d\theta} = 1$ (i.e., $\xi = \theta + \xi_0$). Here, the prime symbol denotes derivative with respect to θ.

We let $\theta = \varepsilon\tau$, and we use a dot to denote differentiation with respect to τ. System (2.1)–(2.3) becomes

$$\dot{u} = v, \tag{2.4}$$

$$\dot{v} = (Df(u) - \xi I)v, \tag{2.5}$$

$$\dot{\xi} = \varepsilon. \tag{2.6}$$

System (2.4)–(2.6) is system (2.1)–(2.3) written in the fast time τ. The boundary conditions (1.14) become

$$(u, v, \xi)(-\infty) = (u_L, 0, -\infty), \quad (u, v, \xi)(\infty) = (u_R, 0, \infty). \tag{2.7}$$

Setting $\varepsilon = 0$ in (2.4)–(2.6) yields the fast limit system

$$\dot{u} = v, \tag{2.8}$$

$$\dot{v} = (Df(u) - \xi I)v, \tag{2.9}$$

$$\dot{\xi} = 0. \tag{2.10}$$

System (2.8)–(2.10) has the $(n+1)$-dimensional space of equilibria $v = 0$.

We now restrict to the case $n = 2$. For a small $\delta > 0$, let

$$S_0 = \{(u, v, \xi) : \|u\| \leq \frac{1}{\delta}, v = 0 \text{ and } \xi \leq \lambda_1(u) - \delta\},$$

$$S_1 = \{(u, v, \xi) : \|u\| \leq \frac{1}{\delta}, v = 0 \text{ and } \lambda_1(u) + \delta \leq \xi \leq \lambda_2(u) - \delta\},$$

$$S_2 = \{(u, v, \xi) : \|u\| \leq \frac{1}{\delta}, v = 0, \text{ and } \lambda_2(u) + \delta \leq \xi\}.$$

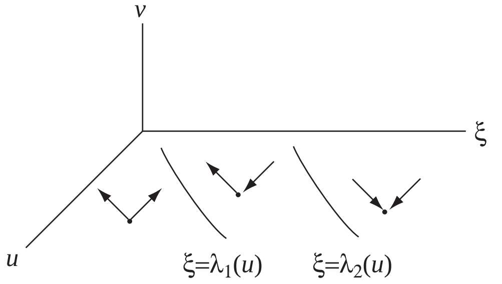

Fig. 1. Phase space for the fast limit system (2.8)–(2.10). The three-dimensional space $v = 0$ consists of equilibria. This space is divided by the surfaces $\xi = \lambda_1(u)$ and $\xi = \lambda_2(u)$ into sets equilibria with two positive eigenvalues, one positive and one negative eigenvalue, and two negative eigenvalues.

For the system (2.8)–(2.10), each S_k is a three-dimensional normally hyperbolic manifold of equilibria [6,7]. Every point of S_k has a stable manifold of dimension k and an unstable manifold of dimension $2-k$. Thus the unstable manifold of S_0 for (2.8)–(2.10), which is the union of the unstable manifolds of the equilibria that comprise S_0, has open interior in $\mathbb{R}^5$. Similarly the stable manifold of S_2 for (2.8)–(2.10), which is the union of the stable manifolds of the equilibria that comprise S_2, has open interior in $\mathbb{R}^5$ (S_1 will not be important to us.) See Fig. 1.

According to [6], for ε near 0, the system (2.4)–(2.6) has normally hyperbolic invariant manifolds near each S_k. Since the three-dimensional space $v = 0$ is invariant under (2.4)–(2.6) for every ε, the perturbed manifolds can be taken to be the S_ks themselves. On S_k, the system (2.4)–(2.6) reduces to

$$\dot{u} = 0, \quad \dot{v} = 0, \quad \dot{\xi} = \varepsilon.$$

For each fixed u_0 in $\mathbb{R}^2$, let $S_k(u_0)$ be the set of point in S_k with $u = u_0$, a (portion of a) line. Then for (2.4)–(2.6), each line $S_0(u)$ has a three-dimensional unstable manifold $W^{\mathrm{u}}_{\varepsilon}(S_0(u))$, and each line $S_2(u)$ has a three-dimensional stable manifold $W^{\mathrm{s}}_{\varepsilon}(S_0(u))$. These manifolds depend smoothly on (u, ε).

Geometrically, for a fixed $\varepsilon > 0$, a solution of the boundary-value problem (2.4)–(2.7) corresponds to a solution of (2.4)–(2.6) that lies in the intersection of $W^{\mathrm{u}}_{\varepsilon}(S_0(u_L))$ and $W^{\mathrm{s}}_{\varepsilon}(S_2(u_R))$. These are three-dimensional manifolds in a five-dimensional space, so they are expected to intersect in isolated curves. See Fig. 2.

In (2.4)–(2.6) we let $w = f(u) - \xi u - v$, i.e., we make the invertible coordinate transformation

$$(u, v, \xi) \to (u, w, \xi) = (u, f(u) - \xi u - v, \xi). \tag{2.11}$$

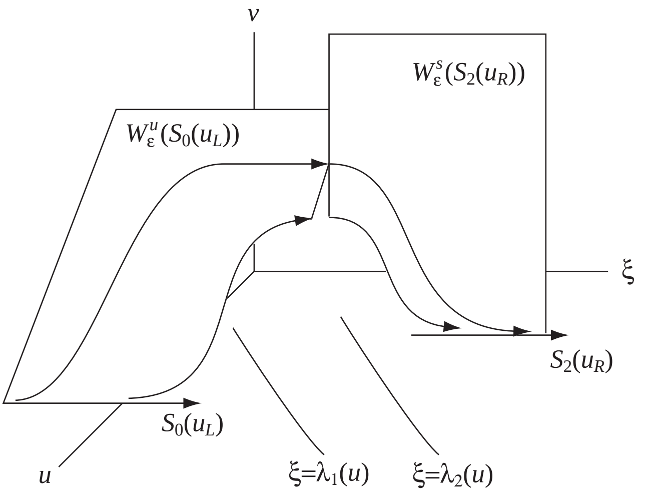

Fig. 2. Phase space for the Dafermos system (2.4)–(2.6) with $\varepsilon > 0$. The three-dimensional space $v = 0$ is invariant but no longer consists of equilibria. A solution in $W^{\mathrm{u}}_\varepsilon(S_0(u_L)) \cap W^{\mathrm{s}}_\varepsilon(S_2(u_R))$ is shown.

Also, from now on we shall treat ε as a state variable. Thus, we obtain the system

$$\dot{u} = f(u) - \xi u - w, \tag{2.12}$$

$$\dot{w} = -\varepsilon u, \tag{2.13}$$

$$\dot{\xi} = \varepsilon, \tag{2.14}$$

$$\dot{\varepsilon} = 0. \tag{2.15}$$

In six-dimensional $uw\xi\varepsilon$-space, each subspace $\varepsilon =$ constant is invariant. Corresponding to the three-dimensional subspace $v = 0$ of $uv\xi$-space, which is invariant under (2.4)–(2.6) for each ε, we have the four-dimensional invariant surface $w = f(u) - \xi u$ in $uw\xi\varepsilon$-space. Corresponding to the three-dimensional subsets S_k of $v = 0$, we have four-dimensional normally hyperbolic subsets T_k of the surface $w = f(u) - \xi u$. T_0 and T_2 (we shall not need T_1) are foliated into invariant lines

$$T_0^\varepsilon(u) = \{(u, w, \xi, \varepsilon) : u \text{ and } \varepsilon \text{ fixed}, \;\; \xi \leq \lambda_1(u) - \delta, w = f(u) - \xi u\},$$

$$T_2^\varepsilon(u) = \{(u, w, \xi, \varepsilon) : u \text{ and } \varepsilon \text{ fixed}, \;\; \lambda_2(u) + \delta \leq \xi, w = f(u) - \xi u\}.$$

From the theory of normally hyperbolic invariant manifolds [6,7], each line $T_0^\varepsilon(u)$ has a three-dimensional unstable manifold $W^{\mathrm{u}}(T_0^\varepsilon(u))$, and each line $T_2^\varepsilon(u)$ has a three-dimensional stable manifold $W^{\mathrm{s}}(T_2^\varepsilon(u))$; these manifolds depend smoothly on (u, ε). In these coordinates, we wish to find, for each small $\varepsilon > 0$, a solution of (2.12)–(2.15) that lies in the intersection of $W^{\mathrm{u}}(T_0^\varepsilon(u_L))$ and $W^{\mathrm{s}}(T_2^\varepsilon(u_R))$.

3. Keyfitz–Kranzer system

For the system of conservation laws (1.1)–(1.2), the corresponding Dafermos system (2.4)–(2.6) is

$$\dot{u}_1 = v_1, \tag{3.1}$$

$$\dot{u}_2 = v_2, \tag{3.2}$$

$$\dot{v}_1 = (2u_1 - \xi)v_1 - v_2, \tag{3.3}$$

$$\dot{v}_2 = (u_1^2 - 1)v_1 - \xi v_2, \tag{3.4}$$

$$\dot{\xi} = \varepsilon. \tag{3.5}$$

The corresponding alternate Dafermos system (2.12)–(2.15) is

$$\dot{u}_1 = u_1^2 - u_2 - \xi u_1 - w_1, \tag{3.6}$$

$$\dot{u}_2 = \frac{1}{3}u_1^3 - u_1 - \xi u_2 - w_2, \tag{3.7}$$

$$\dot{w}_1 = -\varepsilon u_1, \tag{3.8}$$

$$\dot{w}_2 = -\varepsilon u_2, \tag{3.9}$$

$$\dot{\xi} = \varepsilon, \tag{3.10}$$

$$\dot{\varepsilon} = 0. \tag{3.11}$$

Motivated by [10,13], in (3.6)–(3.11) we introduce the new variables

$$y_1 = \varepsilon u_1, \quad y_2 = \varepsilon^2 u_2. \tag{3.12}$$

We multiply the resulting system by ε, i.e., we rescale time by $\tau = \varepsilon\zeta$, and we use a prime to denote derivative with respect to ζ (This differs from the use of prime in Section 2.) We obtain

$$y_1' = y_1^2 - y_2 - \varepsilon\xi y_1 - \varepsilon^2 w_1, \tag{3.13}$$

$$y_2' = \frac{1}{3}y_1^3 - \varepsilon^2 y_1 - \varepsilon\xi y_2 - \varepsilon^3 w_2, \tag{3.14}$$

$$w_1' = -\varepsilon y_1, \tag{3.15}$$

$$w_2' = -y_2, \tag{3.16}$$

$$\xi' = \varepsilon^2, \tag{3.17}$$

$$\varepsilon' = 0. \tag{3.18}$$

Note that this change of variables collapses the five-dimensional subspace $\varepsilon = 0$ of $uw\xi\varepsilon$-space to a three-dimensional subspace E of $yw\xi\varepsilon$-space,

$$E = \{(y, w, \xi, \varepsilon) : y = 0,\ \varepsilon = 0\}.$$

Each two-dimensional set $\{(u, w, \xi, \varepsilon) : w = w_0,\ \xi = \xi_0,\ \varepsilon = 0\}$ collapses to the point $(0, w_0, \xi_0, 0)$ of E. The advantage of this change of variables is that for small $\varepsilon > 0$, some solutions that take on very large u-values take on only moderate y-values. In [10,13] the singular shock profiles consist of two outer solutions, expressed in u, that satisfy the boundary conditions (1.14), and an inner solution, expressed in y, that represents a large excursion in the solution. The difficulty lies in matching them.

In this paper, we shall take system (3.13)–(3.18) *to be the fundamental one to analyze.*

Setting $\varepsilon = 0$ in system (3.13)–(3.18), we obtain

$$y_1' = y_1^2 - y_2, \tag{3.19}$$

$$y_2' = \frac{1}{3}y_1^3, \tag{3.20}$$

$$w_1' = 0, \tag{3.21}$$

$$w_2' = -y_2, \tag{3.22}$$

$$\xi' = 0, \tag{3.23}$$

$$\varepsilon' = 0. \tag{3.24}$$

This five-dimensional system (recall $\varepsilon = 0$) has the three-dimensional space of equilibria E. The equilibria in E have all eigenvalues equal to 0.

The phase portrait of the two-dimensional system (3.19)–(3.20) is shown in Fig. 3. There is a unique equilibrium at the origin. Through it are two invariant parabolas $y_2 = c_\pm y_1^2$ with $c_\pm = \frac{1}{6}(3 \pm \sqrt{3})$. Above $y_2 = c_+ y_1^2$ is a one-parameter family of homoclinic orbits. They are all tangent to $y_2 = c_+ y_1^2$ at both ends; each orbit is represented by a unique solution $(y_1(\zeta), y_2(\zeta))$ with $y_1(0) = 0$; $y_2(\zeta)$ is integrable; and the homoclinic solutions are parameterized by $\gamma = \int_{-\infty}^{\infty} y_2(\zeta)\, d\zeta$, $0 < \gamma < \infty$ [17].

Proposition 3.1. *Let* $q_0 = (0, 0, w_{01}, w_{02}, \xi_0, 0)$ *and* $q_1 = (0, 0, w_{01}, w_{12}, \xi_0, 0)$ *be two points of* E *with* $w_{02} > w_{12}$. *Then there is a unique solution of* (3.19)–(3.24) *that goes from* q_0 *to* q_1 *and has* $y_1(0) = 0$.

Proof. Let $(y_1(\zeta), y_2(\zeta))$ be the unique solution of (3.19)–(3.20) that is homoclinic to the origin, satisfies $y_1(0) = 0$, and has $\int_{-\infty}^{\infty} y_2(\zeta)\, d\zeta = w_{02} - w_{12}$. Then the desired solution of (3.19)–(3.24) is

$$(y_1(\zeta), y_2(\zeta), w_{01}, w_{02} - \int_{-\infty}^{\zeta} y_2(\eta)\, d\eta, \xi_0, 0). \qquad \square$$

We remark that $y_1(\zeta)$ is an odd function and $y_2(\zeta)$ is even.

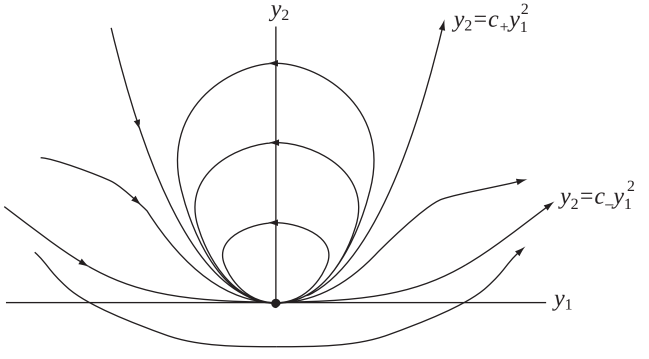

Fig. 3. Phase portrait of $y_1' = y_1^2 - y_2$, $y_2' = \frac{1}{3} y_1^3$.

4. Blow up

Corresponding to the lines $T_0^\varepsilon(u)$ and $T_2^\varepsilon(u)$ in $uw\xi\varepsilon$-space, we have in $yw\xi\varepsilon$-space the lines

$$M_0^\varepsilon(u) = \{(y, w, \xi, \varepsilon) : y_1 = \varepsilon u_1, y_2 = \varepsilon^2 u_2, \xi \leq \lambda_1(u) - \delta, w = f(u) - \xi u, \varepsilon \text{ fixed}\},$$

$$M_2^\varepsilon(u) = \{(y, w, \xi, \varepsilon) : y_1 = \varepsilon u_1, y_2 = \varepsilon^2 u_2, \lambda_2(u) + \delta \leq \xi, w = f(u) - \xi u, \varepsilon \text{ fixed}\},$$

For small $\varepsilon > 0$, we wish to find a solution of (3.13)–(3.18) that lies in the intersection of $W^{\mathrm{u}}(M_0^\varepsilon(u_L))$ and $W^{\mathrm{s}}(M_2^\varepsilon(u_R))$.

Notice that $M_0^0(u_L)$ and $M_2^0(u_R)$ are lines in the three-dimensional space E, which consists entirely of equilibria with all eigenvalues equal to 0. A blow-up is necessary to resolve the behavior of the system near E [15].

We shall blow up E, which is the product of the origin in $y_1y_2\varepsilon$-space with $w_1w_2\xi$-space, to the product of a two-sphere with $w_1w_2\xi$-space. The two-sphere is a blow-up of the origin in $y_1y_2\varepsilon$-space.

The blow-up transformation is a map from $S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$ to $yw\xi\varepsilon$-space defined as follows. Let $((\bar{y}_1, \bar{y}_2, \bar{\varepsilon}), \bar{r}, (w_1, w_2, \xi))$ be a point of $S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$; we have $\bar{y}_1{}^2 + \bar{y}_2{}^2 + \bar{\varepsilon}^2 = 1$. Then the blow-up transformation is

$$y_1 = \bar{r}\bar{y}_1, \tag{4.1}$$

$$y_2 = \bar{r}^2\bar{y}_2, \tag{4.2}$$

$$w_1 = w_1, \tag{4.3}$$

$$w_2 = w_2, \tag{4.4}$$

$$\xi = \xi, \tag{4.5}$$

$$\varepsilon = \bar{r}\bar{\varepsilon}. \tag{4.6}$$

Under this transformation the system (3.13)–(3.18) becomes one for which the five-dimensional set $\bar{r} = 0$, which is the product of S^2 with $w_1 w_2 \xi$-space, consists entirely of equilibria. The system we shall study is this one divided by $\bar{r}$. Division by $\bar{r}$ desingularizes the system on the set $\bar{r} = 0$ but leaves it invariant.

We shall need two charts on $S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$.

4.1. Chart for $\bar{\varepsilon} > 0$

Chart 1 uses the coordinates $u_1 = \frac{\bar{y}_1}{\bar{\varepsilon}}$, $u_2 = \frac{\bar{y}_2}{\bar{\varepsilon}^2}$ and $(w_1, w_2, \xi, \varepsilon)$ on the set of points in $S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$ with $\bar{\varepsilon} > 0$. Thus we have

$$y_1 = \varepsilon u_1, \tag{4.7}$$

$$y_2 = \varepsilon^2 u_2, \tag{4.8}$$

$$w_1 = w_1, \tag{4.9}$$

$$w_2 = w_2, \tag{4.10}$$

$$\xi = \xi, \tag{4.11}$$

$$\varepsilon = \varepsilon. \tag{4.12}$$

After division by ε (equivalent to division by $\bar{r}$ up to multiplication by a positive function), the system (3.13)–(3.18) becomes the system (3.6)–(3.11). This is not surprising; compare (4.7)–(4.8) and (3.12). Thus, in our approach to singular shocks the system (3.6)–(3.11) is a blow-up of the system (3.13)–(3.18) in one-coordinate patch. Also, note that division by ε is equivalent to changing the time coordinate from ζ back to τ.

4.2. Chart for $\bar{y}_2 > 0$

Chart 2 uses the coordinates $a = \frac{\bar{y}_1}{\sqrt{\bar{y}_2}}$, $r = \bar{r}\sqrt{\bar{y}_2}$, $b = \frac{\bar{\varepsilon}}{\sqrt{\bar{y}_2}}$ and (w_1, w_2, ξ) on the set of points in $S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$ with $\bar{y}_2 > 0$). Thus, we have

$$y_1 = ra, \tag{4.13}$$

$$y_2 = r^2, \tag{4.14}$$

$$w_1 = w_1, \tag{4.15}$$

$$w_2 = w_2, \tag{4.16}$$

$$\xi = \xi, \tag{4.17}$$

$$\varepsilon = rb. \tag{4.18}$$

It is the use of this chart that enables the geometric matching of the two parts of the solution (u and y, or outer and inner). It is the key advantage of the blowing-up approach to singular shocks.

We divide by r (equivalent to division by $\bar{r}$ up to multiplication by a positive function), and, by a small abuse of notation, as in chart 1 we use τ to denote the rescaled time variable and a dot to represent derivative with respect to τ. The system (3.13)–(3.18) becomes

$$\dot{a} = a^2 - 1 - \frac{1}{6}a^4 + \frac{1}{2}b\left(-\xi a - 2bw_1 + ba^2 + b^2 a w_2\right), \tag{4.19}$$

$$\dot{r} = \tfrac{1}{6}r\left(a^3 - 3b\xi - 3b^2 a - 3b^3 w_2\right), \tag{4.20}$$

$$\dot{w}_1 = -rab, \tag{4.21}$$

$$\dot{w}_2 = -r, \tag{4.22}$$

$$\dot{\xi} = rb^2, \tag{4.23}$$

$$\dot{b} = -\tfrac{1}{6}b\left(a^3 - 3b\xi - 3b^2 a - 3b^3 w_2\right). \tag{4.24}$$

If we set $b = 0$ in (4.19), we find that $\dot{a} = 0$ at the four points

$$a_1 = -\sqrt{3+\sqrt{3}} < a_2 = -\sqrt{3-\sqrt{3}} < a_3 = \sqrt{3-\sqrt{3}} < a_4 = \sqrt{3+\sqrt{3}}.$$

For $j = 1, \ldots, 4$, let

$$P_j = \{(a, r, w, \xi, b) : a = a_j, r = 0, b = 0\}.$$

Each P_j is a three-dimensional manifold of equilibria of (4.19)–(4.24). These are “corner equilibria": They lie in the intersection of the invariant sets $r = 0$, corresponding to $S^2 \times \{0\} \times \mathbb{R}^3$, and $b = 0$, corresponding to the “plane" $\bar{\varepsilon} = 0$ in $S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$. See Fig. 4.

At the equilibrium $(a, 0, w_1, w_2, \xi, 0)$, there is an eigenvalue 0, with the three-dimensional eigenspace $\dot{a} = \dot{r} = \dot{b} = 0$; an eigenvalue $\frac{2}{3}a(3 - a^2)$ with eigenvector $(1, 0, 0, 0, 0, 0)$; an eigenvalue $\frac{1}{6}a^3$ with eigenvector $(0, \frac{1}{6}a^3, 0, -1, 0, 0)$; and an eigenvalue $-\frac{1}{6}a^3$ with eigenvector $(\frac{2\xi}{4-a^2}, 0, 0, 0, 0, 1)$. Thus the manifolds P_j are normally hyperbolic.

The manifolds P_3 and P_2 will be most important to us.

Each point $(a_3, 0, w_{01}, w_{02}, \xi_0, 0)$ of P_3 has:

- A one-dimensional stable manifold tangent to $(\frac{2\xi}{4-a_3^2}, 0, 0, 0, 0, 1)$. This curve is contained in the two-dimensional invariant plane $\{(a, r, w_1, w_2, \xi, b) : r = 0, w_1 = w_{01}, w_2 = w_{02}, \xi = \xi_0\}$. The union of these curves is $W^s(P_3)$, a four-dimensional manifold contained in the five-dimensional plane $r = 0$.

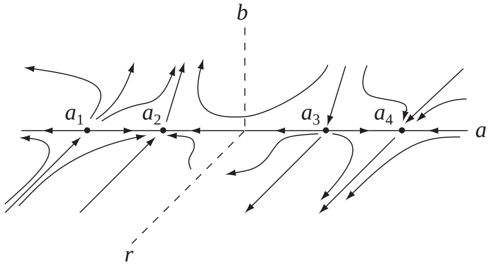

Fig. 4. Phase portrait of (4.19)–(4.24), with w_1, w_2 and ξ coordinates suppressed. For $r=0$ and fixed (w_1, w_2, ξ) we have $\dot{r} = \dot{w}_1 = \dot{w}_2 = \dot{\xi} = 0$; the phase portrait in this two-dimensional space is as shown. For $b=0$ we have $\dot{b} = \dot{w}_1 = \dot{\xi} = 0$ but $\dot{w}_2 \neq 0$ for $r \neq 0$. Thus along the solutions shown in the space $b=0$ with $r>0$, w_2 decreases.

- A two-dimensional unstable manifold tangent to the plane spanned by $(1,0,0,0,0,0)$ and $(0, \frac{1}{6}a_3^3, 0, -1, 0, 0)$. This surface is contained in the three-dimensional invariant plane $\{(a, r, w_1, w_2, \xi, b) : w_1 = w_{01}, \xi = \xi_0, b = 0\}$. The union of these surfaces is $W^{\mathrm{u}}(P_3)$, which is the five-dimensional space $b=0$.
  Each point $(a_2, 0, w_{01}, w_{02}, \xi_0, 0)$ of P_2 has:
- A one-dimensional unstable manifold tangent to $(\frac{2\xi}{4-a_2^2}, 0, 0, 0, 0, 1)$. This curve is contained in the two-dimensional invariant plane $\{(a, r, w_1, w_2, \xi, b) : r = 0, w_1 = w_{01}, w_2 = w_{02}, \xi = \xi_0\}$. The union of these curves is $W^{\mathrm{u}}(P_2)$, a four-dimensional manifold contained in the five-dimensional plane $r=0$.
- A two-dimensional stable manifold tangent to the plane spanned by $(1,0,0,0,0,0)$ and $(0, \frac{1}{6}a_2^3, 0, -1, 0, 0)$. This surface is contained in the three-dimensional invariant plane $\{(a, r, w_1, w_2, \xi, b) : w_1 = w_{01}, \xi = \xi_0, b = 0\}$. The union of these surfaces is $W^2(P_2)$, which is the five-dimensional space $b=0$.

5. Corner Lemma

In blown-up geometric singular perturbation problems, at manifolds of normally hyperbolic corner equilibria such as the P_j of the previous section, the following problem arises: Given a normally hyperbolic manifold P of equilibria and a manifold N that is transverse to $W^{\mathrm{s}}(P)$, track the flow of N past P. At corner equilibria the differential equation cannot be regarded as a parameterized family, so the Exchange Lemma [9,8] is not relevant. The following lemma plays the role of the Exchange Lemma for such points. Like the Exchange Lemma, it is a consequence of a result of Deng [3] about solutions of Silnikov problems near nonhyperbolic points.

(The Exchange Lemma was originally proved using differential forms [9]. The fact that it is a consequence of Deng's result is observed in [14, p. 58]. The paper [1] proves a result similar to Deng's and then gives the argument by which it implies the Exchange Lemma.)

The notation of this section is independent of that of the remainder of the paper.

Consider a differential equation $\dot{w} = f(w)$ on a neighborhood of 0 in $\mathbb{R}^p$ that is C^{r+4} , $r \geq 1$, and:

(1) The origin is an equilibrium.
(2) There are integers $k \geq 0$, $\ell \geq 0$, $m \geq 1$, and $n \geq 1$ such that $Df(0)$ has $k+\ell$ eigenvalues equal to 0, m eigenvalues with negative real part, and n eigenvalues with positive real part, with $k+\ell+m+n=p$.
(3) A codimension one subspace S of $\mathbb{R}^p$ is invariant.
(4) The restriction of $Df(0)$ to S has $k+\ell$ eigenvalues equal to 0, m eigenvalues with negative real part, and $n-1$ eigenvalues with positive real part.
(5) The origin is part of a $k+\ell$-dimensional manifold of equilibria P.

P is a normally hyperbolic manifold of equilibria. Each point of P has a stable manifold of dimension m and an unstable manifold of dimension n. The union of the stable manifolds of points of P is $W^{\mathrm{s}}(P)$, which has dimension $k+\ell+m$; the union of the unstable manifolds of points of P is $W^{\mathrm{u}}(P)$, which has dimension $k+\ell+n$. P and $W^{\mathrm{s}}(P)$ are necessarily contained in S.

Assumption (3) is probably not necessary. However, it holds in the applications we have in mind (in chart 2 of Section 4, S is the set $r=0$), and it simplifies the proof.

Let N be a C^{r+4} manifold of dimension $k+n$ that is transverse to $W^{\mathrm{s}}(P)$ at a point p in $W^{\mathrm{s}}(0)\setminus\{0\}$ and such that $T_pN \cap T_pW^{\mathrm{s}}(0)=\{0\}$. Then the intersection of N and $W^{\mathrm{s}}(P)$ is a manifold of dimension k that projects, along the fibration of $W^{\mathrm{s}}(P)$ by the stable manifolds of points, to a k-dimensional submanifold Q of P. Let y_n be a coordinate on $\mathbb{R}^p$ that vanishes on S, and, for $\delta>0$, let $N_\delta = N\cap\{y_n=\delta\}$, a manifold of dimension $k+n-1$. Let q be a point in $W^{\mathrm{u}}(Q)$ with $y_n(q)>0$. Notice that $W^{\mathrm{u}}(Q)$ has dimension $k+n$. Under the flow of $\dot{w}=f(w)$, N_δ becomes a manifold $\tilde{N}_\delta$ of dimension $k+n$ that passes near q. Let U be a small neighborhood of q.

Theorem 5.1 (Corner Lemma). *As $\delta \to 0$, $\tilde{N}_\delta \cap U \to W^{\mathrm{u}}(Q)\cap U$ in the C^r topology.*

To prove the Corner Lemma, we define coordinates (u,v,x,y) on a neighborhood of 0 in $\mathbb{R}^p$ with $u\in\mathbb{R}^k$, $v\in\mathbb{R}^\ell$, $x\in\mathbb{R}^m$, $y\in\mathbb{R}^n$. The coordinate y_n has already been chosen, and $(u,v,x,y_1,\ldots,y_{n-1})$ are Fenichel coordinates on S. More precisely, and ignoring the fact that we are working locally near the origin, Q is u-space; P is uv-space; $W^{\mathrm{s}}(P)$ is uvx-space; $W^{\mathrm{u}}(P)$ is uvy-space. Moreover, $W^{\mathrm{s}}(u^0,v^0,0,0)=\{(u,v,x,y): u=u^0, v=v^0, y=0\}$, and $W^{\mathrm{u}}(u^0,v^0,0,0)=\{(u,v,x,y): u=u^0, v=v^0, x=0\}$. See Fig. 5. Therefore

$$\dot{u}_i = x^\top A_i y, \quad i=1,\ldots,k, \tag{5.1}$$

$$\dot{v}_i = x^\top B_i y, \quad i=1,\ldots,\ell, \tag{5.2}$$

$$\dot{x} = Cx, \tag{5.3}$$

$$\dot{y} = Dy, \tag{5.4}$$

where A_i and B_i are $m\times n$ matrices, C is $m\times m$ and D is $n\times n$. The entries of these matrices are functions of (u,v,x,y). The eigenvalues of C have negative real part, and

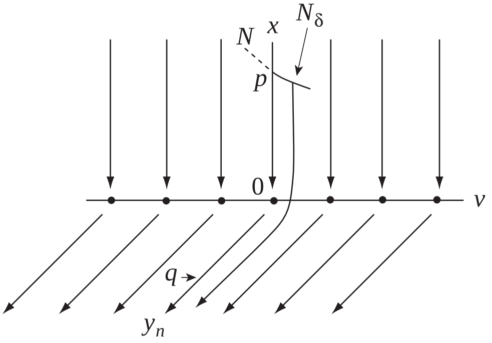

Fig. 5. Phase portrait of (5.1)–(5.4) with $k = 0$ and $\ell = m = n = 1$. Thus $Q = \{0\}$, N is one-dimensional and N_δ is a point. In this simple situation, the Corner Lemma just says that the solution through this point passes near q and is C^r-close to the one-dimensional unstable manifold of the origin near q.

those of D have positive real part. The coordinate change can be chosen to be C^{r+2} [3], so the system (5.1)–(5.4) is C^{r+2}, and the manifold N is now C^{r+2}.

Denote the entries of D by $d_{i,j}$. Since the space $y_n = 0$ is invariant, we may assume that $d_{n,1} = \cdots = d_{n,n-1} = 0$, so that $\dot{y}_n = d_{n,n} y_n$ and $d_{n,n}$ is a function of (u, v, x, y) with $d_{n,n} > 0$. After division by $d_{n,n}$ we may assume that $d_{n,n} = 1$. Since $d_{n,n}$ is C^{r+1}, the system (5.1)–(5.4) is now C^{r+1}, but N is still C^{r+2}.

Let $\tau > 0$. The solution of (5.1)–(5.4) on the interval $0 \le t \le \tau$ with boundary conditions

$$\begin{aligned} u(\tau) &= u^1, \\ v(0) &= v^0, \\ x(0) &= x^0, \\ y(\tau) &= y^1 \end{aligned}$$

is $(u, v, x, y)(t, \tau, u^1, v^0, x^0, y^1)$, $0 \le t \le \tau$. From [3], (u, v, x, y) is a C^r function of $(t, \tau, u^1, v^0, x^0, y^1)$; moreover, there exist $\rho > 0$, $\lambda < 0 < \mu$ and $K > 0$ such that for $\max(|u^1|, |v^0|, |x^0|, |y^1|) \le \rho$ and for any multi-index i with $|i| \le r$,

$$\|D^i x\| \le K e^{\lambda t}, \tag{5.5}$$

$$\|D^i y\| \le K e^{\mu(t-\tau)}, \tag{5.6}$$

$$\|D^i (u - u^1)\| \le K e^{\lambda t + \mu(t-\tau)}, \tag{5.7}$$

$$\|D^i (v - v^0)\| \le K e^{\lambda t + \mu(t-\tau)}. \tag{5.8}$$

Here, D^i represents repeated differentiation $|i|$ times with respect to any sequence of the variables $(t, \tau, u^1, v^0, x^0, y^1)$.

In the remainder of the proof we shall assume for simplicity that $m = 1$. Then N meets $W^s(0)$ at $p = (u, v, x, y) = (0, 0, x^0, 0)$ with x^0 a nonzero real number. We may assume that $0 < |x^0| \leq \rho$, and we fix x^0 for the remainder of the proof. We may assume that N is the set $\{(u, v, x, y) : x = x^0$ and $v = h(u, y)\}$ with h a C^{r+2} function and $h(u, 0) = 0$. Therefore there is an $\ell \times n$ matrix H, whose entries are C^{r+1} functions of (u, y), such that $h(u, y) = H(u, y)y$.

(If $m > 1$, the function h must also give $m - 1$ components of x as functions of (u, y).)

Let

$$A = \{(u^1, v^1, x^1, y^1) : |u^1| \leq \frac{\rho}{2}, \max(|v^1|, |x^1|, |y^1|) \leq \rho, \text{ and } \frac{\rho}{2} \leq y^1_n \leq \rho\},$$

$$B = \{(u^1, y^1) : |u^1| \leq \frac{\rho}{2}, |y^1| \leq \rho, \text{ and } \frac{\rho}{2} \leq y^1_n \leq \rho\},$$

$$C_{u^1} = \{(u^0, v^0) : \max(|u^0 - u^1|, |v^0|) \leq \frac{\rho}{2}\},$$

We may assume that $q \in A$ and $U \subset A$.

Given $(u^1, y^1) \in B$ and a small $\delta > 0$, let $\tau = \ln \frac{y^1_n}{\delta}$ and define $F_{(u^1,y^1,\delta)} : C_{u^1} \to \mathbb{R}^{k+\ell}$ by

$$F_{(u^1,y^1,\delta)}(u^0, v^0) = (u(0, \tau, u^1, v^0, x^0, y^1), h(u^0, y(0, \tau, u^1, v^0, x^0, y^1))).$$

Lemma 5.2. *For $\delta > 0$ sufficiently small independent of $(u^1, y^1) \in B$, $F_{(u^1,y^1,\delta)}$ is a contraction of C_{u^1}. Moreover, there is a constant M independent of $(u^1, y^1) \in B$ such that for all $(u^0, v^0) \in C_{u^1}$, $\|DF_{(u^1,y^1,\delta)}(u^0, v^0)\| \leq M\left(\frac{\rho}{2\delta}\right)^{-\mu}$.*

Proof. In this proof only, to simplify the notation, let $F = F_{(u^1,y^1,\delta)}$ with (u^1, y^1, δ) fixed, $(u^1, y^1) \in B$. By (5.7),

$$|F_1(u^0, v^0) - u^1| \leq Ke^{-\mu\tau} \leq K\left(\frac{y^1_n}{\delta}\right)^{-\mu} \leq K\left(\frac{\rho}{2\delta}\right)^{-\mu}. \tag{5.9}$$

Also, by (5.6), $|y(0, \tau, u^1, v^0, x^0, y^1)| \leq Ke^{-\mu\tau} \leq K\left(\frac{\rho}{2\delta}\right)^{-\mu}$. For δ sufficiently small, this is less than ρ.

Let $L = \max(\|h\|, \|Dh\|, \|H\|, \|DH\|)$ on $\{(u, y) : \max(|u|, |y|) \leq \rho\}$. Then, using $h = Hy$, we see that

$$|F_2(u^0, v^0)| \leq LKe^{-\mu\tau} \leq LK\left(\frac{\rho}{2\delta}\right)^{-\mu}. \tag{5.10}$$

It follows from (5.9)–(5.10) that for δ sufficiently small independent of $(u^1, y^1) \in B$, F maps C_{u^1} into itself.

To estimate $\|DF_{(u^1,y^1,\delta)}(u^0,v^0)\|$, we consider the partial derivatives of F. We have $\frac{\partial F_1}{\partial u^0}=0$, and, using (5.8),

$$\|\frac{\partial F_1}{\partial v^0}(u^0,v^0)\| = \|\frac{\partial u}{\partial v^0}(0,\tau,u^1,v^0,x^0,y^1)\| \leq Ke^{-\mu\tau} \leq K\left(\frac{\rho}{2\delta}\right)^{-\mu}.$$

Also,

$$\begin{aligned}\frac{\partial F_2}{\partial u^0}(u^0,v^0) &= \frac{\partial h}{\partial u}(u^0,y(0,\tau,u^1,v^0,x^0,y^1)) = \frac{\partial (Hy)}{\partial u}(u^0,y(0,\tau,u^1,v^0,x^0,y^1))\\ &= \frac{\partial H}{\partial u}(u^0,y(0,\tau,u^1,v^0,x^0,y^1))y(0,\tau,u^1,v^0,x^0,y^1),\end{aligned}$$

so by (5.6), $\|\frac{\partial F_2}{\partial u^0}(u^0,v^0)\| \leq LKe^{-\mu\tau} \leq LK\left(\frac{\rho}{2\delta}\right)^{-\mu}$. Finally,

$$\frac{\partial F_2}{\partial v^0}(u^0,v^0) = \frac{\partial h}{\partial y}(u^0,y(0,\tau,u^1,v^0,x^0,y^1))\frac{\partial y}{\partial v^0}(0,\tau,u^1,v^0,x^0,y^1),$$

so by (5.6), $\|\frac{\partial F_2}{\partial u^0}(u^0,v^0)\| \leq LKe^{-\mu\tau} \leq LK\left(\frac{\rho}{2\delta}\right)^{-\mu}$. From these estimates, the estimate on $\|DF_{(u^1,y^1,\delta)}(u^0,v^0)\|$ follows, and hence the fact that $F_{(u^1,y^1,\delta)}$ is a contraction of C_{u^1} for $\delta>0$ sufficiently small independent of (u^1,y^1). □

Lemma 5.3. *The fixed point* (u^0,v^0) *of* $F_{(u^1,y^1,\delta)}$ *satisfies the following estimates: There is a constant* M *such that* $|u^0-u^1|$, $|v^0|$, $\|\frac{\partial u^0}{\partial u^1}-I\|$, $\|\frac{\partial u^0}{\partial y^1}\|$, $\|\frac{\partial v^0}{\partial u^1}\|$, *and* $\|\frac{\partial v^0}{\partial y^1}\|$ *are bounded by* $M\left(\frac{\rho}{2\delta}\right)^{-\mu}$ *independent of* $(u^1,y^1)\in B$.

Proof. The estimates on $|u^0-u^1|$ and $|v^0|$ follow from setting (u^0,v^0) equal to the fixed point in (5.9) and (5.10).

To estimate the derivatives, let $z=(u^0,v^0)$, $\rho=(u^1,y^1)$, and let

$$F_\delta(z,\rho)=F_\delta(u^0,v^0,u^1,y^1)=F_{(u^1,y^1,\delta)}(u^0,v^0).$$

The fixed point $z(\rho)$ of $F_\delta(z,\rho)$ satisfies $z(\rho)=F_\delta(z(\rho),\rho)$, so

$$\frac{dz}{d\rho}=\left(I-\frac{\partial F_\delta}{\partial z}(z(\rho),\rho)\right)^{-1}\frac{\partial F_\delta}{\partial \rho}(z(\rho),\rho). \tag{5.11}$$

By Lemma 5.2, $\|\frac{\partial F_\delta}{\partial z}(z(\rho),\rho)\| \le M\left(\frac{\rho}{2\delta}\right)^{-\mu}$, so $\left(I - \frac{\partial F_\delta}{\partial z}(z(\rho),\rho)\right)^{-1} = I + P$ with $\|P\| \le M\left(\frac{\rho}{2\delta}\right)^{-\mu}$ for a possibly larger M. Therefore we can rewrite (5.11) as

$$\begin{pmatrix} \frac{\partial u^0}{\partial u^1} & \frac{\partial u^0}{\partial y^1} \\ \frac{\partial v^0}{\partial u^1} & \frac{\partial v^0}{\partial y^1} \end{pmatrix} = (I+P)\begin{pmatrix} \frac{\partial F_1}{\partial u^1} & \frac{\partial F_1}{\partial y^1} \\ \frac{\partial F_2}{\partial u^1} & \frac{\partial F_2}{\partial y^1} \end{pmatrix}.$$

Calculating as in the proof of Lemma 5.2, we find

$$\begin{aligned} \|\frac{\partial F_1}{\partial u^1} - I\| &\le Ke^{-\mu\tau} \le K\left(\frac{\rho}{2\delta}\right)^{-\mu}, \\ \|\frac{\partial F_1}{\partial y^1}\| &\le Ke^{-\mu\tau} \le K\left(\frac{\rho}{2\delta}\right)^{-\mu}, \\ \|\frac{\partial F_2}{\partial u^1}\| &\le MKe^{-\mu\tau} \le MK\left(\frac{\rho}{2\delta}\right)^{-\mu}, \\ \|\frac{\partial F_2}{\partial y^1}\| &\le MKe^{-\mu\tau} \le MK\left(\frac{\rho}{2\delta}\right)^{-\mu}. \end{aligned}$$

The estimates on the derivatives follow easily, again for a possibly larger M. $\square$

As in Lemma 5.3, let the fixed point be of $F_{(u^1,y^1,\delta)}$ be (u^0, v^0), and let $y^0 = y(0,\tau,u^1,v^0,x^0,y^1)$. Then $v^0 = h(u^0,y^0)$, so $(u^0,v^0,x^0,y^0) \in N$.

Define $g^\delta : B \to \mathbb{R}^{\ell+1}$ by

$$g^\delta(u^1,y^1) = (v,x)(\tau,\tau,u^1,v^0,x^0,y^1) = (v^1,x^1).$$

Then $(u^1,v^1,x^1,y^1) \in A$. Moreover, if we denote the time τ map of $\dot{w} = f(w)$ by ϕ_τ, then $(u^1,v^1,x^1,y^1) = \phi_\tau(u^0,v^0,x^0,y^0)$. Since $\dot{y}_n = y_n$, we have $y_n^1 = e^\tau y_n^0 = \frac{y_n^1}{\delta} y_n^0$, so $y_n^0 = \delta$. Therefore $(u^0,v^0,x^0,y^0) \in N_\delta$ and $(u^1,v^1,x^1,y^1) \in \tilde{N}_\delta$. Therefore, $\tilde{N}_\delta \cap U$ is part of the graph of g^δ. To complete the proof of the Corner Lemma, we need only to show that as $\delta \to 0$, $g^\delta \to 0$ in the C^r-topology.

We consider only g_1^δ. By (5.8) and Lemma 5.3,

$$|g_1^\delta(u^1,y^1)| = |v(\tau,\tau,u^1,v^0,x^0,y^1)| \le |v^0| + Ke^{\lambda\tau} \le M\left(\frac{\rho}{2\delta}\right)^{-\mu} + K\left(\frac{\rho}{2\delta}\right)^{\lambda}.$$

Therefore, g^δ approaches 0 uniformly in (u^1,v^1) as $\delta \to 0$.

Also, by (5.7) and Lemma 5.3,

$$\begin{aligned} \|\frac{\partial g_1^\delta}{\partial u^1}(u^1,y^1)\| &= \|\frac{\partial v}{\partial u^1}(\tau,\tau,u^1,v^0,x^0,y^1) + \frac{\partial v}{\partial v^0}(\tau,\tau,u^1,v^0,x^0,y^1)\frac{\partial v^0}{\partial u^1}(u^1,y^1)\| \\ &\le Ke^{\lambda\tau} + Ke^{\lambda\tau}M\left(\frac{\rho}{2\delta}\right)^{-\mu} \le K\left(\frac{\rho}{2\delta}\right)^{\lambda} + KM\left(\frac{\rho}{2\delta}\right)^{\lambda-\mu}. \end{aligned}$$

Similar estimates hold for $\frac{\partial g_1^\delta}{\partial y^1}$, except that additional terms occur in the partial derivative with respect to y_n^1 because of the dependence of τ on y_n^1. Indeed, in calculating $\frac{\partial g_1^\delta}{\partial y_n^1}$, we must include the terms

$$\frac{\partial v}{\partial t}(\tau, \tau, u^1, v^0, x^0, y^1)\frac{\partial \tau}{\partial y_n^1}(u^1, y^1) + \frac{\partial v}{\partial \tau}(\tau, \tau, u^1, v^0, x^0, y^1)\frac{\partial \tau}{\partial y_n^1}(u^1, y^1).$$

The size of each of these terms is bounded by $Ke^{\lambda\tau}\frac{1}{y_n^1} \leq K\left(\frac{\rho}{2\delta}\right)^\lambda\left(\frac{2}{\rho}\right)$.

Similar estimates hold through order r. This completes the proof of the Corner Lemma.

6. Proof of main result

We return to using the notation of Sections 1–4.

Theorem 6.1. *In the Keyfitz–Kranzer system of conservation laws* (1.1)–(1.2), *let* u_L *and* u_R *be points of* $\mathbb{R}^2$ *with* $u_{L1} \neq u_{R1}$. *Let*

$$\xi_0 = \frac{f_1(u_L) - f_1(u_R)}{u_{L1} - u_{R1}}, \quad \gamma_0 = f_2(u_L) - f_2(u_R) - \xi_0(u_{L2} - u_{R2}). \tag{6.1}$$

Assume

(1) $\xi_0 < \lambda_i(u_L)$ *for* $i = 1, 2$.
(2) $\lambda_i(u_R) < \xi_0$ *for* $i = 1, 2$.
(3) $\gamma_0 > 0$.

Then there is a singular shock with Dafermos profile from u_L *to* u_R. *In other words, for small* $\varepsilon > 0$ *there is a solution* $u_\varepsilon(\xi)$ *of the boundary-value problem* (2.4)–(2.7), *and, as* $\varepsilon \to 0$, $u_\varepsilon(\xi)$ *becomes unbounded.*

Let us make several remarks about this theorem.

1. For $\xi < \xi_0$, $\lim_{\varepsilon\to 0} u_\varepsilon(\xi) = u_L$, and for $\xi > \xi_0$, $\lim_{\varepsilon\to 0} u_\varepsilon(\xi) = u_R$. The limiting function

$$u_0(\xi) = \begin{cases} u_L & \text{for } \xi < \xi_0, \\ u_R & \text{for } \xi > \xi_0, \end{cases}$$

can be regarded as a shock wave with speed ξ_0. Assumptions (1) and (2) say that this shock wave is overcompressive.

2. From (6.1), the first Rankine–Hugoniot condition for the shock wave $u_0(\xi)$,

$$f_1(u_L) - f_1(u_R) - \xi_0(u_{L1} - u_{R1}) = 0,$$

is satisfied. The second, however, is not: From (6.1) and assumption (3),

$$f_2(u_L) - f_2(u_R) - \xi_0(u_{L2} - u_{R2}) = \gamma_0 > 0.$$

The number γ_0 is called the "Rankine–Hugoniot deficit" in [22].

3. For fixed u_L, the set of u_R for which assumptions (1)–(3) hold is an unbounded open set. For a precise description see [10] or [13].

4. Sever [22] observed that the system of conservation laws (1.1)–(1.2) has the convex entropy $\rho = e^{\frac{1}{2}u_1^2 - u_2}$, with entropy flux ρu. However, we shall make no use of this fact.

To prove the theorem, we shall work with the system (3.13)–(3.18) in $yw\xi\varepsilon$-space. As explained at the start of Section 4, we seek solutions in the intersection of $W^{\mathrm{u}}(M_0^\varepsilon(u_L))$ and $W^{\mathrm{s}}(M_2^\varepsilon(u_R))$, $\varepsilon > 0$. In fact, we shall work in the blowup of $yw\xi\varepsilon$-space that was defined in Section 4.

We shall first describe the subset of $S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$ near which the solutions we seek are to lie. The description uses the two charts of Section 4.

In chart 1, the lines $M_0^\varepsilon(u_L)$ of Section 4 correspond to lines $T_0^\varepsilon(u_L)$ described in Section 2. We have

$$W^{\mathrm{u}}(T_0^0(u_L)) = \{(u, w, \xi, \varepsilon) : u \in U_\xi, \xi < \lambda_1(u_L), w = f(u_L) - \xi u_L, \varepsilon = 0\},$$

where U_ξ is an open subset of u-space that depends on ξ (and u_L). Therefore, $W^{\mathrm{u}}(T_0^0(u_L))$ is three-dimensional.

In chart 2, the lines $M_0^\varepsilon(u_L)$ correspond to lines

$$N_0^\varepsilon(u_L) = \{(a, r, w, \xi, b) : a = \frac{u_{L1}}{\sqrt{u_{L2}}}, r = \varepsilon\sqrt{u_{L2}}, w = f(u_L) - \xi u_L, \xi < \lambda_1(u_L),$$
$$b = \frac{1}{\sqrt{u_{L2}}}\}.$$

We have

$$W^{\mathrm{u}}(N_0^0(u_L)) = \{(a, r, w, \xi, b) : (a, b) \in V_\xi, r = 0, w = f(u_L) - \xi u_L, \xi < \lambda_1(u_L)\},$$

where V_ξ is an open subset of ab-space that depends on ξ (and u_L). Therefore, $W^{\mathrm{u}}(N_0^0(u_L))$ is three-dimensional.

In chart 2, let

$$C_3 = \{(a, r, w, \xi, b) : a = a_3, r = 0, w = f(u_L) - \xi u_L, \xi < \lambda_1(u_L), b = 0\},$$

a line of equilibria in the three-dimensional space of equilibria P_3. $W^{\mathrm{s}}(C_3)$ is a two-dimensional surface in the five-dimensional space $r = 0$, the union of the stable manifolds of the points of C_3.

We claim that the intersection of $W^{\mathrm{u}}(N_0^0(u_L))$ and $W^{\mathrm{s}}(P_3)$ is an open subset Q_3 of $W^{\mathrm{s}}(C_3)$, namely the points of $W^{\mathrm{s}}(C_3)$ with $b>0$. To see this, let $\bar{q}=(a_3,0,\bar{w},\bar{\xi},0)$ be a point of C_3, so $\bar{\xi}<\lambda_1(u_L)$ and $\bar{w}=f(u_L)-\bar{\xi}u_L$. In chart 2, the stable manifold of $\bar{q}$ is a solution of (4.19)–(4.24) of the form $(a(\tau),0,\bar{w},\bar{\xi},b(\tau))$ in the two-dimensional invariant plane $\{(a,r,w,\xi,b): r=0, w=\bar{w}, \xi=\bar{\xi}\}$, a copy of ab-space. In chart 1, this solution corresponds to a solution $(u(\tau),\bar{w},\bar{\xi},0)$ of (3.6)–(3.11) in the two-dimensional invariant plane $\{(u,w,\xi,\varepsilon): w=\bar{w}, \xi=\bar{\xi}, \varepsilon=0\}$, a copy of u-space. In [17], Section 3.3, it is shown that in backward time this solution approaches the equilibrium u_L, which is a repeller because $\bar{\xi}<\lambda_1(u_L)$. Therefore, in chart 1 it is contained in $W^{\mathrm{u}}(T_0^0(u_L))$; in chart 2 it is contained in $W^{\mathrm{u}}(N_0^0(u_L))$.

Similarly, in chart 1, the lines $M_2^{\varepsilon}(u_R)$ of Section 4 correspond to lines $T_2^{\varepsilon}(u_R)$ of Section 2. We have

$$W^{\mathrm{s}}(T_2^0(u_R))=\{(u,w,\xi,\varepsilon): u\in U_{\xi}, \lambda_2(u_R)<\xi, w=f(u_R)-\xi u_R, \varepsilon=0\},$$

where U_{ξ} is an open subset of u-space that depends on ξ (and u_R). Therefore, $W^{\mathrm{s}}(T_2^0(u_R))$ is three-dimensional.

In chart 2, the lines $M_2^{\varepsilon}(u_R)$ correspond to lines

$$N_2^{\varepsilon}(u_R)=\{(a,r,w,\xi,b): a=\frac{u_{R1}}{\sqrt{u_{R2}}}, r=\varepsilon\sqrt{u_{R2}}, w=f(u_R)-\xi u_R, \lambda_2(u_R)<\xi, b=\frac{1}{\sqrt{u_{R2}}}\}.$$

We have

$$W^{\mathrm{s}}(N_2^0(u_R))=\{(a,r,w,\xi,b):(a,b)\in V_{\xi}, r=0, w=f(u_R)-\xi u_R, \lambda_2(u_R)<\xi\},$$

where V_{ξ} is an open subset of ab-space that depends on ξ (and u_R). Therefore $W^{\mathrm{s}}(N_2^0(u_R))$ is three-dimensional.

In chart 2, let

$$C_2=\{(a,r,w,\xi,b): a=a_2, r=0, w=f(u_R)-\xi u_R, \lambda_2(u_R)<\xi, b=0\},$$

a curve of equilibria in the three-dimensional space of equilibria P_2. $W^{\mathrm{u}}(C_2)$ is two-dimensional, the union of the stable manifolds of its points. The intersection of $W^{\mathrm{u}}(P_2)$ and $W^{\mathrm{s}}(N_2^0(u_R)$ is an open subset Q_2 of $W^{\mathrm{u}}(C_2)$, namely the points of $W^{\mathrm{u}}(C_2)$ with $b>0$.

Let

$$w_L=f(u_L)-\xi_0 u_L, \quad w_R=f(u_R)-\xi_0 u_R.$$

From (6.1), $w_{R1} = w_{L1}$ and $w_{R2} = w_{L2} - \gamma_0$. Also, let

$$q_L = (a_3, 0, w_{L1}, w_{L2}, \xi_0, 0), \quad q_R = (a_2, 0, w_{R1}, w_{R2}, \xi_0, 0).$$

By assumption (1), $q_L \in C_3$, and by assumption (2), $q_R \in C_2$.

Since $\gamma_0 > 0$ by assumption (3), Proposition 3.1 yields a unique solution

$$(y_1(\zeta), y_2(\zeta), w_{L1}, w_{L2} - \int_{-\infty}^{\zeta} y_2(\eta)\, d\eta, \xi_0, 0) \tag{6.2}$$

of (3.19)–(3.24) that goes from $(0, 0, w_{L1}, w_{L2}, \xi_0, 0)$ to $(0, 0, w_{R1}, w_{R2}, \xi_0, 0)$ and has $y_1(0) = 0$.

In chart 2, (6.2) corresponds to a solution

$$q(\tau) = (a(\tau), r(\tau), w_{L1}, w_{L2} - \int_{-\infty}^{\tau} r(\sigma)\, d\sigma, \xi_0, 0). \tag{6.3}$$

As $\tau \to \pm\infty$, $r(\tau) \to 0$. Also, recall that as $\zeta \to \pm\infty$,

$$\frac{y_2(\zeta)}{y_1(\zeta)^2} \to c_+.$$

Therefore

$$\lim_{\tau \to -\infty} a(\tau) = \lim_{\zeta \to -\infty} \frac{y_1(\zeta)}{\sqrt{y_2(\zeta)}} = \frac{1}{\sqrt{c_+}} = a_3,$$

$$\lim_{\tau \to \infty} a(\tau) = \lim_{\zeta \to \infty} \frac{y_1(\zeta)}{\sqrt{y_2(\zeta)}} = -\frac{1}{\sqrt{c_+}} = a_2.$$

Hence, $q(\tau)$ approaches q_L as $\tau \to -\infty$ and q_R as $\tau \to \infty$. From the remark after Proposition 3.1 and (4.2)–(4.3), we see that $r(\tau)$ is an even function and $a(\tau)$ is odd.

In $S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$, we search for solutions near the union of the following five curves: (1) the portion of $N_0^0(u_L)$ with $\xi < \xi_0$; (2) the branch of the stable manifold of q_L in $b > 0$, (3) the solution (6.3) from q_L to q_R, (4) the branch of the unstable manifold of q_R in $b > 0$; (5) the portion of $N_2^0(u_R)$ with $\xi > \xi_0$. As we have seen, curve (2) is in $W^{\mathrm{u}}(N_0^0(u_L))$, and curve (4) is in $W^{\mathrm{s}}(N_2^0(u_R))$.

The solutions we seek are to lie in the intersection of $W^{\mathrm{u}}(N_0^\varepsilon(u_L))$ and $W^{\mathrm{s}}(N_2^\varepsilon(u_R))$ for $\varepsilon > 0$. They correspond to solutions of (3.13)–(3.18) that lie in the intersection of $W^{\mathrm{u}}(M_0^\varepsilon(u_L))$ and $W^{\mathrm{s}}(M_2^\varepsilon(u_R))$.

Let $N_0(u_L)$ be the union of the $N_0^\varepsilon(u_L)$ with $0 \le \varepsilon \le \varepsilon_0$, a two-dimensional set. Its unstable manifold $W^{\mathrm{u}}(N_0(u_L))$ is the union of the $W^{\mathrm{u}}(N_0^\varepsilon(u_L))$ and is four-dimensional. We have $W^{\mathrm{u}}(N_0(u_L)) \cap W^{\mathrm{s}}(P_3) = Q_3$.

Similarly let $N_2(u_R)$ be the union of the $N_2^{\varepsilon}(u_R)$ with $0 \le \varepsilon \le \varepsilon_0$, a two-dimensional set. Its stable manifold $W^{\mathrm{s}}(N_2(u_R))$ is the union of the $W^{\mathrm{s}}(N_2^{\varepsilon}(u_R))$ and is four-dimensional. We have $W^{\mathrm{s}}(N_0(u_R)) \cap W^{\mathrm{u}}(P_2) = Q_2$.

Proposition 6.2. *$W^{\mathrm{u}}(N_0(u_L))$ is transverse to $W^{\mathrm{s}}(P_3)$ along Q_3. Similarly, $W^{\mathrm{s}}(N_2(u_R))$ is transverse to $W^{\mathrm{u}}(P_2)$ along Q_2.*

Proof. We prove only the first statement. At a point of Q_3, the tangent space to $W^{\mathrm{u}}(N_0(u_L))$ is spanned by $(1,0,0,0,0,0)$, $(0,0,-u_{L1},-u_{L2},1,0)$, $(0,0,0,0,0,1)$ (all tangent vectors to $W^{\mathrm{u}}(N_0^0(u_L))$), and a vector with nonzero r-component. Among the tangent vectors to $W^{\mathrm{s}}(P_3)$ at that point are $(*,0,1,0,0,*)$ and $(*,0,0,1,0,*)$, where the values of the starred entries are unimportant. These six vectors are linearly independent. □

Proposition 6.3. *Within the five-dimensional space $b = 0$, $W^{\mathrm{u}}(C_3)$ and $W^{\mathrm{s}}(C_2)$ meet transversally along $q(\tau)$.*

Proof. We work in the space $b = 0$, with coordinates (a, r, w_1, w_2, ξ). The differential equation is therefore (4.19)–(4.23) with $b = 0$. Let $g(a) = a^2 - 1 - \frac{1}{6}a^4$. The linearization along $q(\tau)$ is

$$\frac{d}{dt}\begin{pmatrix} \bar{a} \\ \bar{r} \\ \bar{w}_1 \\ \bar{w}_2 \\ \bar{\xi} \end{pmatrix} = \begin{pmatrix} g'(a(\tau)) & 0 & 0 & 0 & 0 \\ \frac{1}{2}a(\tau)^2 r(\tau) & \frac{1}{6}a(\tau)^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{r} \\ \bar{w}_1 \\ \bar{w}_2 \\ \bar{\xi} \end{pmatrix}. \tag{6.4}$$

The adjoint equation is therefore

$$\frac{d}{dt}\begin{pmatrix} \tilde{a} \\ \tilde{r} \\ \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} -g'(a(\tau)) & -\frac{1}{2}a(\tau)^2 r(\tau) & 0 & 0 & 0 \\ 0 & -\frac{1}{6}a(\tau)^3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{r} \\ \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{\xi} \end{pmatrix}. \tag{6.5}$$

$T_{q_L} W^{\mathrm{u}}(C_3)$ is spanned by the vectors $(1,0,0,0,0)$, $(0, \frac{1}{6}a_3^3, 0, -1, 0)$ and $(0, 0, -u_{L1}, -u_{L2}, 1)$. Since $T_{q(\tau)} W^{\mathrm{u}}(C_3)$ approaches $T_{q_L} W^{\mathrm{u}}(C_3)$ as $\tau \to -\infty$, the orthogonal complement of $T_{q(\tau)} W^{\mathrm{u}}(C_3)$ approaches the space spanned by $q_1 = (0,0,1,0,u_{L1})$ and $q_2 = (0, 1, 0, \frac{1}{6}a_3^3, \frac{1}{6}a_3^3 u_{L2})$ as $\tau \to -\infty$. As $\tau \to -\infty$, the unique solution of

(6.5) that approaches q_1 is the constant solution q_1; and the unique solution of (6.5) that approaches q_2 is

$$(\tilde{a}(\tau), \tilde{r}(\tau), 0, \frac{1}{6}a_3^3, \frac{1}{6}a_3^3 u_{L2})$$

where

$$\tilde{r}(\tau) = 1 - \int_{-\infty}^{\tau} e^{-\int_\sigma^\tau \frac{1}{6}a(\rho)^3\,d\rho} \frac{1}{6}(a(\sigma)^3 - a_3^3)\,d\sigma,$$

$$\tilde{a}(\tau) = -\int_{-\infty}^{\tau} e^{-\int_\sigma^\tau g'(a(\rho))\,d\rho} \frac{1}{2} r(\sigma)a(\sigma)^2\tilde{r}(\sigma)\,d\sigma.$$

Therefore these two solutions of (6.5) span the orthogonal complement of $T_{q(\tau)}W^{\mathrm{u}}(C_3)$.

Similarly, $T_{q_R}W^{\mathrm{s}}(C_2)$ is spanned by the vectors $(1, 0, 0, 0, 0)$, $(0, \frac{1}{6}a_2^3, 0, -1, 0)$ and $(0, 0, -u_{R1}, -u_{R2}, 1)$. Thus its orthogonal complement is spanned by $q_3 = (0, 0, 1, 0, u_{R1})$ and $q_4 = (0, 1, 0, \frac{1}{6}a_2^3, \frac{1}{6}a_2^3 u_{R2})$. As $\tau \to \infty$, the unique solution of (6.5) that approaches q_3 is the constant solution q_3. The unique solution of (6.5) that approaches q_4 as $\tau \to \infty$ is

$$(\hat{a}(\tau), \hat{r}(\tau), 0, \frac{1}{6}a_2^3, \frac{1}{6}a_2^3 u_{R2})$$

where

$$\hat{r}(\tau) = 1 + \int_{\tau}^{\infty} e^{-\int_\sigma^\tau \frac{1}{6}a(\rho)^3\,d\rho} \frac{1}{6}(a(\sigma)^3 - a_2^3)\,d\sigma,$$

$$\hat{a}(\tau) = \int_{\tau}^{\infty} e^{-\int_\sigma^\tau g'(a(\rho))\,d\rho} \frac{1}{2} r(\sigma)a(\sigma)^2\bar{r}(\sigma)\,d\sigma.$$

Therefore these two solutions of (6.5) span the orthogonal complement of $T_{q(\tau)}W^{\mathrm{s}}(C_2)$.

We wish to check that $T_{q(0)}W^{\mathrm{u}}(C_3)$ and $T_{q(0)}W^{\mathrm{s}}(C_2)$ are transverse. It suffices to check that the four vectors $(0, 0, 1, 0, u_{L1})$, $(\tilde{a}(0), \tilde{r}(0), 0, \frac{1}{6}a_3^3, \frac{1}{6}a_3^3 u_{L2})$, $(0, 0, 1, 0, u_{R1})$ and $(\hat{a}(0), \hat{r}(0), 0, \frac{1}{6}a_2^3, \frac{1}{6}a_2^3 u_{R2})$ that span their orthogonal complements are linearly independent. Using the last four components of these vectors and the fact that $a_2 = -a_3$, we have

$$\det \begin{pmatrix} 0 & 1 & 0 & u_{L1} \\ \tilde{r}(0) & 0 & \frac{1}{6}a_3^3 & \frac{1}{6}a_3^3 u_{L2} \\ 0 & 1 & 0 & u_{R1} \\ \hat{r}(0) & 0 & \frac{1}{6}a_2^3 & \frac{1}{6}a_2^3 u_{R2} \end{pmatrix} = -\frac{1}{6}(\tilde{r}(0) + \hat{r}(0))a_3^3(u_{R1} - u_{L1}).$$

Since $a(\tau)$ is an odd function and $a_2 = -a_3$, we see that

$$\tilde{r}(0) + \hat{r}(0) = 2 - \int_{-\infty}^{0} e^{-\int_\sigma^0 \frac{1}{6} a(\rho)^3 \, d\rho} \, \frac{1}{6}(a(\sigma)^3 - a_3^3) \, d\sigma + \int_{0}^{\infty} e^{-\int_\sigma^0 \frac{1}{6} a(\rho)^3 \, d\rho} \, \frac{1}{6}(a(\sigma)^3 - a_2^3) \, d\sigma = 2.$$

Also, $u_{R1} - u_{L1} \neq 0$ by assumption. Therefore, the determinant is nonzero. □

Proof of Theorem 6.1. Let $\varepsilon > 0$ be small and choose $T >> 0$. In chart 2, by Proposition 6.2 and the Corner Lemma, $W^{\mathrm{u}}(N_0^\varepsilon(u_L))$ passes q_L and arrives near $q(-T)$ C^1 close to $W^{\mathrm{u}}(C_3)$. (In using the Corner Lemma, take the origin at q_L, take N to be a codimension one slice of $W^{\mathrm{u}}(N_0(u_L))$ transverse to the vector field, take y_n to be r, and take Q to be C_3.) Similarly, $W^{\mathrm{s}}(N_0^\varepsilon(u_R))$ passes q_R (in backward time) and arrives near $q(T)$ C^1 close to $W^{\mathrm{s}}(C_2)$. Both $W^{\mathrm{u}}(N_0^\varepsilon(u_L))$ and $W^{\mathrm{s}}(N_0^\varepsilon(u_R))$ lie in the five-dimensional space $rb = \varepsilon$. With the aid of Proposition 6.3 we see that $W^{\mathrm{u}}(N_0^\varepsilon(u_L))$ and $W^{\mathrm{s}}(N_0^\varepsilon(u_R))$ meet transversally within that space. The result follows.

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