

## Oxidation heat pulses in two-phase expansive flow in porous media

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**Abstract.** When air or oxygen is injected into a petroleum reservoir, and low temperature oxidation or combustion is induced, traveling oxidation fronts are observed to form.

We construct a two-phase model for oxidation, involving air or oxygen and oil, that includes heat loss to the rock formation. We use Darcy's law and energy and mass balance equations. We take into account heat generation and the oil's ensuing viscosity reduction, volume and flow rate changes due to the reaction, capillary pressure, and heat diffusion. The main simplifications are (1) we neglect volume changes of gases due to pressure variations, since the latter are assumed to be small; (2) we assume that liquid water is absent; and (3) we assume that the same Arrhenius chemical reaction rate governs the oxidation of all hydrocarbons in the oil.

Our analysis shows: (1) For small heat loss, the oxidation front is actually the lead part of a traveling pulse, while the trailing part of the pulse is a slow cooling process. The overall pulse has a triangular saw-tooth shape and may be long. (2) If the heat loss is too small, the pulse may be so long that only its lead front fits in an actual reservoir of finite length, so just this part of the pulse is observed. On the other hand, if the cooling is faster, not all oxygen is burned, a fact with deleterious consequences when the unburned oxygen reaches the producing well. (3) If the heat loss is too large, the medium does not sustain oxidation pulses. This last result is proved under the further simplifying assumption of constant effective thermal conductivity.

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### 1. Introduction

Low temperature oxidation is a method of oil recovery that uses a chemical reaction to cause a temperature increase, thereby reducing oil viscosity and allowing the oil to flow more readily. It has been successfully used in various oil fields (for example, Total Oil's Horse Creek Project, North Dakota, U.S.A. [15]), and has been the subject of a number of papers in the petroleum engineering literature [2],[5],[9],[13],[14], [16],[19] and the mathematics literature [3],[4],[10],[11]. The

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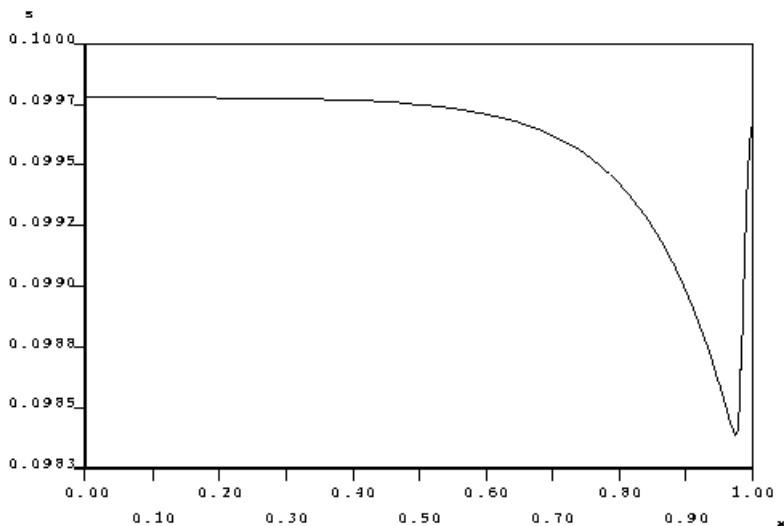
mathematical theory of low temperature oxidation is an aspect of the theory of combustion in multiphase flow in a porous medium. The latter subject is little developed, especially by comparison to the theory of combustion in gases.

In this work, we identify conditions under which there exists a traveling oxidation pulse in a model for one-dimensional two-phase flow in a porous medium. These conditions are strict inequalities. We show that such a pulse has a triangular shape. Its lead part is a steep front where the reaction takes place and the temperature quickly increases. Its trailing part has a mild slope; the main process in this part of the wave is cooling by heat loss to the overburden rock formation. A pulse with this shape was observed numerically in the work of Crookston *et al.*; see Fig. 1 in [8]. Along such a pulse, the temperature is higher, and so is the oil mobility. Thus the pressure drop required to move the fluid between the injection and production wells is decreased by the pulse.

We have simplified somewhat the equations used in petroleum engineering [8], [6]. We assume chemical reactions with a single reaction rate for the several hydrocarbons contained in the oil, instead of the multiple reaction rates that actually occur in low temperature oxidation. We also do not allow the presence of liquid water, which is reasonable only if the temperature is relatively high, or if there was no liquid water initially. Thus we treat a two-phase flow (oleic and gaseous), and avoid difficulties associated with the analysis of three-phase flow. These difficulties should be irrelevant to the key aspect of low temperature oxidation, the reduction of oil viscosity by temperature increase. In work on oil recovery by steam injection, for example, similar effects occur in both two-phase and three-phase models [4].

No other unrealistic simplifications are made in the model that we present.

Our analysis is motivated by the following physical situation. Initially, there is a uniform distribution of oil and air or oxygen in a porous rock. The fluids

(a)  $x$  vs.  $s$

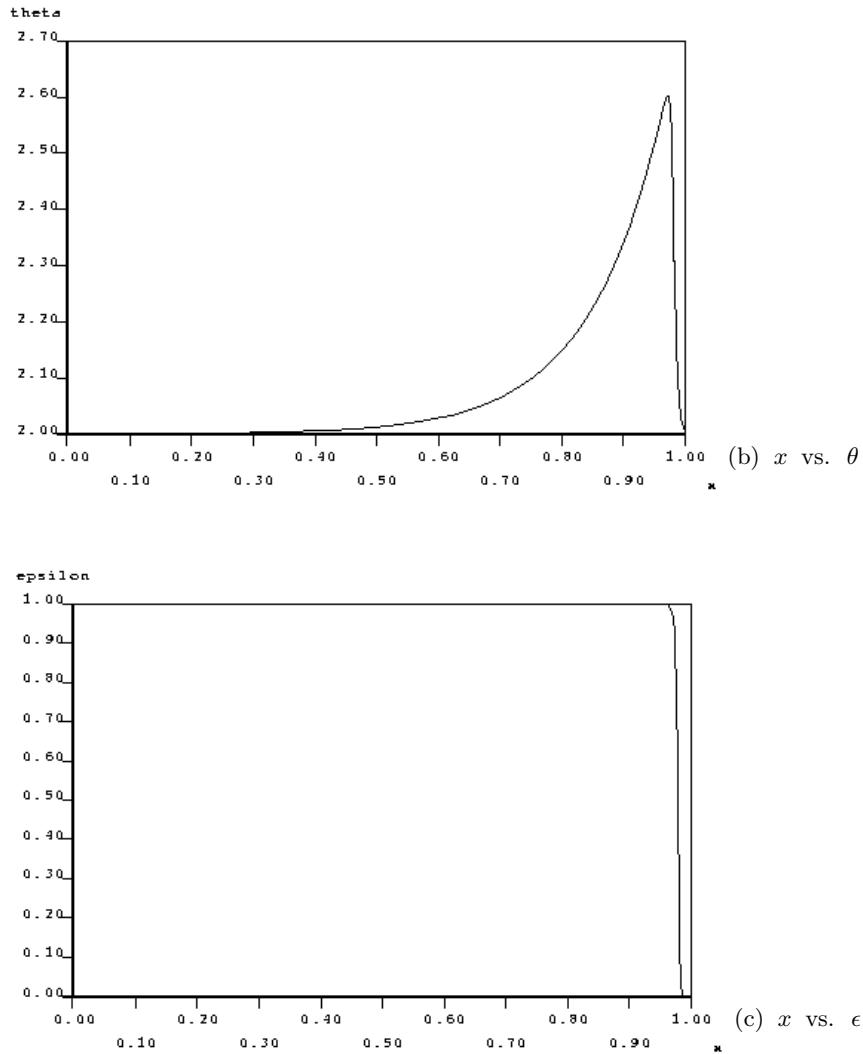


Figure 1. The traveling wave at a fixed time from [20]. The variables  $s$ ,  $\theta$ , and  $\epsilon$  represent gaseous phase saturation, temperature, and fraction of burned gas respectively.

are being displaced as a whole by proper injection at the left end. At a certain time and location, ignition starts. Is it possible that an oxidation pulse forms and propagates as a traveling wave?

We find, under the further simplifying assumption of constant effective thermal conductivity, that if the heat loss to the overburden is too large, no such oxidation

pulse can exist. On the other hand, if the heat loss is small enough, a traveling oxidation pulse indeed exists as a solution to the equations. Proving the existence of such a pulse is an important step toward solving the ignition problem for flow in a porous medium.

In [20], the same questions were considered in a more simplified model. (The gaseous phase density was assumed to be constant independent of temperature. Thus the total seepage velocity was constant and could be factored out of the flow solutions. In addition, a single oversimplified chemical reaction was assumed.) Analogous results were obtained. A numerically computed oxidation pulse for that model is shown in Fig. 1. The shape is similar to that of the pulse in [8].

The more realistic model of the present work is made tractable by introducing auxiliary variables representing cumulative mass and heat production along the flow direction. We are thereby able to change the form of the mass and heat terms in the traveling wave equations so that they can be integrated once. The total seepage velocity becomes another unknown in the traveling wave equations.

The paper is organized as follows. In Sec. 2, the basic conservation laws are described. There are source terms due to the chemical reaction. We construct a model with two phases (oleic and gaseous), and three components (the gaseous phase is divided into unburned and burned fractions).

A key physical assumption is that physical properties such as densities are pressure-independent. This approximation is valid for large pressure difference between injection production and wells, provided the width of the pulse is small relative to the distance between wells, so that the pressure change within the pulse is small relative to the background pressure. It is also valid for pressure differences that are small relative to the overall pressures.

In Sec. 3, capillary pressure diffusion terms are expressed in more detail, and the equations are manipulated to introduce the total seepage velocity. In the standard Buckley-Leverett theory, the total seepage velocity is assumed to be constant. Here this assumption is not appropriate, since there are volume changes due to the reaction and to temperature changes. The seepage velocity becomes a state variable whose value is a function of space and time.

In Sec. 4, we derive equations for traveling waves. The essential mathematical manipulation is the introduction of new variables representing the cumulative mass and heat transferred. Thus we are able to change the source terms in the equations so that first integrals can still be found. The problem of finding traveling waves reduces to the problem of finding connecting orbits in a system of ordinary differential equations with a larger number of variables.

Motivated by geometric singular perturbation theory, which is frequently used in the study of traveling waves [17], we first study a reduced system of equations in which heat loss vanishes. Simpler systems with no heat loss were previously studied in [10] and [11]. As in [20], we focus our attention on two curves of equilibria of the reduced system, one consisting of equilibria at which the temperature is that of the surrounding rock and the percentage of gas that has burned varies, the other

consisting of equilibria at which the temperature is above that of the surrounding rock and none of the gas has burned. These curves of equilibria of the reduced system are studied in Sec. 5, and the associated invariant manifolds are found in Sec. 6. The curve of high-temperature equilibria is normally hyperbolic; the curve of low-temperature equilibria is not, due to the form of Arrhenius's law, which governs the rate of the chemical reaction. For small heat loss, the low-temperature equilibria remain equilibria; the high-temperature equilibria do not.

In Sec. 7, we state precisely the main result of this paper, which asserts the existence of certain connecting orbits for the ordinary differential equation with small heat loss. As in [20], the speed of the traveling wave is approximately the speed for which the reduced system has a connection with a special structure between certain high- and low-temperature equilibria: it is a connection between the unstable manifold of a hyperbolic equilibrium and the stable manifold of a nonhyperbolic equilibrium, rather than a connection that arrives at the nonhyperbolic equilibrium tangent to its center direction. The traveling wave with small heat loss is approximately this connection followed by slow drift along the curve of high-temperature equilibria of the reduced system. The temperature gradually falls along this curve of equilibria until it meets the curve of low-temperature equilibria. The traveling wave terminates near this point of intersection, which is a further degeneracy.

In Sec. 8 we reinterpret the results of Secs. 5 and Sec. 6 to make them more useful in studying the ordinary differential equation with small heat loss. For small heat loss, the front end of the traveling pulse is studied in Sec. 9, and the back end in Secs. 10 and 11. The latter section treats the termination of the traveling wave using center manifold reduction at the intersection of the two curves of equilibria of the reduced system.

The proof of the existence result is completed in Sec. 12. In Sec. 13, we show that large heat loss prevents the existence of a traveling pulse.

Our arguments assume that the produced seepage velocity far in front of the traveling pulse is given, but it is more natural to prescribe the injected seepage velocity. This issue is discussed in Section 14.

Concluding discussion is presented in Sec. 15. An appendix summarizes nomenclature used throughout the paper.

## 2. Physical assumptions and conservation laws

The system of equations that we consider expresses the conservation of mass of each phase, taking into account fluid and gas mass exchanges as well as the conservation of internal energy. It also expresses Darcy's law of force.

In the current work, pressure variations are considered small. Volumetric changes associated with the combustion process are considered only in the sense that the gaseous phase density is assumed to depend on temperature and composi-

tion. The dependence of oleic and gaseous phase densities on pressure is neglected, which is justified when pressure variations are small relative to the total pressure. Lateral heat losses are taken into account. Longitudinal heat conduction, as well as diffusion effects due to the capillary pressure present in displacement of multiphase fluid in porous media, are also taken into account. They contribute through parabolic terms, which affect the speed and shape of the traveling wave solution in the finite width combustion zone. We assume that the combustion process is described by a single chemical reaction. We restrict our attention to flow in one spatial dimension, as is customary in initial analysis in petroleum engineering.

Throughout this section, the indices  $g$  and  $o$  refer to the gaseous and oleic phases, respectively, and  $r$  to the porous rock phase. The flow is described by state quantities depending on  $(x, t)$ , the space and time coordinates. They are denoted as follows. The saturation of phase  $j$  in the fluid,  $j = g, o$ , i.e., the fraction of the porous volume occupied by phase  $j$ , is denoted by  $s_j = s_j(x, t)$ ; the temperature, in Kelvin, which is assumed to be the same for gaseous, oleic and rock phases at each  $(x, t)$ , is denoted by  $\theta = \theta(x, t)$ ; and the burned volume fraction of the gaseous phase is denoted by  $\epsilon = \epsilon(x, t)$ .

We denote by  $K$  the (constant) absolute permeability of the rock, which measures the porous medium's capability of allowing fluid flow. The relative permeability of phase  $j$ , denoted by  $k_j$ , is a dimensionless function of saturation  $s_j$ , measured in the laboratory;  $Kk_j$  is the porous medium's capability of allowing the flow of phase  $j$ .

The capillary pressure is

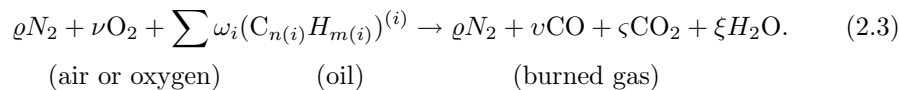
$$p_c = p_o(s_o) - p_g(s_g), \quad (2.1)$$

where  $p_j$ ,  $j = g, o$ , is the pressure in the phase  $j$ . Since  $s_o = 1 - s_g$ , we obtain  $p_c$  as a function of  $s_g$ . This function is obtained empirically and is known to be decreasing [1], so we assume

$$\frac{dp_c}{ds_g} < 0 \quad \text{for} \quad 0 < s_g < 1. \quad (2.2)$$

Other relevant quantities are  $\rho_j$ ,  $j = g, o, r$ , the density of phase  $j$ ;  $\phi$ , the (constant) rock porosity, i.e., the fraction of the total volume occupied by the fluid phases;  $v_j$ ,  $j = g, o$ , the seepage velocity of phase  $j$ ; and  $\mu_j$ ,  $j = g, o$ , the viscosity of phase  $j$ , i.e., its intrinsic resistance to motion. The gaseous phase density  $\rho_g$  is assumed to depend on temperature and composition. The densities  $\rho_o$  and  $\rho_r$  as well as the gas viscosity  $\mu_g$  are assumed to be constant. The oil viscosity  $\mu_o$  may change with temperature  $\theta$ .

The chemical reaction (where nitrogen is inert) is



In Eq. (2.3)  $\varrho$ ,  $\nu$ ,  $\omega_i$ ,  $v$ ,  $\varsigma$ , and  $\xi$  are known constants, determined by the composition of the oil and by stoichiometry. For simplicity we have neglected the

formation of coke C. For  $\varrho$  and  $\nu$  moles of unburned gas, there correspond  $(\varrho + \nu + \varsigma + \xi)$  moles of burned gas, so the gaseous phase mole (or volume) relative change due to the reaction is

$$c = \left( \frac{\varrho + \nu + \varsigma + \xi}{\varrho + \nu} - 1 \right). \quad (2.4)$$

Counting oxygen O on both sides of Eq. (2.3), we obtain

$$2\nu = \nu + 2\varsigma + \xi, \quad (2.5)$$

The quantity  $c$  in Eq. (2.4) can be written as a fraction with denominator  $\varrho + \nu$  and numerator  $\nu + \varsigma + \xi - \nu$ . Now, using Eq. (2.5):

$$\nu + \varsigma + \xi - \nu = \frac{\nu + \xi}{2} > 0, \quad (2.6)$$

so we see that  $c > 0$ .

Let  $d_i$  be the molecular weight of  $(C_{n(i)}H_{m(i)})^{(i)}$ . For simplicity we will use rounded values for atomic masses: 12 for C, 14 for N, 16 for O. Now,  $a' = 2\varrho \cdot 14 + 2\nu \cdot 16$  mass units of oxygen/nitrogen mixture generate  $b' = 2\varrho \cdot 14 + \nu(12 + 16) + \varsigma(12 + 2 \cdot 16) + \xi(2 \cdot 1 + 16)$  mass units of burned gas. Let  $d' = \sum \omega_i d_i$ . Then one mass unit of gaseous mixture reacts with  $d = \frac{d'}{a'}$  mass units of oil to generate  $b = \frac{b'}{a'}$  mass units of burned gas. Notice that by conservation of mass

$$1 + d = b. \quad (2.7)$$

Thus, if pressure and temperature were constant, when a fraction  $\epsilon$  of the reaction has taken place, the density would change by the positive factor  $\frac{1+\epsilon d}{1+\epsilon c}$ . Let  $\rho_0$  be the density of injected gas at the reservoir temperature  $\theta_0$ . Then, taking into account temperature variations but ignoring compressibility, the ideal law of gases yields:

$$\rho_g = \rho_g(\theta, \epsilon) = \rho_0 \frac{\theta_0}{\theta} \left( \frac{1 + \epsilon d}{1 + \epsilon c} \right), \quad (2.8)$$

If coke C were produced, its main effect would be to reduce the values of  $c$  and  $d$ , so that instead of Eq. (2.7), we would have  $1 + d < b$ .

A chemical reaction rate function  $q$  will be used to describe the fraction of oxygen-nitrogen mixture that reacts (or burns) per unit time. It will be assumed to have the following form (see Eq. 95.02 of [7]), a version of Arrhenius's law,

$$q(\theta, \epsilon) = \begin{cases} (1 - \epsilon)K_\infty e^{-\frac{L}{\theta - \theta_0}} & \text{if } \theta > \theta_0, \\ 0 & \text{if } 0 < \theta \leq \theta_0, \end{cases} \quad (2.9)$$

where  $K_\infty$  and  $L$  are positive constants, related to the asymptotic rate and to the activation energy. The same constants  $K_\infty$  and  $L$  are used for each of the hydrocarbons in the oil. This artificial assumption ensures that the oil composition is not affected by the chemical reaction. It justifies regarding the oil density  $\rho_o$  and the oil viscosity  $\mu_o$  as functions of temperature only. However, in contrast

to the density of the gaseous phase, the density of the oleic phase depends only slightly on the temperature. We therefore assume that  $\rho_o$  is constant. Using the same constants  $K_\infty$  and  $L$  for each of the hydrocarbons also justifies writing the thermodynamic properties as in Eqs. (2.17) and (2.18) below.

The exponential term in Arrhenius's original law was  $e^{-L/\theta}$ . The introduction of  $\theta_0$  in the exponent denominator in Eq. (2.9) facilitates the mathematical analysis of traveling waves by ensuring  $q(\theta_0) = 0$  exactly rather than approximately.

The solubility of the reaction product  $\text{CO}_2$  in oil, which occurs on a slow time scale, is neglected. This is valid if the seepage velocity is large enough.

We now write the equations governing the flow. First, the equation for *conservation of mass of burned gas* is

$$\frac{\partial}{\partial t}(\phi\rho_g s_g \epsilon) = -\frac{\partial}{\partial x}(\rho_g v_g \epsilon) + b\phi\rho_g s_g q, \quad (2.10)$$

where the first term represents accumulation of burned gas mass. The second term represents transport of the burned gas mass, and the third term represents the rate of gain of burned gas mass due to chemical reaction. In this third term,  $q = q(\theta, \epsilon)$  is the fraction of the total gas burned per unit time. The value for the stoichiometric constant  $b$  was found in the discussion of the chemical reaction.

Similarly, the equation for *conservation of mass of unburned gas* is

$$\frac{\partial}{\partial t}(\phi\rho_g s_g (1 - \epsilon)) = -\frac{\partial}{\partial x}(\rho_g v_g (1 - \epsilon)) - \phi\rho_g s_g q, \quad (2.11)$$

where the third term represents the rate of loss of unburned gas mass (oxygen) in the chemical reaction.

The equation for *conservation of oil mass* is

$$\frac{\partial}{\partial t}(\phi\rho_o s_o) = -\frac{\partial}{\partial x}(\rho_o v_o) - d\phi\rho_g s_g q. \quad (2.12)$$

Notice that the equation for conservation of total mass resulting from adding Eqs. (2.10), (2.11), and (2.12) must have a vanishing source term; this corresponds to the fact that  $b = 1 + d$ .

Finally, denoting by  $E_j$ ,  $j = g, o, r$ , the internal energy of phase  $j$ , the equation for *conservation of energy* is

$$\begin{aligned} & \frac{\partial}{\partial t}(\phi\rho_g s_g E_g + \phi\rho_o s_o E_o + \rho_r E_r) \\ &= -\frac{\partial}{\partial x}(\rho_g v_g E_g + \rho_o v_o E_o) + \frac{\partial}{\partial x}\left(\phi\kappa\frac{\partial\theta}{\partial x}\right) - \kappa_l(\theta - \theta_0). \end{aligned} \quad (2.13)$$

The first term represents thermal energy accumulation in the gaseous, oleic and rock phases. The second term represents energy transport by convection of the gaseous and oleic phases. The third term represents longitudinal heat conduction;  $\phi\kappa$  is the effective thermal conductivity of the composite material formed by the several fluid and rock components. The expression for  $\kappa$  is

$$\kappa = \kappa(\theta, \epsilon) = \rho_g(\theta, \epsilon)s_g\kappa_g(\epsilon) + \rho_o s_o \kappa_o + \frac{1}{\phi}\rho_r \kappa_r, \quad (2.14)$$



where  $\kappa_g$ ,  $\kappa_o$  and  $\kappa_r$  are the thermal conductivities of gas, oil, and rock. Notice that  $\kappa_o$  and  $\kappa_r$  are constant, while  $\kappa_g$  depends (slightly) on  $\epsilon$ , because the composition of the gaseous phase changes. Of course,  $\kappa_g(\epsilon)$  is continuous and positive for  $0 \leq \epsilon \leq 1$ .

The last term in (2.13) represents lateral heat loss to the adjacent rock formation according to Newton's law. This treatment of heat loss is analogous to that in [6]. The (constant) lateral rock conductivity  $\kappa_l$  has magnitude and units different from that of other  $\kappa$ 's. We shall treat  $\kappa_l$  as a small parameter. This is correct if the thickness of the rock formation containing oil is large, or if the speed of fluid flow is not too small.

From Darcy's force law in porous media, which is accurate for many relevant flow regimes in petroleum engineering, the seepage velocity of phase  $j$  is given by

$$v_j = -K\lambda_j \frac{\partial p_j}{\partial x}, \quad j = g, o, \quad (2.15)$$

where  $K$ , we recall, is the (constant) absolute permeability of the rock, and  $\lambda_j$  is the relative mobility of phase  $j$ . The quantity  $\lambda_j$  (mobility) is a function of saturation  $s_j$  and temperature  $\theta$ :

$$\lambda_j = \lambda_j(s_j, \theta) = \frac{k_j(s_j)}{\mu_j(\theta)}, \quad j = g, o. \quad (2.16)$$

Typically, for each  $j$ , the relative permeabilities  $k_j(s_j) = 0$  for  $0 \leq s_j \leq s_j^r$ , and  $k_j(s_j)$  are positive and monotone increasing for  $s_j^r < s_j \leq 1$ . (Relative permeabilities and viscosities are measured in the laboratory.) The values  $s_j^r$ ,  $j = g, o$ , are called reduced gaseous and oleic phase saturations. For simplicity the reduced saturations are set to 0 in this work.

The (somewhat) simplified thermodynamical equations of state that we will employ in the current work are

$$E_g = C_g(\epsilon)\theta - \epsilon H, \quad (2.17)$$

$$E_o = C_o\theta, \quad (2.18)$$

$$E_r = C_r\theta. \quad (2.19)$$

Here  $C_g$  is the thermal capacity of the gaseous phase at constant pressure. It depends on composition, and therefore is written as  $C_g = C_g(\epsilon)$ .  $C_o$  and  $C_r$  are the thermal capacities of oil and rock per unit mass, and  $H$  is the heat released by the combustion per unit mass of oxygen/nitrogen mixture at constant pressure. We make the simplifying assumptions that  $H$  and  $C_j$ ,  $j = o, r$ , are constant. Eqs. (2.18) and (2.19) are reasonably accurate on a substantial span of temperatures and pressures. (2.17) is only accurate for not too large changes in temperature at an essentially fixed pressure. We will use it anyway, because it is simple and it captures the main aspects of the dependence of  $E_g$  on  $\epsilon$  and  $\theta$ .

**Remark.** Since the heat capacity of oil exceeds substantially that of the gaseous mixture in situations of practical interest,  $C_o \gg C_g(\epsilon)$  for  $0 \leq \epsilon \leq 1$ . Also,

the dependence of  $C_g(\epsilon)$  on  $\epsilon$  is weak: the function  $C_g(\epsilon)$  has small derivative. We will use these observations to motivate some of the assumptions that we shall make.

### 3. The model

Having described the basic conservation laws, we are now ready to derive the equations to be analyzed. The total flow of the fluid is given by

$$v = v_g + v_o, \tag{3.1}$$

while the total mobility of the fluid, which depends on saturation and temperature, is

$$\lambda = \lambda_g + \lambda_o. \tag{3.2}$$

Hence, from (2.15),

$$v = -K\lambda_g \frac{\partial p_g}{\partial x} - K\lambda_o \frac{\partial p_o}{\partial x}. \tag{3.3}$$

Let us introduce  $f_j$ , the fractional flow function of phase  $j$ ,  $j = g, o$ . The functions  $f_j$  depend on  $s_j$  and  $\theta$  as follows:

$$f_j(s_j, \theta) = \frac{\lambda_j(s_j, \theta)}{\lambda}. \tag{3.4}$$

From Eqs. (2.1), (3.2), (3.3), and (3.4), we have

$$\frac{\partial p_j}{\partial x} = -f_j \frac{\partial p_c}{\partial x} - \frac{v}{K\lambda} = -f_o \frac{dp_c}{ds_j} \frac{\partial s_j}{\partial x} - \frac{v}{K\lambda}, \quad \text{for } j = g, o. \tag{3.5}$$

We introduce the function

$$h(s_g, \theta, \epsilon) = -\lambda_o f_g p'_c = -\lambda_g f_o p'_c \tag{3.6}$$

to simplify the following formulas. By (2.2),  $h$  is positive for  $0 < s_g < 1$ .

Substituting Eqs. (2.15) and (3.5) into Eqs. (2.10)–(2.13), and using the equalities  $\lambda_g f_o = \lambda_o f_g$  and  $s_o = 1 - s_g$ , we have, respectively,

$$\frac{\partial}{\partial t}(\phi \rho_g s_g \epsilon) + \frac{\partial}{\partial x}(\rho_g \epsilon f_g v) = \frac{\partial}{\partial x} \left( K \rho_g \epsilon h \frac{\partial s_g}{\partial x} \right) + b \phi \rho_g s_g q, \tag{3.7}$$

$$\frac{\partial}{\partial t}(\phi \rho_g s_g (1 - \epsilon)) + \frac{\partial}{\partial x}(\rho_g (1 - \epsilon) f_g v) = \frac{\partial}{\partial x} \left( K \rho_g (1 - \epsilon) h \frac{\partial s_g}{\partial x} \right) - \phi \rho_g s_g q, \tag{3.8}$$

$$\frac{\partial}{\partial t}(\phi \rho_o s_o) + \frac{\partial}{\partial x}(\rho_o f_o v) = \frac{\partial}{\partial x} \left( K \rho_o h \frac{\partial s_o}{\partial x} \right) - d \phi \rho_g s_g q, \tag{3.9}$$

$$\begin{aligned} & \frac{\partial}{\partial t}(\phi\rho_g s_g \mathbf{E}_g + \phi\rho_o s_o \mathbf{E}_o + \rho_r \mathbf{E}_r) + \frac{\partial}{\partial x}((\rho_g \mathbf{E}_g f_g + \rho_o \mathbf{E}_o f_o)v) \\ &= \frac{\partial}{\partial x} \left( \mathbf{K}(\rho_g \mathbf{E}_g - \rho_o \mathbf{E}_o)h \frac{\partial s_g}{\partial x} \right) + \frac{\partial}{\partial x} \left( \phi\kappa \frac{\partial \theta}{\partial x} \right) - \kappa_l(\theta - \theta_0). \end{aligned} \quad (3.10)$$

Let  $R$  be a typical reservoir length scale. We introduce scaled  $\tilde{x}$ ,  $\tilde{t}$ , and  $\tilde{v}$  in Eqs. (3.7) – (3.10) as follows

$$\tilde{x} = \frac{x}{R}, \quad \tilde{t} = \frac{Kt}{\phi R^2}, \quad \tilde{v} = \frac{Rv}{K}. \quad (3.11)$$

This scaling makes the advection terms (that is, the second terms) in each of the equations (3.7)–(3.10) balance the diffusion terms (that is, the third terms). It is appropriate for the study of traveling waves, which are formed by the competition between converging characteristics and diffusion.

We define

$$\begin{aligned} \tilde{a} &= \frac{\phi R^2}{K}, \quad \tilde{b} = \frac{\phi R^2}{K}b, \quad \tilde{c} = \frac{\phi R^2}{K}c, \quad \tilde{d} = \frac{\phi R^2}{K}d, \\ \tilde{\rho}_r &= \frac{1}{\phi}\rho_r, \quad \tilde{\kappa} = \frac{\phi}{K}\kappa, \quad \tilde{\delta} = \frac{R^2}{K}\kappa_l. \end{aligned} \quad (3.12)$$

To further simplify the notation, let  $s = s_g$ ,  $1 - s = s_o$ ,  $f = f_g$ , and  $1 - f = f_o$ . Thus, from now on, the gaseous phase saturation is denoted by  $s$ , the fractional flow function of the gaseous phase is denoted by  $f$ , and the coefficient of lateral heat conduction to the rock formation is denoted by  $\delta$ .

After dropping the tildes in all quantities, (3.7)–(3.9) become:

$$\frac{\partial}{\partial t}(\rho_g s \epsilon) + \frac{\partial}{\partial x}(\rho_g \epsilon f v) = \frac{\partial}{\partial x} \left( \rho_g \epsilon h \frac{\partial s}{\partial x} \right) + b \rho_g s q, \quad (3.13)$$

$$\frac{\partial}{\partial t}(\rho_g s(1 - \epsilon)) + \frac{\partial}{\partial x}(\rho_g(1 - \epsilon)f v) = \frac{\partial}{\partial x} \left( \rho_g(1 - \epsilon)h \frac{\partial s}{\partial x} \right) - a \rho_g s q, \quad (3.14)$$

$$\frac{\partial}{\partial t}(\rho_o(1 - s)) + \frac{\partial}{\partial x}(\rho_o(1 - f)v) = -\frac{\partial}{\partial x} \left( \rho_o h \frac{\partial s}{\partial x} \right) - d \rho_g s q. \quad (3.15)$$

Making the same changes, and using Eqs. (2.17), (2.18), and (2.19) as well, (3.10) becomes

$$\begin{aligned} & \frac{\partial}{\partial t} \left( ((-C_o \rho_o + C_g \rho_g)s + C_o \rho_o + C_r \rho_r)\theta - \epsilon H \rho_g s \right) \\ & + \frac{\partial}{\partial x} \left( (((-C_o \rho_o + C_g \rho_g)f + C_o \rho_o)\theta - \epsilon H \rho_g f)v \right) \\ &= \frac{\partial}{\partial x} \left( ((-C_o \rho_o + C_g \rho_g)\theta - \epsilon H \rho_g)h \frac{\partial s}{\partial x} \right) + \frac{\partial}{\partial x} \left( \kappa \frac{\partial \theta}{\partial x} \right) - \delta(\theta - \theta_0). \end{aligned} \quad (3.16)$$

The system of equations to be solved is (3.13)–(3.16), expressing the conservation of unburned gaseous mass, burned gaseous mass, oil mass, and energy.

Note that Eqs. (2.7) and (2.8) must be replaced by

$$a + d = b, \tag{3.17}$$

$$\rho_g = \rho_g(\theta, \epsilon) = \rho_0 \frac{\theta_0}{\theta} \left( \frac{a + \epsilon d}{a + \epsilon c} \right). \tag{3.18}$$

#### 4. Traveling waves

We shall look for traveling wave solutions  $(s(z), \theta(z), \epsilon(z), v(z))$  of (3.13)–(3.16), with  $z = x - \sigma t$ , where  $\sigma$  is the traveling wave speed, or shock speed in the zero viscosity limit. Substituting this ansatz into the system, we obtain

$$-\sigma \frac{d}{dz}(\rho_g s \epsilon) + \frac{d}{dz}(\rho_g \epsilon f v) = \frac{d}{dz} \left( \rho_g \epsilon h \frac{ds}{dz} \right) + b \rho_g s q, \tag{4.1}$$

$$-\sigma \frac{d}{dz}(\rho_g s(1 - \epsilon)) + \frac{d}{dz}(\rho_g(1 - \epsilon) f v) = \frac{d}{dz} \left( \rho_g(1 - \epsilon) h \frac{ds}{dz} \right) - a \rho_g s q, \tag{4.2}$$

$$-\sigma \frac{d}{dz}(\rho_o(1 - s)) + \frac{d}{dz}(\rho_o(1 - f)v) = -\frac{d}{dz} \left( \rho_o h \frac{ds}{dz} \right) - d \rho_g s q, \tag{4.3}$$

$$\begin{aligned} & -\sigma \frac{d}{dz} \left( \left( (-C_o \rho_o + C_g \rho_g) s + C_o \rho_o + C_r \rho_r \right) \theta - \epsilon H \rho_g s \right) \\ & \quad + \frac{d}{dz} \left( \left( (-C_o \rho_o + C_g \rho_g) f + C_o \rho_o \right) \theta - \epsilon H \rho_g f \right) v \\ & = \frac{d}{dz} \left( \left( (-C_o \rho_o + C_g \rho_g) \theta - \epsilon H \rho_g \right) h \frac{ds}{dz} \right) + \frac{d}{dz} \left( \kappa \frac{d\theta}{dz} \right) - \delta(\theta - \theta_0). \end{aligned} \tag{4.4}$$

In this system,  $(s, \theta, \epsilon, v)$  is the vector of unknowns, and we shall regard  $(\sigma, \delta)$  as a vector of parameters. We will usually use a dot to indicate  $\frac{d}{dz}$ .

Let

$$\mathcal{X} = \{g : \mathbb{R} \rightarrow \mathbb{R} : g \text{ is } C^1, \lim_{z \rightarrow \pm\infty} g(z) \text{ exists, and } \lim_{z \rightarrow \pm\infty} \dot{g}(z) = 0\}.$$

Let  $\mathcal{X}^n = \mathcal{X} \times \dots \times \mathcal{X}$  ( $n$  times).

Given  $(s, \theta, \epsilon)$  in  $\mathcal{X}^3$ , let

$$s^\pm = \lim_{z \rightarrow \pm\infty} s(z), \quad \theta^\pm = \lim_{z \rightarrow \pm\infty} \theta(z), \quad \epsilon^\pm = \lim_{z \rightarrow \pm\infty} \epsilon(z).$$

Let  $s_0$  denote the initial gaseous phase saturation, and recall that  $\theta_0$  is the initial reservoir temperature. We assume  $0 < s_0 < 1$ . We shall look for solutions  $(s(z), \theta(z), \epsilon(z), v(z))$  of (4.1)–(4.4) in  $\mathcal{X}^4$  with

$$s^+ = s_0, \quad \theta^\pm = \theta_0, \quad \epsilon^+ = 0. \tag{4.5}$$

The values of  $s^-$  and  $\epsilon^-$  must be determined. It is natural when the oxygen-nitrogen mixture is injected on the left to assume that  $v^- = \lim_{z \rightarrow -\infty} v(z)$  is

given and that  $\lim_{z \rightarrow \infty} v(z)$  must be determined. However, we shall assume that  $v^+ = \lim_{z \rightarrow \infty} v(z)$  is given and that the other limit must be found. We discuss this point further in Sec. 14.

We define

$$\rho_g^+ = \rho_g(\theta^+, \epsilon^+) = \rho_g(\theta_0, 0) = \rho_0, \quad (4.6)$$

$$f^+ = f(s^+, \theta^+) = f(s_0, \theta_0), \quad (4.7)$$

$$C_g^+ = C_g(\epsilon^+) = C_g(0). \quad (4.8)$$

Let a number  $v^+ > 0$  be given, which will turn out to be  $\lim_{z \rightarrow \infty} v(z)$ . It will be convenient to make the following definitions. In these definitions, recall that  $\rho_g = \rho_g(\theta, \epsilon)$  is given by (3.18).

$$A(\sigma) = \sigma s^+ - f^+ v^+, \quad (4.9)$$

$$B(\sigma) = \sigma((-C_o \rho_o + C_g^+ \rho_g^+) s^+ + C_o \rho_o + C_r \rho_r) - ((-C_o \rho_o + C_g^+ \rho_g^+) f^+ + C_o \rho_o) v^+, \quad (4.10)$$

$$C(\epsilon) = C_g(\epsilon) - C_g^+ + \frac{d\epsilon}{b - d\epsilon} (C_o - C_g(\epsilon)), \quad (4.11)$$

$$D(\theta, \epsilon, \sigma) = \rho_g^+ A \left( \frac{b\epsilon}{b - d\epsilon} H + C(\epsilon)\theta \right), \quad (4.12)$$

$$T(s, \theta, \epsilon, \sigma) = \frac{s \rho_g (b - d\epsilon)^2}{\rho_g^+ A b}. \quad (4.13)$$

$$Q(\epsilon, \sigma) = -\rho_g^+ A \frac{\epsilon}{b - d\epsilon}, \quad (4.14)$$

$$v(\theta, \epsilon, \sigma) = -\sigma \left( \frac{\rho_g^+}{\rho_g} - 1 \right) s^+ + \left( \left( \frac{\rho_g^+}{\rho_g} - 1 \right) f^+ + 1 \right) v^+ + d \left( \frac{1}{\rho_g} - \frac{1}{\rho_o} \right) Q. \quad (4.15)$$

We remark that since by definition the particle speed in a conservation law  $s_t + (fv)_x = 0$  is  $fv/s$ , the positive or negative sign of  $A(\sigma)$  corresponds to wave speeds  $\sigma$  larger or smaller than particle speed.

We now define a first-order system of ordinary differential equations with variables  $(s, \theta, \epsilon, \Psi)$  and parameters  $(\sigma, \delta)$ . This system will be our focus of attention for the remainder of this paper.

$$\dot{s} = G_1(s, \theta, \epsilon, \Psi, \sigma) = \frac{1}{\rho_g h} (\rho_g^+ A - \rho_g(\sigma s - fv) - dQ), \quad (4.16)$$

$$\dot{\theta} = G_2(s, \theta, \epsilon, \Psi, \sigma) = \frac{1}{\kappa} (-B(\theta - \theta_0) + D + \Psi), \quad (4.17)$$

$$\dot{\epsilon} = G_3(s, \theta, \epsilon, \Psi, \sigma) = -Tq, \quad (4.18)$$

$$\dot{\Psi} = \delta(\theta - \theta_0). \quad (4.19)$$

We note that  $G_1$  and  $G_3$  are independent of  $\Psi$ , and  $G_2$  is independent of  $s$ .

We shall always assume that  $0 < s < 1$  and  $0 \leq \epsilon \leq 1$ . Only the intervals  $0 \leq s \leq 1$  and  $0 \leq \epsilon \leq 1$  are physically meaningful. We shall avoid  $s = 0$  and  $s = 1$  since the function  $h$  in (4.16) vanishes at these points. We remark that since  $h$  vanishes at  $s = 0$  and  $s = 1$ , the viscous terms in (3.13)–(3.15), and the first viscous term in (3.16), vanish at these points.

We shall refer to (4.16)–(4.19) as the *Traveling Wave System*. The reason for this terminology is given by the following result.

**Theorem 4.1.** *Let  $(s(z), \theta(z), \epsilon(z)) \in \mathcal{X}^3$  with  $s^+ = s_0$ ,  $\theta^\pm = \theta_0$ ,  $\epsilon^+ = 0$ . Let  $v^+ \in \mathbb{R}$ . Then following are equivalent:*

- (1) *There is a function  $v(z) \in \mathcal{X}$ , with  $\lim_{z \rightarrow \infty} v(z) = v^+$ , such that  $(s(z), \theta(z), \epsilon(z), v(z))$  is a solution of (4.1)–(4.4).*
- (2) *There is a function  $\Psi(z) \in \mathcal{X}$ , with  $\lim_{z \rightarrow \infty} \Psi(z) = 0$ , such that  $(s(z), \theta(z), \epsilon(z), \Psi(z))$  is a solution of (4.16)–(4.19).*

Moreover, we have the following formulas:

$$v(z) = v(\theta(z), \epsilon(z), \sigma), \quad (4.20)$$

where  $v(\theta, \epsilon, \sigma)$  is given by (4.15); and

$$\Psi(z) = -\delta \int_z^\infty (\theta(\tau) - \theta_0) d\tau. \quad (4.21)$$

*Proof.* First we show that (1) implies (4.20) and (2), with  $\Psi(z)$  given by (4.21).

Assume that  $(s(z), \theta(z), \epsilon(z), v(z))$  is a solution of (4.1)–(4.4) in  $\mathcal{X}^4$  with  $s^+ = s_0$ ,  $\theta^\pm = \theta_0$ ,  $\epsilon^+ = 0$  and  $\lim_{z \rightarrow \infty} v(z) = v^+$ . Integrating (4.1) from  $z$  to  $\infty$ , we obtain

$$\sigma \rho_g s \epsilon - \rho_g \epsilon f v = -\rho_g \epsilon h \dot{s} + b \int_z^\infty \rho_g s q d\tau. \quad (4.22)$$

The integral is finite because the other terms are finite. Therefore we can define

$$Q(z) = - \int_z^\infty \rho_g s q d\tau, \quad (4.23)$$

so that

$$\dot{Q} = \rho_g s q, \quad (4.24)$$

and we can rewrite (4.22) as

$$-\rho_g \epsilon h \dot{s} = \sigma \rho_g s \epsilon - \rho_g \epsilon f v + bQ. \quad (4.25)$$

Similarly, integrating (4.2) from  $z$  to  $\infty$  yields

$$-\rho_g (1 - \epsilon) h \dot{s} = -\sigma \rho_g^+ s^+ + \sigma \rho_g s (1 - \epsilon) + \rho_g^+ f^+ v^+ - \rho_g (1 - \epsilon) f v - aQ. \quad (4.26)$$

Adding Eqs. (4.25) and (4.26) yields

$$-\rho_g h \dot{s} = -\sigma(\rho_g^+ s^+ - \rho_g s) + (\rho_g^+ f^+ v^+ - \rho_g f v) + dQ, \quad (4.27)$$

which is equivalent to (4.16).

Multiplying (4.25) by  $1 - \epsilon$  and (4.26) by  $-\epsilon$  and adding, we obtain

$$\epsilon(\sigma\rho_g^+s^+ - \rho_g^+f^+v^+) + (b - d\epsilon)Q = 0, \quad (4.28)$$

so

$$Q(z) = -\rho_g^+A \frac{\epsilon(z)}{b - d\epsilon(z)}. \quad (4.29)$$

Thus

$$Q(z) = Q(\epsilon(z), \sigma), \quad (4.30)$$

where  $Q(\epsilon, \sigma)$  was defined by (4.14). Differentiating (4.29) with respect to  $z$ , and using (4.24) and

$$\frac{d}{dz} \left( \frac{\epsilon}{b - d\epsilon} \right) = \frac{b}{(b - d\epsilon)^2} \dot{\epsilon}, \quad (4.31)$$

we obtain (4.18).

Integrating (4.3) from  $z$  to  $\infty$  yields

$$\rho_o h \dot{s} = \sigma \rho_o (s^+ - s) + \rho_o (1 - f^+) v^+ - \rho_o (1 - f) v - dQ. \quad (4.32)$$

We divide (4.32) by  $\rho_o$ , and add (4.27) divided by  $\rho_g$ . We obtain

$$-\sigma \left( \frac{\rho_g^+}{\rho_g} - 1 \right) s^+ + \left( \frac{\rho_g^+}{\rho_g} f^+ + (1 - f^+) \right) v^+ - v + d \left( \frac{1}{\rho_g} - \frac{1}{\rho_o} \right) Q = 0, \quad (4.33)$$

which is equivalent to (4.15). Thus (4.20) holds.

Multiplying (4.32) by  $C_o$  and (4.27) by  $C_g$  and adding, we obtain:

$$\begin{aligned} (C_o \rho_o - C_g \rho_g) h \dot{s} &= \sigma \left( (C_o \rho_o - C_g \rho_g^+) s^+ + (-C_o \rho_o + C_g \rho_g) s \right) \\ &\quad + \left( (-C_o \rho_o + C_g \rho_g^+) f^+ + C_o \rho_o \right) v^+ \\ &\quad + \left( (C_o \rho_o - C_g \rho_g) f - C_o \rho_o \right) v - dQ (C_o - C_g). \end{aligned} \quad (4.34)$$

Integrating (4.4) from  $z$  to  $\infty$  yields

$$\begin{aligned} &-\sigma \left( \left( (-C_o \rho_o + C_g^+ \rho_g^+) s^+ + C_o \rho_o + C_r \rho_r \right) \theta_0 \right. \\ &\quad \left. - \left( (-C_o \rho_o + C_g \rho_g) s + C_o \rho_o + C_r \rho_r \right) \theta + \epsilon H \rho_g s \right) \\ &+ \left( (-C_o \rho_o + C_g^+ \rho_g^+) f^+ + C_o \rho_o \right) \theta_0 v^+ - \left( \left( (-C_o \rho_o + C_g \rho_g) f + C_o \rho_o \right) \theta - \epsilon H \rho_g f \right) v \\ &= \left( (C_o \rho_o - C_g \rho_g) \theta + \epsilon H \rho_g \right) h \dot{s} - \kappa \dot{\theta} - \delta \int_z^\infty (\theta(\tau) - \theta_0) d\tau. \end{aligned} \quad (4.35)$$

The integral is finite because the other terms are. Thus we can define  $\Psi(z)$  by (4.21), so (4.19) is satisfied. Clearly  $\lim_{z \rightarrow \infty} \Psi(z) = \lim_{z \rightarrow \infty} \dot{\Psi}(z) = 0$ .

By integrating (4.4) from  $-\infty$  to  $\infty$  we see that  $\lim_{z \rightarrow -\infty} \Psi(z)$  is finite and  $\lim_{z \rightarrow -\infty} \dot{\Psi}(z) = 0$ .

We add Eq. (4.34) multiplied by  $\theta$  to Eq. (4.35) and subtract Eq. (4.27) multiplied by  $\epsilon H$ . We obtain

$$\begin{aligned} & \left( -\sigma((-C_o\rho_o + C_g^+\rho_g^+)s^+ + C_o\rho_o + C_r\rho_r) + ((-C_o\rho_o + C_g^+\rho_g^+)f^+ + C_o\rho_o)v^+ \right) (\theta_0 - \theta) \\ & + \sigma(C_g - C_g^+)\rho_g^+s^+\theta - (C_g - C_g^+)\rho_g^+f^+v^+\theta - \sigma\epsilon H\rho_g^+s^+ + \epsilon H\rho_g^+f^+v^+ \\ & + dQ\epsilon H + dQ(C_o - C_g)\theta - \Psi = -\kappa\dot{\theta}, \end{aligned} \tag{4.36}$$

or

$$\begin{aligned} & \kappa\dot{\theta} = \\ & \left( \sigma((-C_o\rho_o + C_g^+\rho_g^+)s^+ + C_o\rho_o + C_r\rho_r) - ((-C_o\rho_o + C_g^+\rho_g^+)f^+ + C_o\rho_o)v^+ \right) (\theta_0 - \theta) \\ & + \rho_g^+(\sigma s^+ - f^+v^+)(C_g - C_g^+)\theta + \epsilon H\rho_g^+(\sigma s^+ - f^+v^+) - dQ(\epsilon H + (C_o - C_g)\theta) + \Psi, \end{aligned} \tag{4.37}$$

which is equivalent to (4.17).

To show that (2) implies (1), assume that  $(s(z), \theta(z), \epsilon(z), \Psi(z))$  is a solution of (4.16)–(4.19) in  $\mathcal{X}^4$  with  $s^+ = s_0$ ,  $\theta^\pm = \theta_0$ ,  $\epsilon^+ = 0$  and  $\lim_{z \rightarrow \infty} \Psi(z) = 0$ . From (4.16) we obtain (4.27). From (4.14) we obtain (4.28). Adding (4.28) to (4.27) multiplied by  $\epsilon$  yields (4.22), which we differentiate to obtain (4.1). Multiplying (4.27) by  $1 - \epsilon$  and subtracting (4.28) yields (4.26), which we differentiate to obtain (4.2).

Define  $v$  by (4.15). From the definitions of  $\rho_g$  and  $Q$ , we see that  $v(z) \in \mathcal{X}$  and that  $\lim_{z \rightarrow \infty} v(z) = v(\theta_0, 0, \sigma) = v^+$ . From (4.15) we subtract (4.27) divided by  $\rho_g$ , and multiply the result by  $\rho_o$ . We obtain (4.32). Differentiation yields (4.3).

Finally, we multiply (4.27) by  $\epsilon H$ , and subtract the sum of (4.34) and (4.37), which yields (4.35). We differentiate to obtain (4.4).  $\square$

In the Traveling Wave System, the parameter  $\delta$  appears only in the equation for  $\dot{\Psi}$ . Moreover, when  $\delta = 0$ ,  $\dot{\Psi} = 0$ , so  $\Psi$  is constant.

To bring out this structure, let

$$u = (s, \theta, \epsilon), \quad w = (u, \Psi) = (s, \theta, \epsilon, \Psi).$$

We shall write the Traveling Wave System as

$$\dot{w} = F(w, \sigma, \delta), \tag{4.38}$$

with parameters  $\sigma$  and  $\delta$ . (4.38) can also be written

$$\dot{u} = G(u, \Psi, \sigma), \tag{4.39}$$

$$\dot{\Psi} = \delta(\theta - \theta_0). \tag{4.40}$$



The definitions of  $G_1$ ,  $G_2$ , and  $G_3$  were chosen with this form in mind. We can think of (4.39) as defining a system of ordinary differential equations on  $(s, \theta, \epsilon)$ -space with parameters  $\Psi$  and  $\sigma$ .

Motivated by Theorem 4.1, we shall look for a solution  $(s(z), \theta(z), \epsilon(z), \Psi(z))$  of the Traveling Wave System that is in  $\mathcal{X}^4$ , and that satisfies the boundary conditions (4.5) and the additional boundary condition  $\lim_{z \rightarrow \infty} \Psi(z) = 0$ . We also assume  $v^+$  is given. The values of  $s^-$ ,  $\epsilon^-$ , and  $\lim_{z \rightarrow -\infty} \Psi(z)$  must be determined.

## 5. Equilibria of $\dot{u} = G(u, \Psi, \sigma)$

In this section we shall determine the equilibria of  $\dot{u} = G(u, \Psi, \sigma)$ . We first define three sets. The definitions use assumptions (I1)–(I6) to be given shortly, and a positive function  $e(\sigma)$  to be defined shortly.

$$\begin{aligned} \text{I} &= \{\sigma : \text{conditions (I1)–(I7) are satisfied}\}, \\ \text{J} &= \{(\Psi, \sigma) : -e(\sigma) \leq \Psi \leq 0, \sigma \in \text{I}\}, \\ \text{S} &= \{(s, \theta, \epsilon, \Psi, \sigma) : 0 < s < 1, \theta_0 \leq \theta, 0 \leq \epsilon \leq 1, (\Psi, \sigma) \in \text{J}\}. \end{aligned}$$

We shall restrict our attention to equilibria in  $\text{S}$ .

The equilibria of  $\dot{u} = G(u, \Psi, \sigma)$  satisfy

$$G_1(s, \theta, \epsilon, \Psi, \sigma) = G_2(s, \theta, \epsilon, \Psi, \sigma) = G_3(s, \theta, \epsilon, \Psi, \sigma) = 0.$$

In this work, we will consider *forward-moving oxidation waves*, which move faster than gas particle velocity. They are the waves of interest when the oxidation starts at the well where oxygen is injected. More precisely, we assume (I1)  $A(\sigma) > 0$ .

Then the function  $T$  is positive on the set  $\text{S}$ , so from  $G_3 = 0$ , we see that  $\theta = \theta_0$  or  $\epsilon = 1$ .

### 5.1. Equilibria with $\theta = \theta_0$

Substituting  $\theta = \theta_0$  into the equation  $G_2 = 0$ , we obtain

$$\Psi = \Psi_0(\epsilon, \sigma) = -D(\theta_0, \epsilon, \sigma). \quad (5.1)$$

$\Psi_0$  is smooth for  $0 \leq \epsilon \leq 1$ .

Let

$$e(\sigma) = D(\theta_0, 1, \sigma) = \rho_g^+ A(\sigma) \left( \frac{b}{a} H + C(1)\theta_0 \right).$$

We assume:

$$(I2) \quad \frac{b}{a} H + C(1)\theta_0 > 0.$$

$$(I3) \quad \frac{b^2}{(b-d\epsilon)^2} H + C'(\epsilon)\theta_0 > 0. \quad \text{for } 0 \leq \epsilon \leq 1.$$

These assumptions are motivated by the observation in Sec. 2 that  $C_o \gg C_g(\epsilon)$  and  $C_g(\epsilon)$  has small derivative.

Assumptions (I1) and (I2) imply that  $e(\sigma) > 0$ . Assumptions (I1) and (I3) imply that  $\frac{\partial D}{\partial \epsilon}(\theta_0, \epsilon, \sigma) > 0$  for  $0 \leq \epsilon \leq 1$ . Therefore  $\frac{\partial \Psi_0}{\partial \epsilon} < 0$  for  $0 \leq \epsilon \leq 1$ .

It follows that, for each fixed  $\sigma \in I$ , the function  $\Psi_0(\epsilon, \sigma)$  can be inverted on the interval  $0 \leq \epsilon \leq 1$ . We obtain  $\epsilon = \epsilon_0(\Psi, \sigma)$  defined for  $(\Psi, \sigma) \in J$ . We see that

$$\frac{\partial \epsilon_0}{\partial \Psi} < 0, \tag{5.2}$$

$$\epsilon_0(-e(\sigma), \sigma) = 1, \tag{5.3}$$

$$\epsilon_0(0, \sigma) = 0. \tag{5.4}$$

From the definition of  $G_1$ , we see that

$$G_1(s_0, \theta_0, 0, 0, \sigma) = \frac{1}{\rho_g^+ h} (\rho_g^+ A - \rho_g^+ (\sigma s^+ - f^+ v^+) - d \cdot 0) = 0, \tag{5.5}$$

and

$$\frac{\partial G_1}{\partial s}(s_0, \theta_0, 0, 0, \sigma) = \frac{1}{\rho_g^+ h} (-\rho_g^+) (\sigma - f_s(s_0, \theta_0) v^+) = -\frac{1}{h} (\sigma - f_s(s_0, \theta_0) v^+). \tag{5.6}$$

We make the following assumption, which says that the saturation wave characteristic speeds ahead of the oxidation wave are slower than the oxidation wave itself. This is Lax's classical condition for the traveling wave to give rise to a shock [21] in the zero-diffusion limit. It is one of Lax's entropy inequalities.

$$(I4) \quad \sigma - f_s(s_0, \theta_0) v^+ > 0.$$

We also assume:

(I5) The equation  $G_1(s, \theta_0, \epsilon_0(\Psi, \sigma), \Psi, \sigma) = 0$  can be solved for  $s$  on  $J$  using the Implicit Function Theorem. The resulting function  $s = s_0(\Psi, \sigma)$  is required to satisfy  $s_0(0, \sigma) = s_0$ .

Assumption (I4) implies that

$$\frac{\partial G_1}{\partial s}(s_0, \theta_0, 0, 0, \sigma) < 0. \tag{5.7}$$

Since  $\epsilon_0(0, \sigma) = 0$  by (5.4), Eqs. (5.5) and (5.7) imply that  $s_0(\Psi, \sigma)$  can be defined near the line  $\{0\} \times I \subset J$ . (I5) is the assumption that the Implicit Function Theorem can be used to extend this function to all of  $J$ .

We define the following equilibria of  $\frac{du}{dz} = G(u, \Psi, \sigma)$  with  $(\Psi, \sigma) \in J$ :

$$m(\Psi, \sigma) = (s, \theta, \epsilon) \text{ where } s = s_0(\Psi, \sigma), \theta = \theta_0, \text{ and } \epsilon = \epsilon_0(\Psi, \sigma). \tag{5.8}$$

As  $\Psi$  increases from  $-e(\sigma)$  to 0, the  $\epsilon$ -coordinate of  $m(\Psi, \sigma)$  decreases from 1 to 0.

Because of (5.7), implicit in (I5) is the following assumption for all  $(\Psi, \sigma) \in J$ :

$$\frac{\partial G_1}{\partial s}(m(\Psi, \sigma), \Psi, \sigma) < 0. \tag{5.9}$$

## 5.2. Equilibria with $\epsilon = 1$

Let

$$\tilde{B}(\sigma) = B(\sigma) - \rho_g^+ A(\sigma) C(1).$$

Substituting  $\epsilon = 1$  into the equation  $G_2 = 0$ , and recalling the definition of  $e(\sigma)$ , we obtain

$$-\tilde{B}(\sigma)(\theta - \theta_0) + e(\sigma) + \Psi = 0. \quad (5.10)$$

We assume

$$(I6) \quad B(\sigma) - \rho_g^+ A(\sigma) C(\epsilon) > 0 \quad \text{for } 0 \leq \epsilon \leq 1.$$

In particular, assumption (I6) implies that  $\tilde{B}(\sigma) > 0$ , which says that the thermal wave characteristic speeds ahead of the oxidation wave are slower than the oxidation wave itself. Together with (I4), this is part of the Lax condition for the oxidation wave to become a 2-shock [21] in the zero-diffusion limit. Since  $\tilde{B}(\sigma) > 0$ , (5.10) implies that

$$\theta = \theta_1(\Psi, \sigma) = \theta_0 + \frac{1}{\tilde{B}(\sigma)} (\Psi + e(\sigma)).$$

From the definition of  $\theta_1(\Psi, \sigma)$ , (5.3), and the definitions of  $s_0(\Psi, \sigma)$  and  $\epsilon_0(\Psi, \sigma)$ , we have

$$\begin{aligned} G_1(s_0(-e(\sigma), \sigma), \theta_1(-e(\sigma), \sigma), 1, -e(\sigma), \sigma) \\ = G_1(s_0(-e(\sigma), \sigma), \theta_0, \epsilon_0(-e(\sigma), \sigma), -e(\sigma), \sigma) = 0. \end{aligned} \quad (5.11)$$

Moreover, (5.9) implies that

$$\begin{aligned} \frac{\partial G_1}{\partial s}(s_0(-e(\sigma), \sigma), \theta_1(-e(\sigma), \sigma), 1, -e(\sigma), \sigma) \\ = \frac{\partial G_1}{\partial s}(s_0(-e(\sigma), \sigma), \theta_0, \epsilon_0(-e(\sigma), \sigma), -e(\sigma), \sigma) < 0. \end{aligned} \quad (5.12)$$

We assume:

(I7) The equation  $G_1(s, \theta_1(\Psi, \sigma), 1, \Psi, \sigma)$  can be solved for  $s$  on  $J$  using the Implicit Function Theorem. The resulting function  $s = s_1(\Psi, \sigma)$  is required to satisfy  $s_1(-e(\sigma), \sigma) = s_0(-e(\sigma), \sigma)$ .

Eqns. (5.11) and (5.12) imply that  $s_1(\Psi, \sigma)$  can be defined near the line  $\{-e(\sigma)\} \times I \subset J$ . (I7) is the assumption that the Implicit Function Theorem can be used to extend this function to all of  $J$ . (Assumption (I7) plays the same role here as assumption (I4) in [20]).

We define the following equilibria of  $\frac{du}{dz} = G(u, \Psi, \sigma)$  with  $(\Psi, \sigma) \in J$ :

$$n(\Psi, \sigma) = (s, \theta, \epsilon) \text{ where } s = s_1(\Psi, \sigma), \theta = \theta_1(\Psi, \sigma), \text{ and } \epsilon = 1. \quad (5.13)$$

As  $\Psi$  increases from  $-e(\sigma)$  to 0, the  $\theta$ -coordinate of  $n(\Psi, \sigma)$  increases linearly from  $\theta_0$  to  $\theta_1(0, \sigma)$ .

Implicit in (I7) is the following assumption for all  $(\Psi, \sigma) \in J$ :

$$\frac{\partial G_1}{\partial s}(n(\Psi, \sigma), \Psi, \sigma) < 0. \tag{5.14}$$

The equilibria  $m(\Psi, \sigma)$  and  $n(\Psi, \sigma)$  are sketched in Figure 2. Notice that  $m(\Psi, \sigma) = n(\Psi, \sigma)$  at  $\Psi = -e(\sigma)$ .

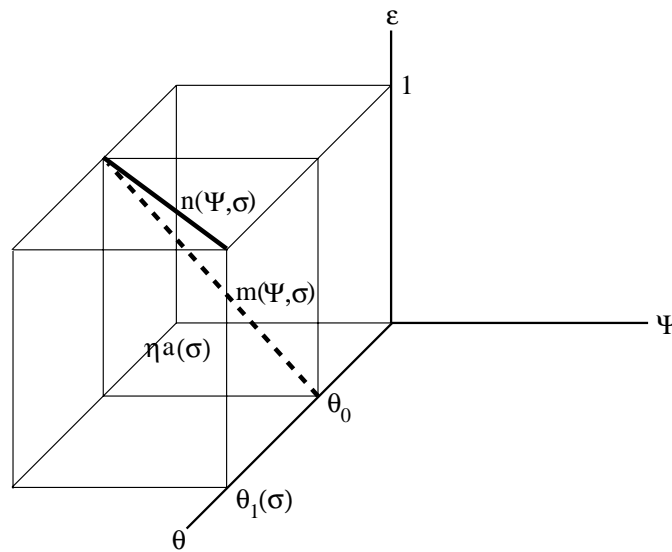


Figure 2. Equilibria of  $\dot{u} = G(u, \Psi, \sigma)$  for fixed  $\sigma$ . The picture has been projected onto  $(\theta, \epsilon, \Psi)$ -space.

For fixed  $(\Psi, \sigma)$ , the linearization of  $\dot{u} = G(u, \Psi, \sigma)$  at an equilibrium in  $S$  is given by the  $3 \times 3$  matrix  $M(u, \Psi, \sigma) = (M_{ij}(u, \Psi, \sigma))$ , with

$$\begin{aligned} M_{11} &= \frac{\partial G_1}{\partial s} = -\frac{1}{h}(\sigma - f_s v), \\ M_{12} &= \frac{\partial G_1}{\partial \theta} = \frac{1}{\rho_g h} \left( -\frac{\partial \rho_g}{\partial \theta}(\sigma s - f v) + \rho_g(f v_\theta + f_\theta v) \right), \\ M_{13} &= \frac{\partial G_1}{\partial \epsilon} = \frac{1}{\rho_g h} \left( -\frac{\partial \rho_g}{\partial \epsilon}(\sigma s - f v) + \rho_g f v_\epsilon \right), \\ M_{21} &= \frac{\partial G_2}{\partial s} = 0, \\ M_{22} &= \frac{\partial G_2}{\partial \theta} = -\frac{1}{\kappa}(B - \rho_g^+ AC(\epsilon)), \\ M_{23} &= \frac{\partial G_2}{\partial \epsilon} = \frac{1}{\kappa} D_\epsilon, \end{aligned}$$

$$\begin{aligned} M_{31} &= \frac{\partial G_3}{\partial s} = -K_\infty T_s (1 - \epsilon) e^{-\frac{L}{\theta - \theta_0}}, \\ M_{32} &= \frac{\partial G_3}{\partial \theta} = -K_\infty (1 - \epsilon) \left( T_\theta e^{-\frac{L}{\theta - \theta_0}} + T \frac{d}{d\theta} e^{-\frac{L}{\theta - \theta_0}} \right), \\ M_{33} &= \frac{\partial G_3}{\partial \epsilon} = K_\infty (T - T_\epsilon (1 - \epsilon)) e^{-\frac{L}{\theta - \theta_0}}. \end{aligned}$$

(Compare Eq. 5.6 of [10].)

At the equilibria  $m(\Psi, \sigma)$ , where  $\theta = \theta_0$ , the last row of  $M$  is all zeros. The eigenvalues are

$$\begin{aligned} M_{11}(m(\Psi, \sigma), \Psi, \sigma) &= \frac{\partial G_1}{\partial s}(m(\Psi, \sigma), \Psi, \sigma), \\ M_{22}(m(\Psi, \sigma), \Psi, \sigma) &= -\frac{1}{\kappa} (B(\sigma) - \rho_g^+ A(\sigma) C(\epsilon)) \text{ with } \epsilon = \epsilon_0(\Psi, \sigma), \end{aligned}$$

and 0. The first of these is negative by (5.9). The second is negative by assumption (I6).

At the equilibria  $n(\Psi, \sigma)$ , where  $\epsilon = 1$ , the the eigenvalues are

$$M_{11}(n(\Psi, \sigma), \Psi, \sigma) = \frac{\partial G_1}{\partial s}(n(\Psi, \sigma), \Psi, \sigma), \quad (5.15)$$

which is negative by (5.14);

$$M_{22}(n(\Psi, \sigma), \Psi, \sigma) = -\frac{1}{\kappa} (B(\sigma) - \rho_g^+ A(\sigma) C(1)) = -\frac{1}{\kappa} \tilde{B}, \quad (5.16)$$

which is negative by (I6); and

$$K_\infty T e^{-\frac{L}{\theta - \theta_0}} \text{ with } \theta = \theta_1(\Psi, \sigma), \quad (5.17)$$

which is positive for  $\Psi > -e(\sigma)$ . When  $\Psi = -e(\sigma)$ , we have  $\theta = \theta_0$ ,  $m(\Psi, \sigma)$  and  $n(\Psi, \sigma)$  coincide, and the positive eigenvalue at  $n(\Psi, \sigma)$  becomes 0.

## 6. Invariant manifolds of $\dot{u} = G(u, \Psi, \sigma)$

We now consider invariant manifolds of  $\dot{u} = G(u, \Psi, \sigma)$  in three-dimensional  $u$ -space for fixed  $\Psi$  and fixed  $\sigma \in I$ .

We first note that the plane  $\epsilon = 1$  is invariant for each  $(\Psi, \sigma)$ .

The point  $m(\Psi, \sigma)$  has a two-dimensional stable manifold tangent at  $m(\Psi, \sigma)$  to the plane  $\epsilon = \epsilon_0(\Psi, \sigma)$ . The point  $m(\Psi, \sigma)$  also has a (nonunique) one-dimensional center manifold. The flow of  $\dot{u} = G(u, \Psi, \sigma)$  near  $m(\Psi, \sigma)$  is determined by the flow on its center manifold.

An eigenvector for the eigenvalue 0 at  $m(\Psi, \sigma)$  is  $(X(\Psi, \sigma), Y(\Psi, \sigma), 1)$  with

$$Y(\Psi, \sigma) = -\frac{M_{23}(m(\Psi, \sigma), \Psi, \sigma)}{M_{22}(m(\Psi, \sigma), \Psi, \sigma)} > 0 \quad (6.1)$$

because, by assumptions (I3) and (I6),  $M_{23} > 0$  and  $M_{22} < 0$ . The sign of  $X$  will not be important.

Since  $\dot{\epsilon} < 0$  for  $\theta_0 < \theta$ , the flow on the branch of the center manifold of  $m(\Psi, \sigma)$  in the region  $\theta_0 < \theta$  is toward  $m(\Psi, \sigma)$ . Thus  $m(\Psi, \sigma)$  attracts nearby points that are on or above its stable manifold. See Figure 3.

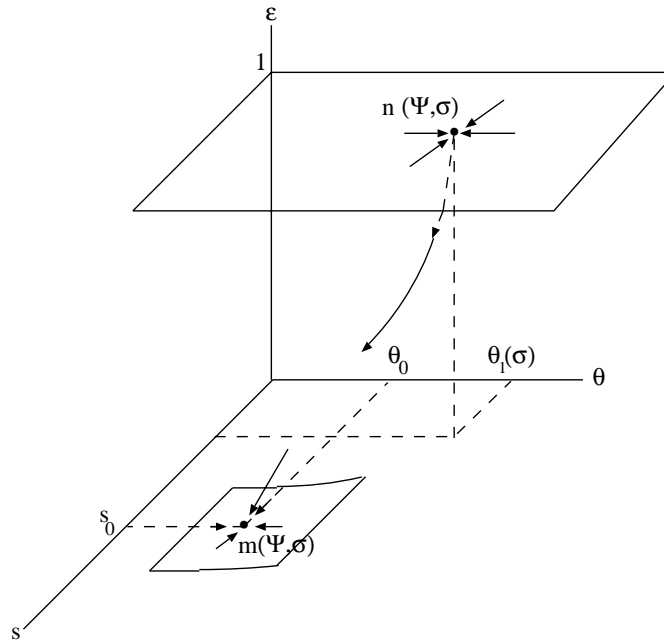


Figure 3. Flow of  $\dot{u} = G(u, \Psi, \sigma)$  for  $(\Psi, \sigma)$  fixed. Notice the two-dimensional stable manifold of  $m(\Psi, \sigma)$  and the one-dimensional center manifold.

Each point  $n(\Psi, \sigma)$  has a two-dimensional stable manifold, which is an open subset of the invariant plane  $\epsilon = 1$ .

For  $\Psi > -e(\sigma)$ , the point  $n(\Psi, \sigma)$  has a one-dimensional unstable manifold. Whether the unstable manifold of  $n(\Psi, \sigma)$  arrives at  $m(\Psi, \sigma)$ , and how it does so, depends on  $\Psi$  and  $\sigma$ . For  $d = 0$  and  $\Psi = 0$  this question is studied in [10].

We shall assume:

- (A1) There exists  $\sigma_0$  in  $I$  such that the lower branch of the unstable manifold of  $n(0, \sigma_0)$  lies in the stable manifold of  $m(0, \sigma_0)$ .

See [10] for a discussion of the generality with which this assumption holds.

We shall further assume:

- (A2) The connection of  $\dot{u} = G(u, \Psi, \sigma_0)$  between the unstable manifold of  $n(0, \sigma)$  and the stable manifold of  $m(0, \sigma)$  breaks in a nondegenerate manner as  $\sigma$  varies.

This assumption is equivalent to the nonvanishing of a Melnikov integral [20].

## 7. Main result

At this point we can state precisely the main result of this paper.

**Theorem 7.1.** *Let  $(s_0, \theta_0)$  and  $v^+$  be given. Let  $I$  be a  $\sigma$ -interval that satisfies (I1)–(I6), and assume there is a speed  $\sigma_0 \in I$  that satisfies (A1)–(A2). Then for small  $\delta > 0$ , the Traveling Wave System (4.16)–(4.19) has a solution in  $\mathcal{X}^4$  with speed  $\sigma(\delta) = \sigma_0 + O(\delta)$  that satisfies the boundary conditions (4.5) and the additional boundary condition  $\lim_{z \rightarrow \infty} \Psi(z) = 0$ . As  $z \rightarrow -\infty$ , the solution approaches  $(n(\Psi^-(\delta), \sigma(\delta)), \Psi^-(\delta))$  with  $\Psi^-(\delta) = -e(\sigma(\delta)) + O(e^{-\frac{k}{\delta}})$ . In particular,  $\epsilon^- = 1 - O(e^{-\frac{k}{\delta}})$ . As  $z \rightarrow \infty$ , the solution approaches  $(s_0, \theta_0, 0, 0)$  exponentially at a rate that is independent of  $\delta$ . As  $z \rightarrow -\infty$ , the solution approaches  $(n(\Psi^-(\delta), \sigma(\delta)), \Psi^-(\delta))$  exponentially at a rate that is  $O(\delta)$ .*

**Remark.** The expression for  $\epsilon^-$  shows that not all the oxygen is burned. The remaining oxygen may be significant for larger  $\delta$ .

The next five sections are devoted to completing the proof of this result.

## 8. Flow of $\dot{w} = F(w, \sigma, \delta)$ for $\delta = 0$

In this section we analyze the flow of  $\dot{w} = F(w, \sigma, \delta)$ ,  $w = (u, \Psi) = (s, \theta, \epsilon, \Psi)$ , for  $\sigma$  near  $\sigma_0$  and  $\delta = 0$ . The Traveling Wave System reduces to

$$\begin{aligned} \dot{u} &= G(u, \Psi, \sigma), \\ \dot{\Psi} &= 0. \end{aligned}$$

Thus the flow is that described in the Secs. 5 and 6, except that  $\Psi$  is regarded as a state variable rather than a parameter.

We have the following structures:

(1) Let  $\tilde{m}(\Psi, \sigma) = (m(\Psi, \sigma), \Psi)$  and  $\tilde{n}(\Psi, \sigma) = (n(\Psi, \sigma), \Psi)$ . We define the following curves of equilibria in  $w$ -space, each parameterized by  $\Psi$ :

$$\begin{aligned} M(\sigma) &= \{\tilde{m}(\Psi, \sigma) : -e(\sigma) \leq \Psi \leq 0\}, \\ N(\sigma) &= \{\tilde{n}(\Psi, \sigma) : -e(\sigma) \leq \Psi \leq 0\}. \end{aligned}$$

For small  $\nu > 0$ , we also define  $N_\nu(\sigma)$  to be the subset of  $N(\sigma)$  with  $-e(\sigma) + \nu \leq \Psi \leq 0$ .

(2) The plane  $\epsilon = 1$  in  $w$ -space is invariant under  $\dot{w} = F(w, \sigma, \delta)$  for each  $(\sigma, \delta)$ . For  $\delta = 0$ , within this three-dimensional plane, the curve of equilibria

$N(\sigma)$  is a normally hyperbolic (in fact, attracting) manifold. For an exposition of the theory of normally hyperbolic invariant manifolds, see [17].

(3) For each  $\tilde{m}(\Psi, \sigma)$  in  $M(\sigma)$ , define  $W^s(\tilde{m}(\Psi, \sigma))$  to be the set of all  $(u, \Psi)$  such that  $u$  is in the stable manifold of  $m(\Psi, \sigma)$ , and define  $W^c(\tilde{m}(\Psi, \sigma))$  to be the set of all  $(u, \Psi)$  such that  $u$  is in the center manifold of  $m(\Psi, \sigma)$ . The former is a two-dimensional manifold, and the latter is a one-dimensional manifold. Then the curve of equilibria  $M(\sigma)$ , regarded as a one-dimensional invariant manifold of  $\dot{w} = F(w, \sigma, 0)$ ,  $\sigma$  fixed, has a three-dimensional stable manifold  $W^s(M(\sigma))$ , which is the union of all  $W^s(\tilde{m}(\Psi, \sigma))$  as  $\Psi$  varies, and a two-dimensional center manifold  $W^c(M(\sigma))$ , which is the union of all  $W^c(\tilde{m}(\Psi, \sigma))$  as  $\Psi$  varies.

(4) For each  $\tilde{n}(\Psi, \sigma)$  in  $N(\sigma)$ , define  $W^s(\tilde{n}(\Psi, \sigma))$  to be the set of all  $(u, \Psi)$  such that  $u$  is in the stable manifold of  $n(\Psi, \sigma)$ .  $W^s(\tilde{n}(\Psi, \sigma))$  is an open subset of the two-dimensional plane  $\epsilon = 1$ ,  $\Psi$  fixed. Thus the curve of equilibria  $N(\sigma)$ , regarded as a one-dimensional invariant manifold of  $\dot{w} = F(w, \sigma, 0)$ ,  $\sigma$  fixed, is normally attracting within the plane  $\epsilon = 1$ .

(5) For each  $\tilde{n}(\Psi, \sigma)$  in  $N(\sigma)$  with  $-e(\sigma) < \Psi$  define  $W^u(\tilde{n}(\Psi, \sigma))$  to be the set of all  $(u, \Psi)$  such that  $u$  is in the unstable manifold of  $n(\Psi, \sigma)$ . This set is a one-dimensional manifold. Then for each small  $\nu > 0$ , the curve of equilibria  $N_\nu(\sigma)$ , regarded as a one-dimensional invariant manifold of  $\dot{w} = F(w, \sigma, 0)$ ,  $\sigma$  fixed, is normally hyperbolic. It has a three-dimensional stable manifold  $W^s(N_\nu(\sigma))$ , which is the union of all  $W^s(\tilde{n}(\Psi, \sigma))$  with  $-e(\sigma) + \nu \leq \Psi \leq 0$ , and a two-dimensional unstable manifold  $W^u(N_\nu(\sigma))$ , which is the union of all  $W^u(\tilde{n}(\Psi, \sigma))$  with  $-e(\sigma) + \nu \leq \Psi \leq 0$ .

We shall now consider the intersection of the two-dimensional manifold  $W^u(N_\nu(\sigma))$  and the two-dimensional manifold  $W^s(\tilde{m}(0, \sigma))$  in four-dimensional  $w$ -space. Notice that  $W^u(N_\nu(\sigma_0))$  and  $W^s(\tilde{m}(0, \sigma_0))$  meet along the connecting orbit from  $\tilde{n}(0, \sigma_0)$  to  $\tilde{m}(0, \sigma_0)$ .

**Proposition 8.1.** *The intersection of the two-dimensional manifold  $W^u(N_\nu(\sigma))$  and the two-dimensional manifold  $W^s(\tilde{m}(0, \sigma))$  in four-dimensional  $w$ -space, which for  $\sigma = \sigma_0$  is the connecting orbit from  $\tilde{n}(0, \sigma_0)$  to  $\tilde{m}(0, \sigma_0)$ , breaks in a nondegenerate manner as  $\sigma$  varies.*

This proposition follows from assumption (A2) and is proved in [20].

## 9. Flow of $\dot{w} = F(w, \sigma, \delta)$ for $\delta > 0$ : fast connection

We now begin to analyze the flow of  $\dot{w} = F(w, \sigma, \delta)$  for  $\sigma$  near  $\sigma_0$  and small  $\delta > 0$ .

(1)  $M(\sigma)$  remains as a curve of equilibria. Linearization shows that for  $\delta > 0$ ,  $M(\sigma)$  has a three-dimensional stable manifold and a two-dimensional unstable



manifold. The former is close to the stable manifold of  $M(\sigma)$  for  $\delta = 0$ ; the latter is close to the center manifold of  $M(\sigma)$  for  $\delta = 0$ . (These facts follow from the Center Manifold Theorem [22].)

(2) The plane  $\epsilon = 1$  remains invariant. Thus near  $N(\sigma)$  there is, in the plane  $\epsilon = 1$ , an invariant curve  $N(\sigma, \delta)$  which is hyperbolically attracting within that plane.

(3) Near  $N_\nu(\sigma)$  is a normally hyperbolic invariant curve  $N_\nu(\sigma, \delta)$ , which can be taken to be the set of points in  $N(\sigma, \delta)$  with  $-e(\sigma) + \nu \leq \Psi$ .  $N_\nu(\sigma, \delta)$  has a three-dimensional stable manifold  $W^s(N_\nu(\sigma, \delta))$  and a two-dimensional unstable manifold  $W^u(N_\nu(\sigma, \delta))$ . The former is the closure of an open subset of the plane  $\epsilon = 1$ . The latter is close to  $W^u(N_\nu(\sigma))$ . (This fact follows from the theory of normally hyperbolic invariant manifolds [17].) From (4.19), the flow along  $N_\nu(\sigma)$  is in the direction of increasing  $\Psi$ .

**Proposition 9.1.** *For each small  $\delta \geq 0$  there is a unique speed  $\sigma(\delta)$  near  $\sigma_0$  such that  $W^u(N_\nu(\sigma(\delta), \delta))$  contains a solution  $w_\delta(z)$  that approaches  $m(0, \sigma(\delta))$  as  $z \rightarrow \infty$ . The function  $\sigma(\delta)$  is smooth, and  $\sigma(0) = \sigma_0$ .*

This proposition is an easy consequence of Proposition 8.1 and is proved in [20].

The solution  $w_\delta(z)$  is shown in Figure 4. At  $\Psi \approx 0$ , in backward time, the solution quickly moves from the point  $\hat{m}(0, \sigma(\delta))$  toward the invariant curve  $N_\nu(\sigma(\delta), \delta)$ . The solution then drifts slowly along  $N_\nu(\sigma(\delta), \delta)$  in the direction of decreasing  $\Psi$ .

## 10. Flow of $\dot{w} = F(w, \sigma, \delta)$ for $\delta > 0$ : slow drift

As the solution  $w_\delta(z)$  drifts slowly along  $N_\nu(\sigma(\delta), \delta)$  in backward time, it is attracted to this curve. The result of this attraction is described by the following proposition.

**Proposition 10.1.** *There is a constant  $k > 0$  such that for small  $\delta > 0$ , the solution  $w_\delta(z)$  arrives at  $\Psi = -e(\sigma) + \nu$  within  $O(e^{-\frac{k}{\delta}})$  of  $N_\nu(\sigma(\delta), \delta)$ .*

For the proof see [20]. The proof shows that the width of the cooling zone is at least of order  $\frac{1}{\delta}$ .

## 11. Behavior of the connecting orbit as $z \rightarrow -\infty$

To see how the connecting orbit behaves as  $z \rightarrow -\infty$ , we shall use center manifold reduction.

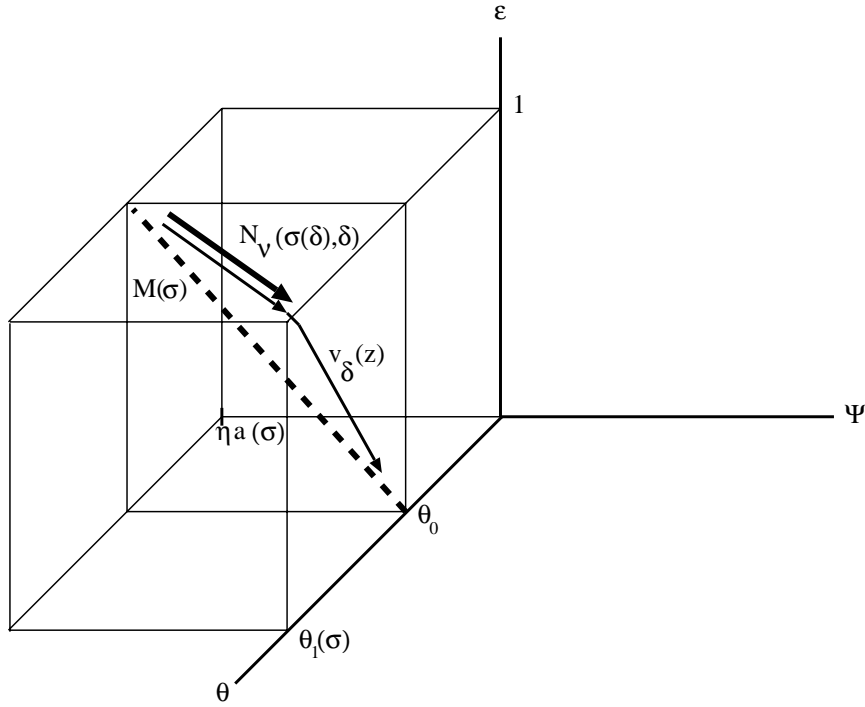


Figure 4. The solution  $w_\delta(z)$ .

We first make a parameter-dependent shift of coordinates in  $w$ -space. Let

$$r = s - s_0(\Psi, \sigma), \tag{11.1}$$

$$\eta = \theta - \theta_0, \tag{11.2}$$

$$\gamma = \epsilon - \epsilon_0(\Psi, \sigma), \tag{11.3}$$

$$\Phi = \Psi + e(\sigma). \tag{11.4}$$

This transformation takes

$$\{(u, \Psi, \sigma, \delta) : u = m(\Psi, \sigma)\} \tag{11.5}$$

to the subspace  $r = \eta = \gamma = 0$  in  $(r, \eta, \gamma, \Phi, \sigma, \delta)$ -space, and takes  $\{(u, \Psi, \sigma, \delta) : u = m(\Psi, \sigma), \Psi = -e(\sigma)\}$  to the subspace  $r = \eta = \gamma = \Phi = 0$ .

Since we are studying the connecting orbit, we shall assume throughout this section that  $\sigma = \sigma(\delta)$ . Then in  $(r, \eta, \gamma, \Phi, \delta)$ -coordinates, with  $\sigma = \sigma(\delta)$ , the Traveling Wave System becomes

$$\dot{r} = G_1 - \frac{\partial s_0}{\partial \Psi} \delta \eta, \tag{11.6}$$

$$\dot{\eta} = G_2, \tag{11.7}$$

$$\dot{\gamma} = G_3 - \frac{\partial \epsilon_0}{\partial \Psi} \delta \eta, \quad (11.8)$$

$$\dot{\Phi} = \delta \eta, \quad (11.9)$$

where the  $G_i$  are evaluated at

$$s = s_0(\Psi, \sigma) + r,$$

$$\theta = \theta_0 + \eta,$$

$$\epsilon = \epsilon_0(\Psi, \sigma) + \gamma,$$

$$\Psi = -e(\sigma) + \Phi,$$

$$\sigma = \sigma(\delta),$$

and  $\frac{\partial s_0}{\partial \Psi}(\Psi, \sigma)$  and  $\frac{\partial \epsilon_0}{\partial \Psi}(\Psi, \sigma)$  are evaluated at

$$\Psi = -e(\sigma) + \Phi,$$

$$\sigma = \sigma(\delta).$$

We add the equation

$$\dot{\delta} = 0, \quad (11.10)$$

and we regard (11.6)–(11.10) as a five-dimensional system.

There are equilibria where  $r = \eta = \gamma = 0$ ,  $\Phi$  and  $\delta$  arbitrary; these correspond to the set (11.5). The manifold  $\gamma = 1 - \epsilon_0(\Psi, \sigma)$ , with  $\Psi = -e(\sigma) + \Phi$  and  $\sigma = \sigma(\delta)$ , is invariant. This corresponds to invariance of the plane  $\epsilon = 1$  under  $\dot{w} = F(w, \sigma, \delta)$ .

Since  $\epsilon_0(-e(\sigma), \sigma) = 1$  by (5.3), we can write

$$\epsilon_0(-e(\sigma(\delta)) + \Phi, \sigma(\delta)) = 1 - \Phi \tilde{\epsilon}(\Phi, \delta), \quad (11.11)$$

where  $\tilde{\epsilon}$  is a smooth function. From (11.11),

$$\frac{\partial}{\partial \Phi}(\Phi \tilde{\epsilon}(\Phi, \delta)) = -\frac{\partial \epsilon_0}{\partial \Psi}(-e(\sigma(\delta)) + \Phi, \sigma(\delta)) > 0. \quad (11.12)$$

In particular, setting  $(\Phi, \delta) = (0, 0)$  and recalling that  $\sigma(0) = \sigma_0$ , we find that

$$\tilde{\epsilon}(0, 0) = -\frac{\partial \epsilon_0}{\partial \Psi}(-e(\sigma_0), \sigma_0) > 0 \quad (11.13)$$

by (5.2).

Let us linearize (11.6)–(11.10) around the equilibrium at the origin. We have

$$\frac{d}{dz} \begin{pmatrix} r \\ \eta \\ \gamma \\ \Phi \\ \delta \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & 0 & 0 \\ 0 & M_{22} & M_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ \eta \\ \gamma \\ \Phi \\ \delta \end{pmatrix}, \quad (11.14)$$

where the  $M_{ij}$  are evaluated at  $(u, \Psi, \sigma) = (m(-e(\sigma_0), \sigma_0), -e(\sigma_0), \sigma_0)$ . The eigenvalues of this matrix are the negative numbers

$$M_{11}(m(-e(\sigma_0), \sigma_0), -e(\sigma_0), \sigma_0) \text{ and } M_{22}(m(-e(\sigma_0), \sigma_0), -e(\sigma_0), \sigma_0),$$

each with multiplicity one, and 0 with algebraic and geometric multiplicity three. A basis for the eigenspace of the eigenvalue 0 is

$$\{(X(-e(\sigma_0), \sigma_0), Y(-e(\sigma_0), \sigma_0), 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}, \tag{11.15}$$

where  $X$  and  $Y$  were defined in Sec. 6. The three-dimensional center manifold of (11.6)–(11.10) at the origin is therefore parameterized by  $(\gamma, \Phi, \delta)$ , and is given by

$$r = \gamma \tilde{X}(\gamma, \Phi, \delta), \quad \eta = \gamma \tilde{Y}(\gamma, \Phi, \delta), \tag{11.16}$$

with  $\tilde{X}(0, 0, 0) = X(-e(\sigma_0), \sigma_0)$  and  $\tilde{Y}(0, 0, 0) = Y(-e(\sigma_0), \sigma_0)$ . The factor  $\gamma$  in (11.16) is due to the family of equilibria  $r = \eta = \gamma = 0$ , which must lie in the center manifold.

Substitution of (11.16) and (11.11) into (11.8)–(11.10) yields the flow on the center manifold in  $(\gamma, \Phi, \delta)$ -coordinates, which we shall refer to as *center manifold coordinates*:

$$\dot{\gamma} = K_\infty T(\gamma - \Phi \tilde{\epsilon}) e^{-L/\gamma \tilde{Y}} + \frac{\partial}{\partial \Phi}(\Phi \tilde{\epsilon}) \delta \gamma \tilde{Y}, \tag{11.17}$$

$$\dot{\Phi} = \delta \gamma \tilde{Y}, \tag{11.18}$$

$$\dot{\delta} = 0. \tag{11.19}$$

The plane  $\gamma = 0$  consists of equilibria, and the surface  $\gamma = \Phi \tilde{\epsilon}(\Phi, \delta)$  is invariant. The latter fact corresponds to invariance of the plane  $\epsilon = 1$  for the Traveling Wave System.

If we restrict to the plane  $\delta = 0$ , then the line  $\gamma = 0$  consists of equilibria with two zero eigenvalues, and the curve  $\gamma = \Phi \tilde{\epsilon}(\Phi, 0)$  also consists of equilibria. For  $\Phi > 0$  these equilibria have one zero eigenvalue and one positive eigenvalue. The lines  $\Phi = c$  are invariant. The flow on the two-dimensional slice of the center manifold with  $\delta = 0$ , near  $(\gamma, \Phi) = (0, 0)$ , is shown in Figure 5(a).

The flow for fixed  $\delta > 0$  is shown in Figure 5(b). The line  $\gamma = 0$  still consists of equilibria, but now one eigenvalue is 0 and the other is

$$\frac{\partial \dot{\gamma}}{\partial \gamma}(0, \Phi, \delta) = \frac{\partial}{\partial \Phi}(\Phi \tilde{\epsilon})(0, \delta) \delta \tilde{Y}, \tag{11.20}$$

which is positive by Eqs. (11.12) and (6.1). The curve  $\gamma = \Phi \tilde{\epsilon}(\Phi, \delta)$  is now the unstable manifold of the origin. This line corresponds, under the coordinate changes, to part of the invariant manifold  $N(\sigma(\delta), \delta)$  for the Traveling Wave System. The portion of this line with  $\Phi \geq \nu$  corresponds to part of  $N_\nu(\sigma(\delta), \delta)$ . The region  $\gamma > 0, \Phi \geq \nu$  corresponds to part of the unstable manifold of  $N_\nu(\sigma(\delta), \delta)$ . Notice that for  $\gamma > 0, \dot{\Phi}$  is positive, so the flow in Figure 5(b) is upward.

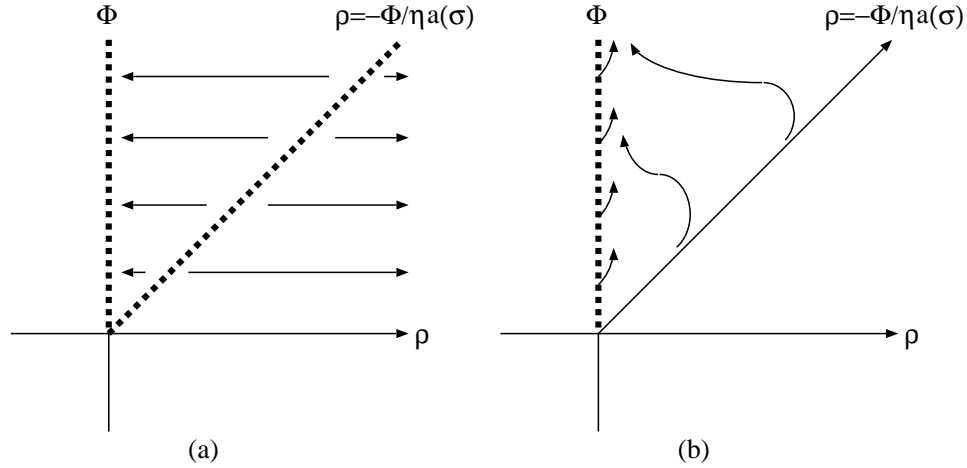


Figure 5. Flow on the center manifold for (a)  $\delta = 0$  and (b)  $\delta > 0$ .

In center manifold coordinates, the solution  $w_\delta(z)$  of the Traveling Wave System that is given by Proposition 9.1 meets the plane  $\Phi = \nu$  at the point  $(\gamma, \Phi, \delta)$  with

$$(\gamma, \Phi) = (\nu\tilde{\epsilon}(\nu, \delta) - p(\delta), \nu). \quad (11.21)$$

By Proposition 10.1,  $p(\delta)$  is  $O(e^{-\frac{k}{\delta}})$ .

**Proposition 11.1.** *As  $z \rightarrow -\infty$ ,  $w_\delta(z)$  approaches, in center manifold coordinates, a point  $(0, \Phi_0(\delta))$  with  $0 < \Phi_0(\delta)$  and  $\Phi_0(\delta) = O(e^{-\frac{k}{\delta}})$ .*

*Proof.* Since  $\dot{\Phi} > 0$ , and  $\gamma = 0$  and  $\gamma = \Phi\tilde{\epsilon}(\Phi, \delta)$  are invariant,  $\Phi$  decreases in backward time to a limit. Since the only invariant sets with  $\Phi$  constant are points on the  $\Phi$ -axis, it follows that the solution converges in backward time to a point  $(0, \Phi_0)$  with  $\Phi_0 > 0$ .

From (11.17)–(11.18), in the region  $0 < \gamma < \Phi\tilde{\epsilon}(\Phi, \delta)$ ,  $0 < \Phi$ ,

$$\frac{d\gamma}{d\Phi} < \frac{\frac{\partial}{\partial\Phi}(\Phi\tilde{\epsilon})\delta\gamma\tilde{Y}}{\delta\gamma\tilde{Y}} = \frac{\partial}{\partial\Phi}(\Phi\tilde{\epsilon}). \quad (11.22)$$

Now the solution of

$$\frac{d\gamma}{d\Phi} = \frac{\partial}{\partial\Phi}(\Phi\tilde{\epsilon}) \quad (11.23)$$

that passes through the point (11.21) is the curve

$$\gamma = \Phi\tilde{\epsilon}(\Phi, \delta) - p(\delta),$$

which meets the  $\Phi$ -axis where

$$\Phi \tilde{\epsilon}(\Phi, \delta) - p(\delta) = 0. \quad (11.24)$$

Now at  $(\Phi, \delta) = (0, 0)$ , the left-hand side of (11.24) equals 0, and its partial derivative with respect to  $\Phi$  is  $\tilde{\epsilon}(0, 0)$ , which is positive by (11.13). Therefore (11.24) can be solved for  $\Phi$  near  $(\Phi, \delta) = (0, 0)$  by the Implicit Function Theorem as  $\Phi = \Phi_1(\delta)$  with  $\Phi_1(0) = 0$ . More precisely,

$$\Phi_1(\delta) = \frac{p(\delta)}{\tilde{\epsilon}(\Phi_1(\delta), \delta)} = O(e^{-\frac{\kappa}{\delta}})$$

since the denominator is positive at  $\delta = 0$ . The inequality (11.22) implies that  $\Phi_0(\delta) < \Phi_1(\delta)$ .  $\square$

## 12. Completion of the proof of the main result

In this section we complete the proof of Theorem 7.1.

We have constructed a solution  $w_\delta(z) = (s(z), \theta(z), \epsilon(z), \Psi(z))$  of the Traveling Wave System with  $\sigma = \sigma(\delta) = \sigma_0 + O(\delta)$ . Since  $w_\delta(z)$  lies in  $W^s(\tilde{m}(0, \sigma(\delta)), \delta)$ , which is close to  $W^s(\tilde{m}(0, \sigma_0), 0)$ ,  $w_\delta(z)$  approaches  $(s, \theta, \epsilon, \Psi) = (s_0, \theta_0, 0, 0)$  as  $z \rightarrow \infty$  at an exponential rate that is independent of  $\delta$ .

As  $z \rightarrow -\infty$ , in center manifold coordinates,  $w_\delta(z)$  approaches  $(\gamma, \Phi, \delta)$  with  $(\gamma, \Phi) = (0, \Phi_0(\delta))$ . Using (11.1)–(11.4) and (11.16), we see that  $\lim_{z \rightarrow -\infty} w_\delta(z)$  is as stated in the theorem. Since the positive eigenvalue of the system (11.17)–(11.19) at an equilibrium  $(0, \Phi)$  is  $O(\delta)$  by (11.20), the desired solution of the Traveling Wave System is just  $(s(z), \theta(z), \epsilon(z))$ .

## 13. Nonexistence of the traveling wave for large heat loss

In this section we prove the following result. It states that if the quantity  $\kappa$  defined by (2.14) is taken to be a constant, and if the rate of heat loss to the surrounding rock formation is sufficiently large (*i.e.*, if  $\delta$  is sufficiently large), then traveling combustion waves with physical speed (*i.e.*, contained in a given bounded subinterval of  $I$ ) cannot occur.

**Theorem 13.1.** *In the Traveling Wave System take  $\kappa$  to be a positive constant. Let  $(s_0, \theta_0)$  and  $v_+$  be given. Let  $\hat{I}$  be a bounded  $\sigma$ -interval on which (I1)–(I6) are satisfied. Then for  $\delta$  sufficiently large, any solution of the Traveling Wave System in  $\mathcal{X}^4$  that (1) has  $0 < s(z) < 1$  and  $0 \leq \epsilon(z) \leq 1$  for all  $z$ , (2) satisfies the boundary conditions (4.5) and the additional boundary condition  $\lim_{z \rightarrow \infty} \Psi(z) = 0$ , and (3) has  $\sigma \in \hat{I}$ , must have  $(\theta(z), \epsilon(z), \Psi(z)) = (\theta_0, 0, 0)$  for all  $z$ .*

To prove this result, let  $(s(z), \theta(z), \epsilon(z), \Psi(z))$  be a solution of the Traveling Wave System in  $\mathcal{X}^4$  that satisfies (1)–(3). We shall regard  $s(z)$  and  $\epsilon(z)$  as given, and we shall show that for  $\delta$  sufficiently large, it must be the case that  $\theta(z) = \theta_0$  for all  $z$ . Then Eqs. (4.18) and (4.19) imply that  $\epsilon(z)$  and  $\Psi(z)$  are also constant, and the boundary conditions imply that they are 0.

Let

$$\eta = \theta - \theta_0, \quad (13.1)$$

$$\zeta = \dot{\eta}. \quad (13.2)$$

Substituting (13.1) into (4.17), differentiating (4.17) with respect to  $z$ , and writing the resulting second-order differential equation as a system yields

$$\dot{\eta} = \zeta, \quad (13.3)$$

$$\begin{aligned} \dot{\zeta} &= \frac{1}{\kappa} \left( -B\dot{\eta} + \dot{\Psi} + \frac{\partial D}{\partial \theta} \dot{\eta} + \frac{\partial D}{\partial \epsilon} \dot{\epsilon} \right) = \frac{1}{\kappa} \left( -B\zeta + \delta\eta + \frac{\partial D}{\partial \theta} \zeta - \frac{\partial D}{\partial \epsilon} Tq \right) \\ &= \frac{1}{\kappa} (\delta\eta - B\zeta + N(z, \eta, \zeta, \sigma)), \end{aligned} \quad (13.4)$$

where

$$\begin{aligned} N(z, \eta, \zeta, \sigma) &= \\ \frac{\partial D}{\partial \theta} (\theta_0 + \eta, \epsilon(z), \sigma) \zeta - \frac{\partial D}{\partial \epsilon} (\theta_0 + \eta, \epsilon(z), \sigma) T(s(z), \theta_0 + \eta, \epsilon(z), \sigma) q(\theta_0 + \eta, \epsilon(z))). \end{aligned} \quad (13.5)$$

(Recall that  $\kappa$  is taken to be constant, and that  $s(z)$  and  $\epsilon(z)$  are given.)

Since  $0 < s(z) < 1$  and  $0 \leq \epsilon(z) \leq 1$  for all  $z$ , it is easy to see from Eqs. (13.5), (4.12), (4.14), (3.18), and (2.9) that there is a constant  $C_1$  such that

$$\left| \frac{\partial N}{\partial \eta}(z, \eta, \zeta, \sigma) \right| \leq C_1 \quad \text{and} \quad \left| \frac{\partial N}{\partial \zeta}(z, \eta, \zeta, \sigma) \right| \leq C_1$$

for all  $(z, \eta, \zeta, \sigma)$  with  $\sigma \in \hat{\mathbb{I}}$ .

Let  $y = (y_1, y_2) = (\eta, \zeta)$ . Then (13.3)–(13.4) become the system

$$\frac{dy}{dz} = Ay + \left(0, \frac{1}{\kappa} N(z, y, \sigma)\right) \quad (13.6)$$

with

$$A = \begin{pmatrix} 0 & 1 \\ \frac{\delta}{\kappa} & -\frac{B}{\kappa} \end{pmatrix}. \quad (13.7)$$

The eigenvalues of  $A$  are

$$\lambda_{\pm} = \frac{1}{2\kappa} \left( -B \pm \sqrt{B^2 + 4\kappa\delta} \right) \quad (13.8)$$

with  $\lambda_+ > 0$  and  $\lambda_- < 0$ . Corresponding eigenvectors are  $(1, \lambda_{\pm})$ . Let

$$\Lambda = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 \\ \lambda_- & \lambda_+ \end{pmatrix}, \quad U^{-1} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} \lambda_+ & -1 \\ -\lambda_- & 1 \end{pmatrix}, \quad (13.9)$$

so that  $U^{-1}AU = \Lambda$ . Notice that  $A$ ,  $\lambda_{\pm}$ ,  $\Lambda$ , and  $U$  are all functions of  $\sigma$  and  $\delta$ . Let  $y = Ux$ . Then (13.6) becomes

$$\dot{x} = \Lambda x + P(z, x, \sigma, \delta), \tag{13.10}$$

where

$$\begin{aligned} P(z, x_1, x_2, \sigma, \delta) &= U^{-1} \left( 0, \frac{1}{\kappa} N(z, Ux, \sigma) \right) \\ &= \frac{1}{\kappa(\lambda_+ - \lambda_-)} N(z, x_1 + x_2, \lambda_- x_1 + \lambda_+ x_2, \sigma)(-1, 1). \end{aligned}$$

Let  $C(\mathbb{R}, \mathbb{R}^\ell)$  denote the Banach space of bounded continuous functions from  $\mathbb{R}$  to  $\mathbb{R}^\ell$ ,  $\ell = 1, 2$ . In  $C(\mathbb{R}, \mathbb{R})$  the norm is  $\|k\| = \sup(|k(z)| : z \in \mathbb{R})$ . In  $\mathbb{R}^2$  we use the norm  $\|(x_1, x_2)\| = \max(|x_1|, |x_2|)$ , and we use the corresponding  $C(\mathbb{R}, \mathbb{R}^2)$ :

$$\|x\| = \sup(\|x(z)\| : z \in \mathbb{R}) = \sup(\|(x_1(z), x_2(z))\| : z \in \mathbb{R}) = \max(\|x_1\|, \|x_2\|).$$

The following lemma is an easy consequence of the Variation of Constants formula.

**Lemma 13.1.** *Let  $h(z) = (h_1(z), h_2(z)) \in C(\mathbb{R}, \mathbb{R}^2)$ . Then the only bounded solution of  $\dot{x} = \Lambda x + h$  is  $x(z) = (x_1(z), x_2(z))$  with*

$$x_1(z) = \int_{-\infty}^z e^{\lambda_-(z-s)} h_1(s) ds, \quad x_2(z) = \int_{-\infty}^z e^{\lambda_+(z-s)} h_2(s) ds.$$

In addition,  $\|x_1\| \leq -\frac{1}{\lambda_-} \|h_1\|$  and  $\|x_2\| \leq \frac{1}{\lambda_+} \|h_2\|$ .

Using Lemma 13.1, define a linear mapping  $L$  from  $C(\mathbb{R}, \mathbb{R}^2)$  to itself by  $Lh = x$ . We also define a mapping

$$\hat{P} : C(\mathbb{R}, \mathbb{R}^2) \times \hat{I} \times \mathbb{R}_+ \rightarrow C(\mathbb{R}, \mathbb{R}^2)$$

by  $\hat{P}(x, \sigma, \delta)(z) = P(z, x(z), \sigma, \delta)$ . From (13.10), if, for some  $\sigma$ ,  $(s(z), \theta(z), \epsilon(z), \Psi(z))$  is a solution of the Traveling Wave System in  $\mathcal{X}^4$ , and  $x(z)$  is related to  $(s(z), \theta(z), \epsilon(z), \Psi(z))$  as described in this section, then we must have  $x = L\hat{P}(x, \sigma, \delta)$ , i.e.,  $x$  must be a fixed point of the mapping  $L\hat{P}(\cdot, \sigma, \delta)$  for the appropriate  $\sigma$  and  $\delta$ .

Let  $P_i(z, x, \sigma, \delta)$  denote the  $i$ th component of  $P(z, x, \sigma, \delta)$ ,  $i = 1, 2$ . We have, for each  $i$  and for each  $x$  and  $\tilde{x}$  in  $\mathbb{R}^2$ ,

$$\begin{aligned} &|P_i(z, x, \sigma, \delta) - P_i(z, \tilde{x}, \sigma, \delta)| \\ &\leq \frac{1}{\kappa(\lambda_+ - \lambda_-)} (C_1|x_1 + x_2 - \tilde{x}_1 - \tilde{x}_2| + C_1|\lambda_- x_1 + \lambda_+ x_2 - \lambda_- \tilde{x}_1 - \lambda_+ \tilde{x}_2|) \\ &\leq \frac{C_1}{\kappa(\lambda_+ - \lambda_-)} (|x_1 - \tilde{x}_1| + |x_2 - \tilde{x}_2| - \lambda_-|x_1 - \tilde{x}_1| + \lambda_+|x_2 - \tilde{x}_2|) \end{aligned}$$



$$\leq \frac{C_1}{\kappa(\lambda_+ - \lambda_-)} (2 + \lambda_+ - \lambda_-) \|x - \tilde{x}\| = C_1 \left( \frac{2}{\sqrt{B^2 + 4\kappa\delta}} + \frac{1}{\kappa} \right) \|x - \tilde{x}\|.$$

Therefore if  $x$  and  $\tilde{x}$  are in  $C(\mathbb{R}, \mathbb{R}^2)$ ,

$$\|\hat{P}(x, \sigma, \delta) - \hat{P}(\tilde{x}, \sigma, \delta)\| \leq C_1 \left( \frac{2}{\sqrt{B^2 + 4\kappa\delta}} + \frac{1}{\kappa} \right) \|x - \tilde{x}\|.$$

Since  $0 < \lambda_+ < -\lambda_-$ , Lemma 13.1 implies that  $\|L\| = \frac{1}{\lambda_+}$ . Therefore

$$\|LP(x, \sigma, \delta) - LP(\tilde{x}, \sigma, \delta)\| \leq \frac{C_1}{\lambda_+} \left( \frac{2}{\sqrt{B^2 + 4\phi\kappa\delta}} + \frac{1}{\phi\kappa} \right) \|x - \tilde{x}\|.$$

As  $\delta \rightarrow \infty$ ,  $\lambda_+ \rightarrow \infty$  uniformly on  $\hat{I}$ . Therefore, for  $\delta$  sufficiently large,  $LP(\cdot, \sigma, \delta)$  is a contraction of  $C(\mathbb{R}, \mathbb{R}^2)$  for each  $(\sigma, \delta)$ .

Since  $q(\theta_0, \epsilon) = 0$  for all  $\epsilon$ ,  $N(z, 0, 0, \sigma) = 0$  for all  $z$  and  $\sigma$ , so  $P(z, 0, \sigma, \delta) = 0$  for all  $z$ ,  $\sigma$ , and  $\delta$ . Therefore  $x = 0$  is a fixed point of  $LP(\cdot, \sigma, \delta)$  for all  $\sigma$  and  $\delta$ . For  $\sigma \in \hat{I}$  and  $\delta$  sufficiently large,  $LP(\cdot, \sigma, \delta)$  is a contraction, so  $x = 0$  is the only fixed point. Since  $x(z) = 0$  implies  $\theta(z) = \theta_0$ , the result follows.

#### 14. Calculation of flow velocities

In this work we have fixed  $v^+$  and determined  $\sigma(\delta)$  and  $\epsilon^-(\delta)$  (Theorem 7.1). One can then find  $v^-(\delta)$  using (4.15). Recall that  $\theta^- = \theta_0$ .

In practical applications, one usually specifies the injected flow  $v^{inj} = v^-$ . In order to discuss this situation, let us treat  $v^+$  as a variable. Then we can use Theorem 7.1 to find  $\sigma(\delta, v^+)$  and  $\epsilon^-(\delta, v^+)$ . Of course,  $\theta^-(\delta, v^+) = \theta_0$ .

Now think of (4.15) as giving  $v$  as a function of  $\theta$ ,  $\epsilon$ ,  $\sigma$ , and  $v^+$ . Then

$$v^-(\delta, v^+) = v(\theta_0, \epsilon^-(\delta, v^+), \sigma(\delta, v^+), v^+). \quad (14.1)$$

For each  $\delta$  we would like to invert  $v^-(\delta, v^+)$  to obtain  $v^+(\delta, v^-)$ . Then we could find traveling waves with  $v^-$  given instead of  $v^+$ .

The function  $v^-(\delta, v^+)$  can be inverted for fixed  $\delta$  provided  $\frac{\partial v^-}{\partial v^+} > 0$ . We have

$$\frac{\partial v^-}{\partial v^+} = \frac{\partial v}{\partial \epsilon} \frac{\partial \epsilon^-}{\partial v^+} + \frac{\partial v}{\partial \sigma} \frac{\partial \sigma}{\partial v^+} + \frac{\partial v}{\partial v^+}. \quad (14.2)$$

(For  $\delta > 0$ , all terms are differentiable.)

We will not attempt to give precise conditions under which (14.2) is positive. However, we note that if  $c = d = 0$ , (4.15) says that  $v(\theta, \epsilon, \sigma) \equiv v^+$ . Thus, for  $c = d = 0$ ,  $\frac{\partial v}{\partial \epsilon} = 0$ ,  $\frac{\partial v}{\partial \sigma} = 0$ , and  $\frac{\partial v}{\partial v^+} = 1$ . Therefore  $\frac{\partial v^-}{\partial v^+} = 1$ . Hence, for  $\delta > 0$  fixed and  $c$  and  $d$  close to 0,  $\frac{\partial v^-}{\partial v^+} \approx 1$  by continuity.

We also note that from Eqs. (4.15) and (3.18),

$$\frac{\partial v}{\partial v^+}(\theta_0, \epsilon, \sigma, v^+) = \left( \frac{a + c\epsilon}{a + d\epsilon} - 1 \right) f^+ + 1. \quad (14.3)$$

Since  $(a + c\epsilon)/(a + d\epsilon)$  is positive, and  $0 \leq f^+ \leq 1$  by (3.4), we see that the term (14.3) in (14.2) is positive.

If  $v^+(\delta, v^-)$  cannot be inverted for fixed  $\delta$ , then there could exist several different oxidation fronts for the same pre-ignition data.

## 15. Discussion

In this work we have considered the existence of oxidation pulses excited in a petroleum reservoir originally under oxygen or air injection so that a uniform ratio of oil to oxygen is in place initially. We have shown (see the end of Sec. 10) that the width of the slow cooling part of the pulse increases unboundedly as the heat loss to the surrounding rock formation decreases. This is the case, for example, when the thickness of the petroleum-bearing formation increases, or when the total seepage velocity of the fluid increases. When heat loss is very small, only the lead front of the pulse may fit between the injection and the producing wells. When the heat loss vanishes, the triangular oxidation pulse reduces to an oxidation bank, with no decay behind it.

On the other hand, in the case of excessive heat loss to the surroundings, we have shown, under the simplifying assumption of constant thermal conductivity, that no oxidation pulses are supported by the medium. Important open problems are (1) to prove this result without the simplifying assumption, and (2) to understand how the pulse vanishes as the heat loss increases.

Our analysis implies that if simulations of oxidation processes in petroleum engineering are to predict correctly the occurrence of oxidation pulses, they must take into account heat loss to the rock formation.

Roughly speaking, we have assumed that there is a uniform mixture of oil and oxygen or air in place before the oxidation pulse ignites, and that the oil does not burn completely. In our model these assumptions correspond respectively to  $0 < s^+ < 1$  and  $0 < s^- < 1$ . We have thereby avoided treating the cases in which there is only residual oxygen behind of the oxidation pulse ( $s_- = 0$ ), or only residual oil behind the pulse ( $s_- = 1$ ). In these cases, the coefficient  $h$  of the second order derivative term vanishes at certain points. We would thus obtain a degenerate parabolic equation, which would yield a singular boundary value problem for the traveling wave profile. Such singular traveling wave problems have been studied for models without combustion [12].

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