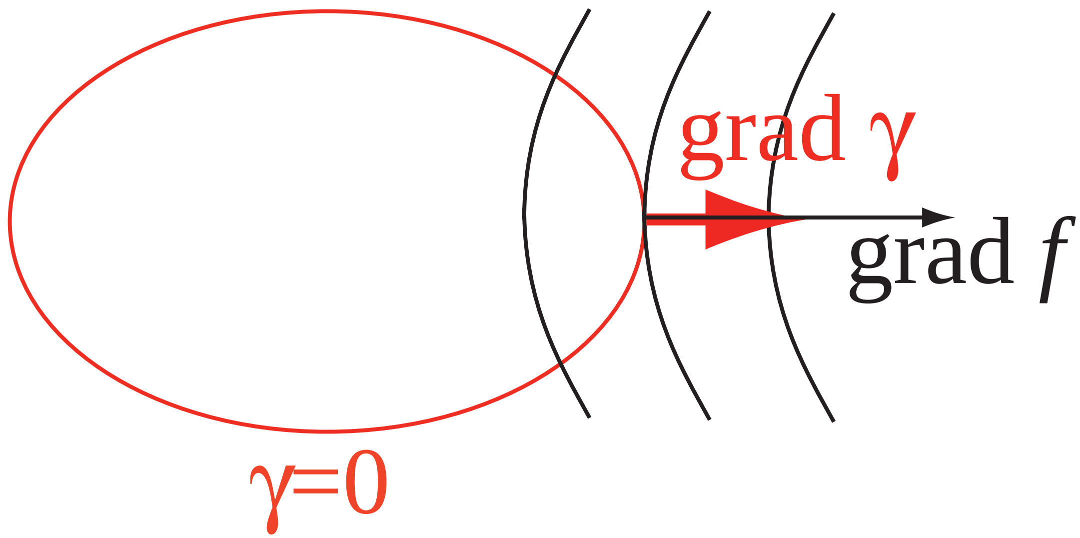
Morse Theory for Lagrange Multipliers



Guangbo Xu
Princeton University

and

Steve Schecter
North Carolina State University

Outline

(1) *History of Mathematics 1950–2050*. Chapter 1. Morse Theory.

(2) Lagrange multipliers with scaled multiplier

(3) The limit $\lambda \to \infty$

(4) The limit $\lambda \to 0$

(5) Etc.

History of Mathematics 1950–2050. Chapter 1. Morse Theory.

Section 1. Classical Morse Theory

Inherited from the 1930's; used from the early 1950's to study differentiable topology. J. Milnor, *Morse Theory*, Princeton University Press, 1963.

Setting:

- $M =$ compact manifold of dimension n.
- $f : M \longrightarrow \mathbb{R}$ is a smooth function.
- x is a critical point if $df(x) = 0$.
- x is a nondegenerate critical point if $d^2 f(x)$ has k negative eigenvalues and $n-k$ positive eigenvalues.
- index $x = k$.
- f is a Morse function if all critical points are nondegenerate.
- $M^a = \{x \in M : f(x) \leq a\}$.

Fundamental Theorem of Morse Theory.

Suppose $a < b$ are regular values of a Morse function f.

- If $f^{-1}[a,b]$ contains **no** critical point of f, then M^b is diffeomorphic to M^a.
- If $f^{-1}[a,b]$ contains **one** nondegenerate critical point of f, with index k, then M^b has the homotopy type of M^a with a k handle-attached.

FIGURE 1. Height function.

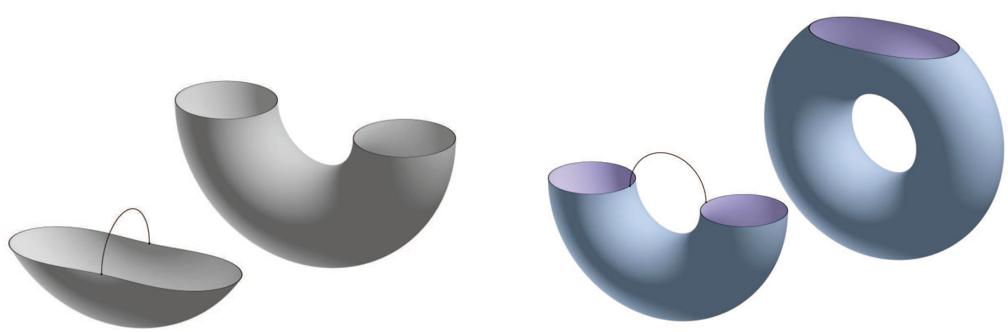


FIGURE 2. Adding handles.

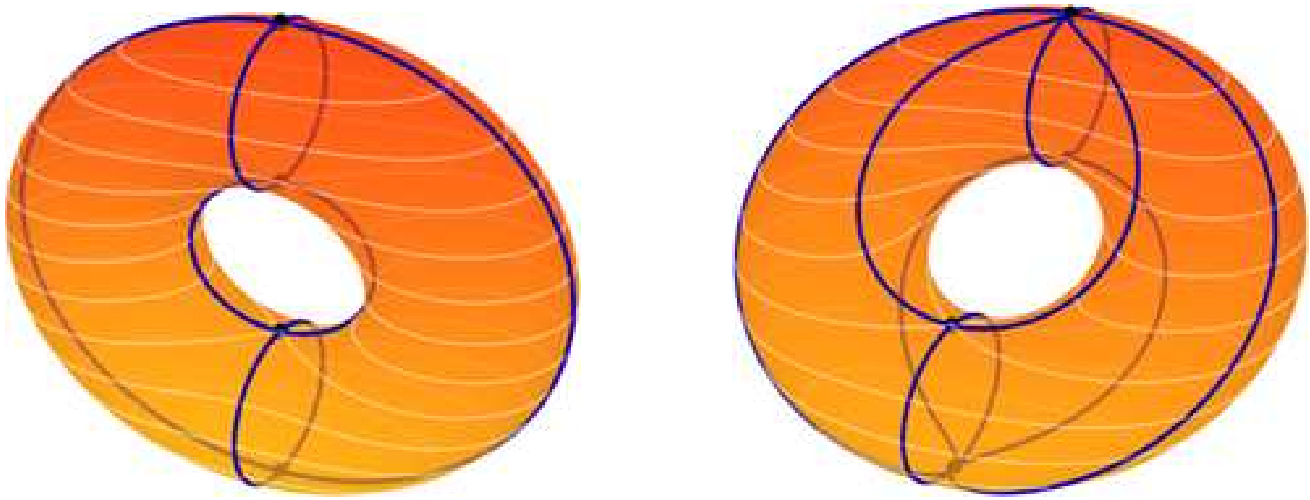
Section 2. Morse-Smale ODEs

Put a metric on M, let f be a Morse function, and consider

$$\dot{x} = -\nabla f(x).$$

If x is a critical point of f of index k, then x is a hyperbolic equilibrium with $\dim W^u(x) = k$.

The ODE is Morse-Smale if the unstable and stable manifolds of different equilibria meet transversally.



(a) Upright torus: not Morse-Smale. (b) Tilted torus: Morse-Smale.

Theorem. Morse-Smale ODE's are structurally stable.

Section 3. Morse homology

Let $f : M \longrightarrow \mathbb{R}$ be a Morse function with $\dot{x} = -\nabla f(x)$ Morse-Smale.

Define the Morse-Smale-Witten chain complex:

$C_k =$ free $\mathbb{Z}_2$-module generated by the critical points of index k.

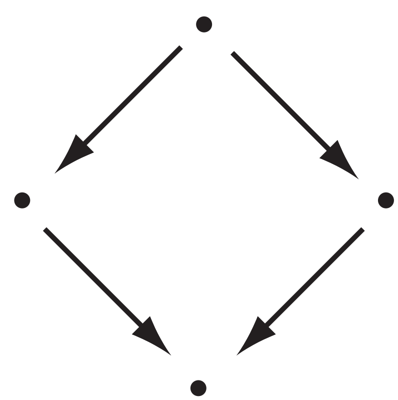
Boundary operator $d : C_k \longrightarrow C_{k-1}$:

$p =$ critical point p of index k, $q =$ critical point of index $k-1$.

$W^u(p) \cap W^s(q) = n(p,q)$ curves.

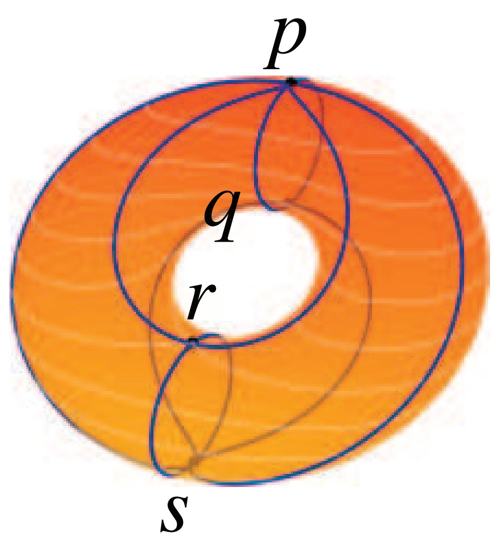
$$d(p) = \sum_{\text{critical points } q \text{ of index } k-1} n(p,q) \cdot q.$$

Proposition. $d \circ d = 0$.

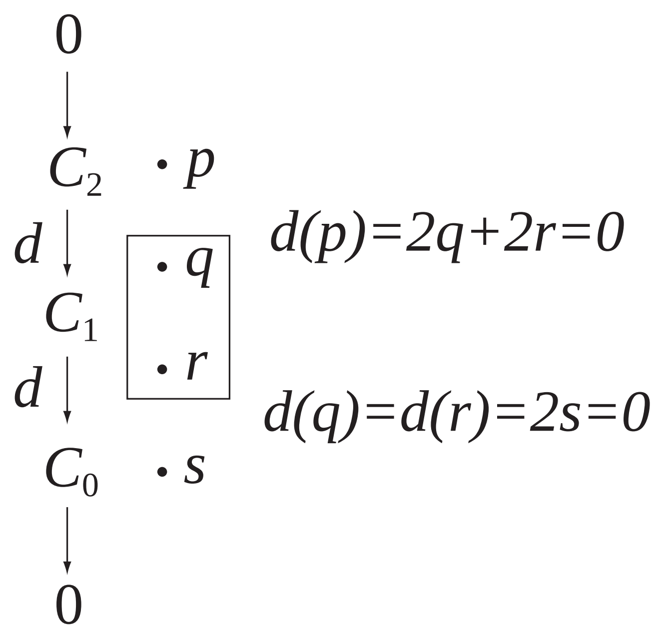


The Morse homology of f is the homology of this chain complex.

Theorem. The Morse homology of f equals the singular homology of M.



(c) Morse-Smale flow.



(d) Chain complex.

History

Apparently known to Thom, Smale, and Milnor.

Rediscovered by Edward Witten in *Supersymmetry and Morse Theory* (J. Diff. Geom., 1982) in which the curves correspond to instantons that represent tunneling to remove spurious degeneracies in a perturbation calculation involving the action of the Laplacian on the deRham complex . . .

Section 4. Floer homology

Method of Andreas Floer, series of papers in 1987-89:

(1) Associate with a manifold M an important infinite-dimensional manifold X (e.g., loop space of a symplectic manifold).

(2) Find a natural functional on X (e.g., symplectic action functional associated to a symplectomorphism) and a natural metric on X.

(3) Calculate the Morse homology. If you encounter infinite indices, try to define a finite index difference.

(4) Prove something about M (e.g., Arnold's conjecture on the number of fixed points of a symplectomorphism).

(5) If you can't do step 4, investigate a simpler Morse homology analog for inspiration.

Motivated by the symplectic vortex equation, we consider . . .

Lagrange multipliers with scaled multiplier

$M =$ compact manifold.

$f : M \to \mathbb{R}$ is a Morse function.

$\gamma : M \to \mathbb{R}$ has 0 as a regular value (so $\gamma^{-1}(0)$ is a manifold).

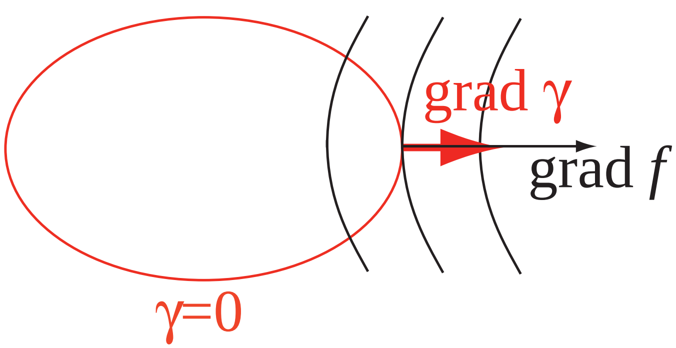
Lagrange function:

$$\mathcal{L} : M \times \mathbb{R} \to \mathbb{R}, \quad \mathcal{L}(x,\eta) = f(x) + \eta\gamma(x).$$

Critical point set of $\mathcal{L}$:

$$\mathrm{Crit}(\mathcal{L}) = \{(x,\eta) \ : \ df(x) + \eta d\gamma(x) = 0, \quad \gamma(x) = 0\},$$

There is a bijection $\mathrm{Crit}(\mathcal{L}) \simeq \mathrm{Crit}\left(f|\gamma^{-1}(0)\right), (x,\eta) \mapsto x$.



We'll investigate the morse homology of $\mathcal{L}$.

$g =$ a Riemannian metric on M.

$e =$ standard metric on $\mathbb{R}$.

$g \oplus e$ is a metric on $M \times \mathbb{R}$.

Gradient vector field of $\mathcal{L}$ with respect to $g \oplus e$:

$$\nabla \mathcal{L}(x, \eta) = (\nabla f + \eta \nabla \gamma, \ \ \gamma(x)).$$

ODE for the negative gradient flow of $\mathcal{L}$

$$\dot{x} = -\left(\nabla f(x) + \eta \nabla \gamma(x)\right),$$
$$\dot{\eta} = -\gamma(x).$$

Rescale the metric on the second factor: $g \oplus \lambda^{-2} e$. Distance $\lambda \to$ distance 1.

New negative gradient ODE:

$$\dot{x} = -\left(\nabla f(x) + \eta \nabla \gamma(x)\right),$$
$$\dot{\eta} = -\lambda^2 \gamma(x).$$

Theorem. Morse homology is unchanged (for λ for which the chain complex C^λ is defined).

Homology of C^λ is not the homology of $M \times \mathbb{R}$.

Limit as $\lambda \to 0$

Lagrange function:

$$\mathcal{L}(x, \eta) = f(x) + \eta\gamma(x).$$

Negative gradient flow with rescaled metric, a fast-slow system for small λ.

$$\begin{aligned}
\dot{x} &= -\left(\nabla f(x) + \eta \nabla \gamma(x)\right), \\
\dot{\eta} &= -\lambda^2 \gamma(x).
\end{aligned}$$

Set $\lambda = 0$:

$$\begin{aligned}
\dot{x} &= -\left(\nabla f(x) + \eta \nabla \gamma(x)\right), \\
\dot{\eta} &= 0.
\end{aligned}$$

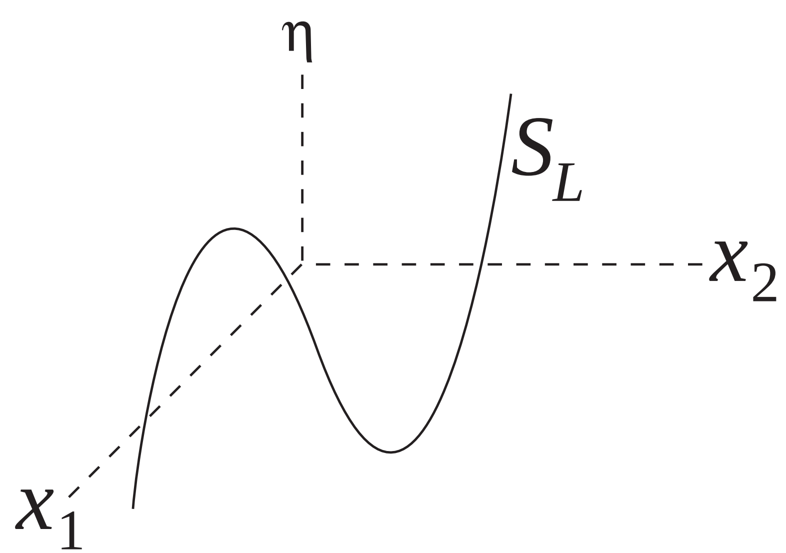
1. The slow manifold is the set of equilibria for $\lambda = 0$:

$$\dot{x} = -\left(\nabla f(x) + \eta \nabla \gamma(x)\right),$$
$$\dot{\eta} = 0.$$

$$\mathcal{S}_{\mathcal{L}} = \left\{(x,\eta) : \nabla f(x) + \eta \nabla \gamma(x) = 0\right\}.$$

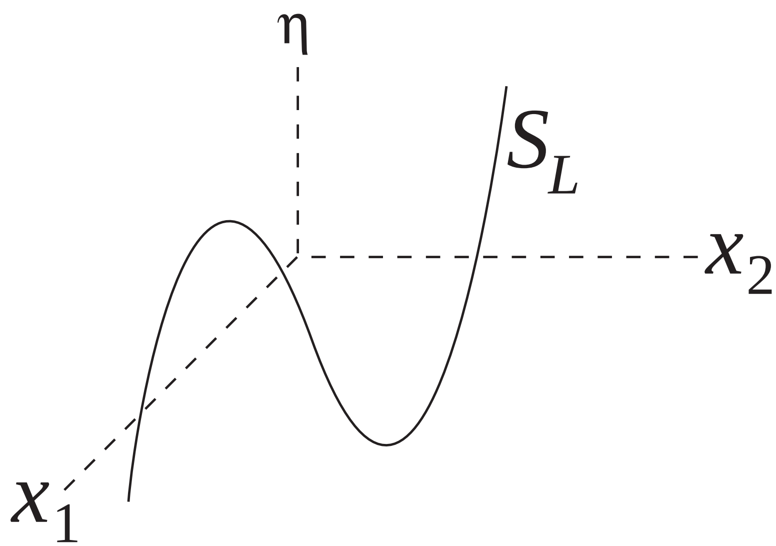
Assume

- 0 is a regular value of $\nabla f(x) + \eta \nabla \gamma(x)$ (so $\mathcal{S}_{\mathcal{L}}$ is a curve).
- $\eta|\mathcal{S}_{\mathcal{L}}$ has nondegenerate critical points.



$\mathcal{S}_{\mathcal{L}}^{sing}$ = critical points of $\eta|\mathcal{S}_{\mathcal{L}}$.

$$\dot{x} = -(\nabla f(x) + \eta \nabla \gamma(x)),$$
$$\dot{\eta} = 0.$$



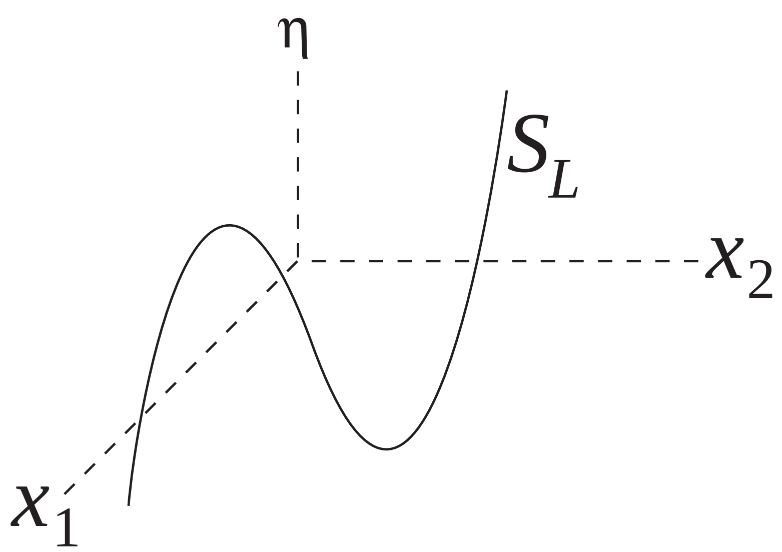
2. The fast equation is $\dot{x} = -(\nabla f(x) + \eta \nabla \gamma(x))$.

Let $f_\eta = f + \eta\gamma : M \to \mathbb{R}$ with η regarded as constant.

- x is a critical point of f_η if and only if $(x, \eta) \in \mathcal{S}_{\mathcal{L}}$.
- x is a degenerate critical point of f_η if and only if $(x, \eta) \in \mathcal{S}_{\mathcal{L}}^{sing}$.
- Assume the fast equation is Morse-Smale except at isolated bifurcation values of η where a single degeneracy occurs (degenerate critical point or nontransverse intersection of stable and unstable manifolds).

3. The slow equation:

$$\dot{x} = -\left(\nabla f(x) + \eta \nabla \gamma(x)\right),$$
$$\dot{\eta} = -\lambda^2 \gamma(x).$$

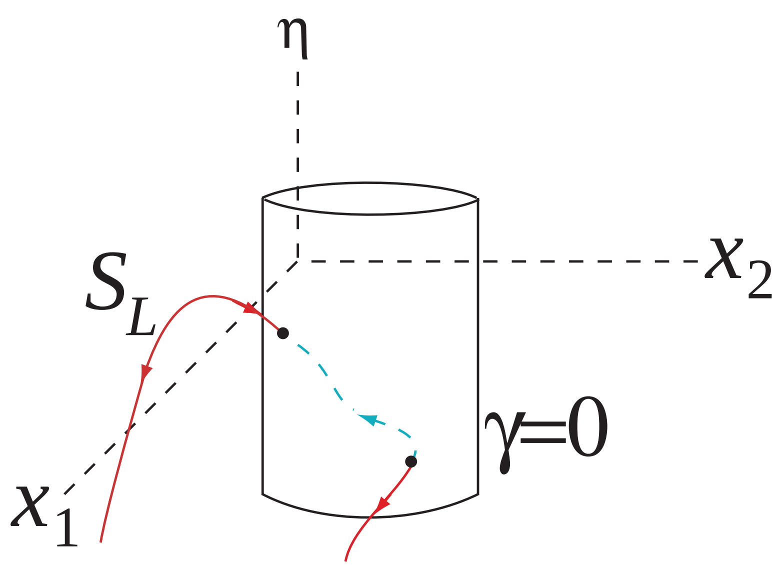


Away from critical points of $\eta|_{\mathcal{S}_\mathcal{L}}$:

- $\mathcal{S}_\mathcal{L}$ is parameterized by η: $x(\eta)$.
- $\mathcal{S}_\mathcal{L}$ is normally hyperbolic.
- Slow equation: $\dot{\eta} = -\gamma(x(\eta))$.

$$\dot{x} = -\left(\nabla f(x) + \eta \nabla \gamma(x)\right),$$
$$\dot{\eta} = -\lambda^2 \gamma(x).$$

Slow equation: $\dot{\eta} = -\gamma(x(\eta))$.



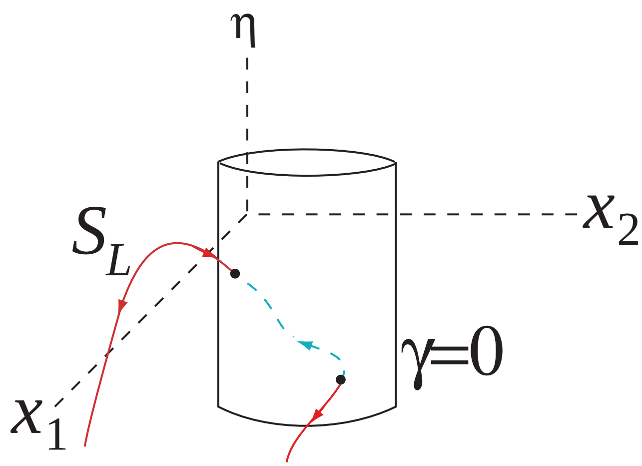
Equilibria of the slow equation are points of $\mathcal{S}_\mathcal{L}$ where $\gamma = 0$.

- They are hyperbolic.
- They are the points of $\mathrm{Crit}(\mathcal{L})$, the critical point set of $\mathcal{L}(x,\eta) = f(x) + \eta\gamma(x)$.
- They are the equilibria of the negative gradient flow of $\mathcal{L}$.

Assume:

- If $(x,\eta) \in \mathcal{S}_\mathcal{L}$ and $\gamma(x) = 0$, η is not a bifurcation value for the fast equation.

$$\dot{x} = -\left(\nabla f(x) + \eta \nabla \gamma(x)\right),$$
$$\dot{\eta} = -\lambda^2 \gamma(x).$$



Three Morse indices:

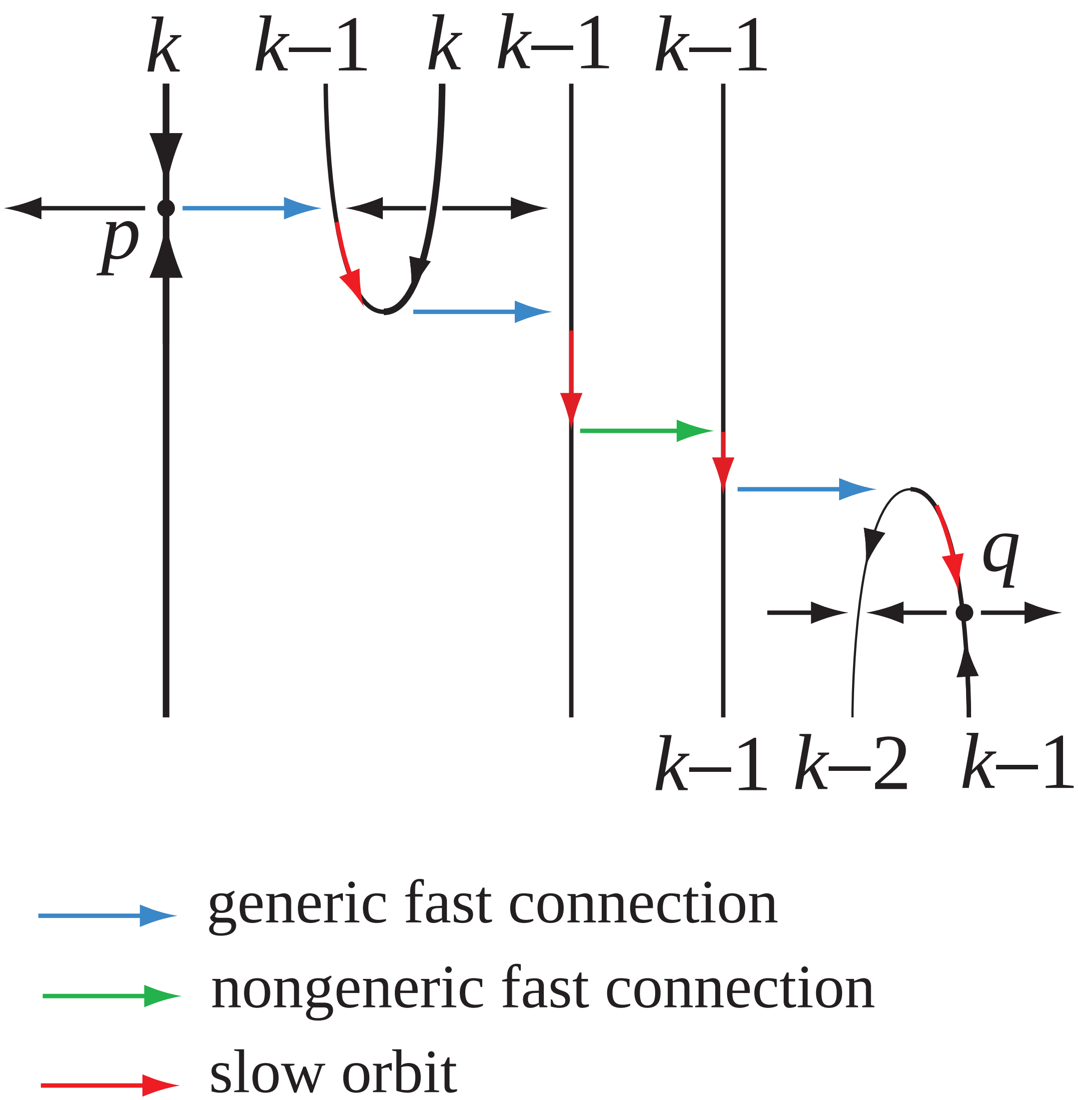
- For any $p = (x,\eta) \in \mathcal{S}_{\mathcal{L}}$, $\mathrm{index}(x, f_\eta)$.
- For $p = (x,\eta) \in \mathrm{Crit}(\mathcal{L})$, $\mathrm{index}\,(p,\mathcal{L})$.
- For $p = (x,\eta) \in \mathrm{Crit}(\mathcal{L})$, $\mathrm{index}\,(x, f|\gamma^{-1}(0))$.

Relation for $p = (x,\eta) \in \mathrm{Crit}(\mathcal{L})$:

- $\mathrm{index}(p,\mathcal{L}) = \mathrm{index}(x, f|\gamma^{-1}(0)) + 1$.
- If p is a repeller of the slow equation, then $\mathrm{index}\,(p,\mathcal{L}) = \mathrm{index}(x, f_\eta) + 1$.
- If p is an attractor of the slow equation, then $\mathrm{index}\,(p,\mathcal{L}) = \mathrm{index}(x, f_\eta)$.

Slow-fast orbits connecting $p \in \mathrm{Crit}(\mathcal{L})$ with $\mathrm{index}\,(p,\mathcal{L}) = k$ to $q \in \mathrm{Crit}(\mathcal{L})$ with $\mathrm{index}\,(q,\mathcal{L}) = k-1$:

Case 1: Both are attractors of the slow equation.



Theorem. Let

$$p \in \mathrm{Crit}(\mathcal{L}) \text{ with index}\,(p,\mathcal{L}) = k,$$
$$q \in \mathrm{Crit}(\mathcal{L}) \text{ with index}\,(q,\mathcal{L}) = k-1$$

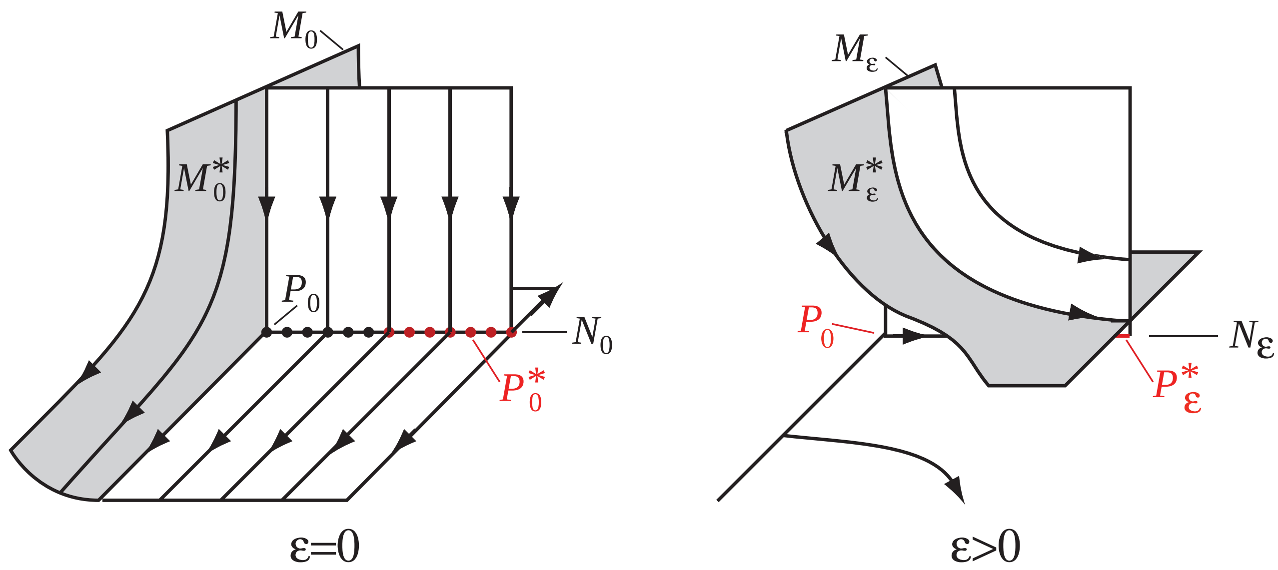
.

The slow-fast orbits from p to q are in one-to-one correspondence with the trajectories from p to q for small λ.

So the slow-fast orbits can be used to define a chain complex C^0 isomorphic to C^λ, the chain complex for small λ.

Proof. Geometric singular perturbation theory plus control afforded by the energy, with one little point.

Consider $\dot{y} = g(y, \varepsilon)$, $\varepsilon \geq 0$. Situation:



(1) $M_\varepsilon =$ cross section of tracked manifold.

(2) $N_\varepsilon =$ normally hyperbolic invariant manifold.

(3) M_ε is transverse to $W^s(N_\varepsilon)$.

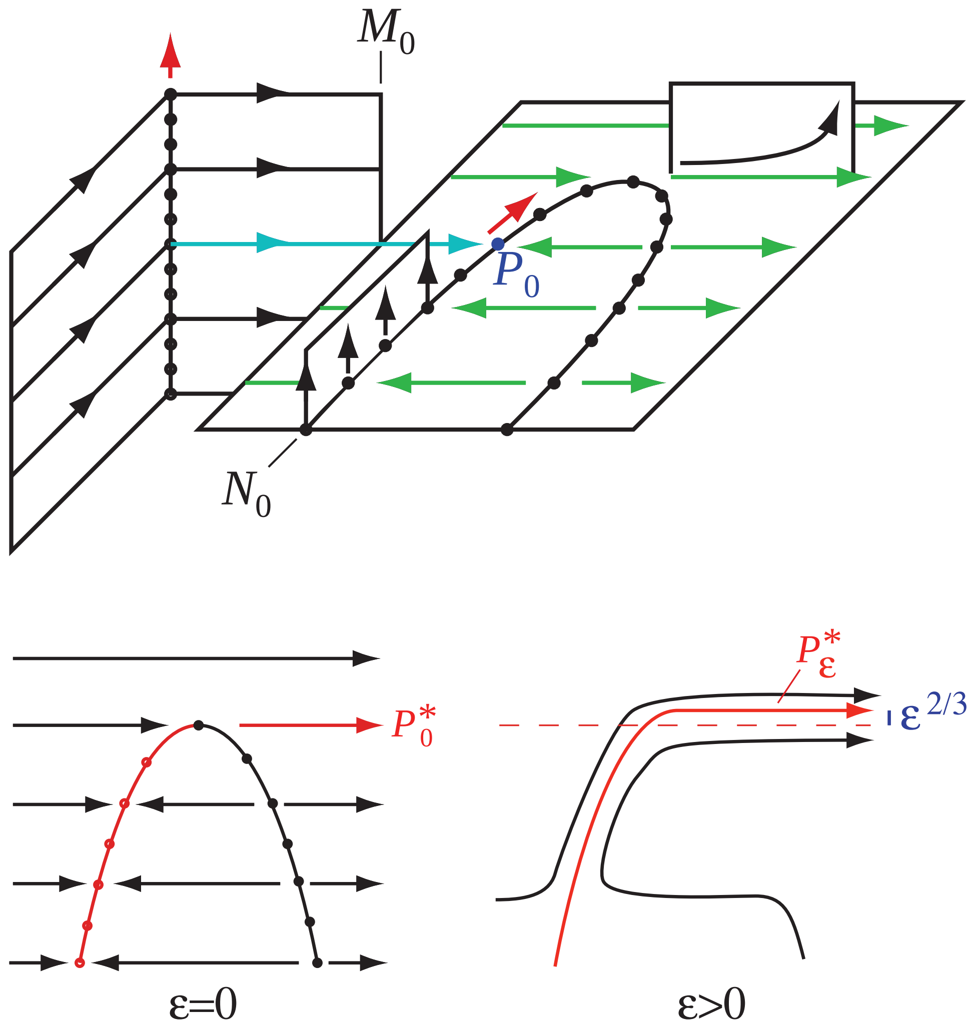
(4) $M_\varepsilon \cap W^s(N_\varepsilon)$ projects diffeomorphically along the stable fibration to $P_\varepsilon \subset N_\varepsilon$.

(5) g is not parallel to P, at least at order ε.

(6) For $\varepsilon > 0$, in time of order $1/\varepsilon$, P_ε becomes P_ε^* with one higher dimension.

(7) P_ε^* is a smooth perturbation of a manifold P_0^*.

General Exchange Lemma. Part of M_ε^*, $\varepsilon > 0$, is a smooth perturbation of $W^u(P_0^*)$.



Replace (7) with:

(7′) P^*_ε is C^r close to a manifold P^*_0.

General Exchange Lemma v.2. M^*_ε, $\varepsilon > 0$, is C^{r-1} close to $W^u(P^*_0)$.

Limit as $\lambda \longrightarrow \infty$

Lagrange function:

$$\mathcal{L}(x,\eta) = f(x) + \eta\gamma(x).$$

Negative gradient flow with rescaled metric:

$$\dot{x} = -\left(\nabla f(x) + \eta\nabla\gamma(x)\right),$$
$$\dot{\eta} = -\lambda^2\gamma(x).$$

Change of variables appropriate for large η:

$$\lambda = \frac{1}{\varepsilon}, \quad \eta = \frac{\rho}{\varepsilon}, \quad t = \varepsilon\tau.$$

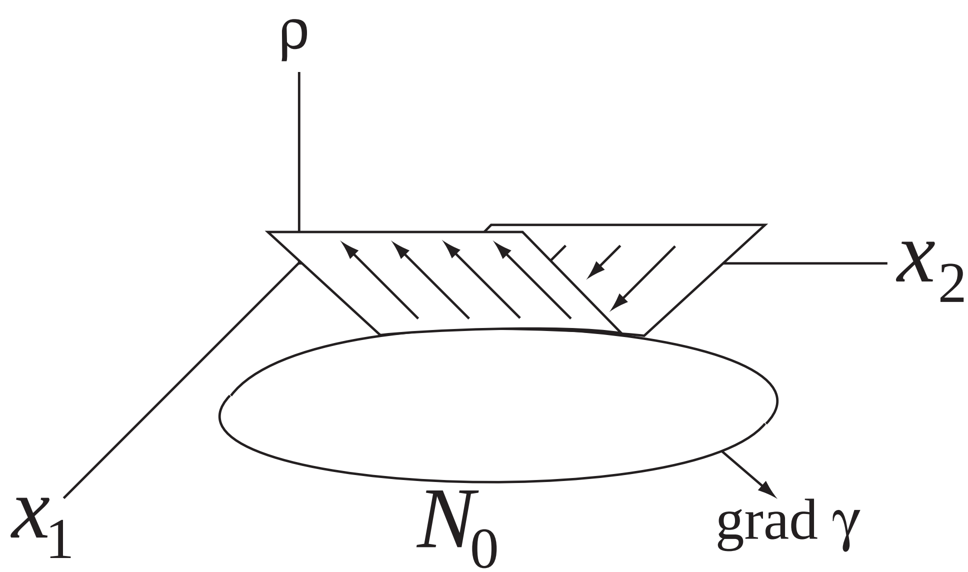
System becomes

$$\frac{dx}{d\tau} = -\left(\varepsilon\nabla f(x) + \rho\nabla\gamma(x)\right)$$
$$\frac{d\rho}{d\tau} = -\gamma(x).$$

Not a slow-fast system, but set $\varepsilon = 0$ (i.e. $\lambda = \infty$):

$$\frac{dx}{d\tau} = -\rho\nabla\gamma(x)$$
$$\frac{d\rho}{d\tau} = -\gamma(x).$$

Set of equilibria for $\varepsilon = 0$: $N_0 = \{(x,\rho) : \gamma(x) = 0 \text{ and } \rho = 0\}$.



N_0 is a compact codimension-two submanifold of $M \times \mathbb{R}$.

For $\varepsilon = 0$, N_0 is normally hyperbolic:

- Eigenvalues are 0 with multiplicity 2 and $\pm\|\nabla\gamma(x)\|_{g(x)}$.

$$\frac{dx}{d\tau} = -\left(\varepsilon\nabla f(x) + \rho\nabla\gamma(x)\right),$$
$$\frac{d\rho}{d\tau} = -\gamma(x).$$

For small $\varepsilon > 0$, there is normally hyperbolic invariant manifold N_ε near N_0.

Locally chose coordinates on M with $\gamma = x_n$.

Let $y = (x_1, \ldots, x_{n-1})$, so $x = (y, x_n)$.

Locally N_ε is parameterized by y.

The system restricted to N_ε, after division by ε, is

$$\dot{y} = -\nabla_y f(y, 0) + O(\varepsilon),$$

where $\nabla_y f(y, x_n)$ denotes the first $n-1$ components of $\nabla f(y, x_n)$.

This is a perturbation of the negative gradient flow of $(f, g)|\gamma^{-1}(0)$.

Assume: $(f, g)|\gamma^{-1}(0)$ is Morse-Smale. Then its negative gradient flow is structurally stable.

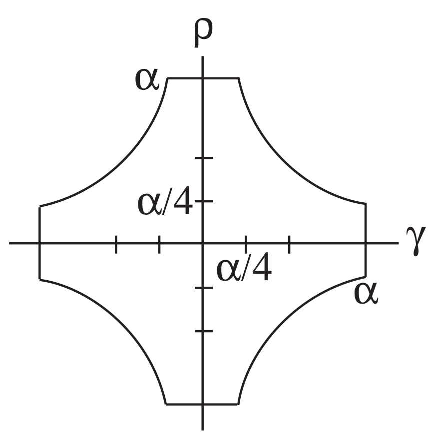
Within N_ε we have the same equilibria and connections. The equilibria have one higher index.

$$\frac{dx}{d\tau} = -\left(\varepsilon \nabla f(x) + \rho \nabla \gamma(x)\right),$$
$$\frac{d\rho}{d\tau} = -\gamma(x).$$

Are there other connections?

- They are the only connections that stay in a neighborhood of N_ε.
- $E_\varepsilon(x,\rho) = \varepsilon f(x) + \rho\gamma(x)$ decreases along solutions.
- Since $\gamma = \mathcal{O}(\varepsilon)$ on N_ε, the energy difference between two equilibria is $\mathcal{O}(\varepsilon)$.
- However, if a solution leaves a neighborhood of N_ε, its energy drops by $\mathcal{O}(1)$.

Let $V = \{(x,\rho) : |\gamma(x)| < \alpha, |\rho| < \alpha, |\rho\gamma(x)|) < \frac{\alpha^2}{4}\}$.



Make a small ε-dependent alteration in V: replace the portions of ∂V on which $\gamma = \pm\alpha$ or $\gamma = \pm\alpha$ by nearby invariant surfaces, so solutions can't cross them.

Theorem. For λ large, the Morse-Smale-Whitten complex of $(\mathcal{L}, g \oplus \lambda^{-2} e)$ equals that of $(f, g)|\gamma^{-1}(0)$ with grading shifted by one.

The slow-fast flow and Morse theory

How does the Morse complex of $\gamma^{-1}(c)$ change as c varies?

Replace $\gamma(x)$ by $\gamma(x) - c$. Then replace $\mathcal{L}(x,\eta)$ by

$$\mathcal{L}_c(x,\eta) = f(x) + \eta(\gamma(x) - c)$$

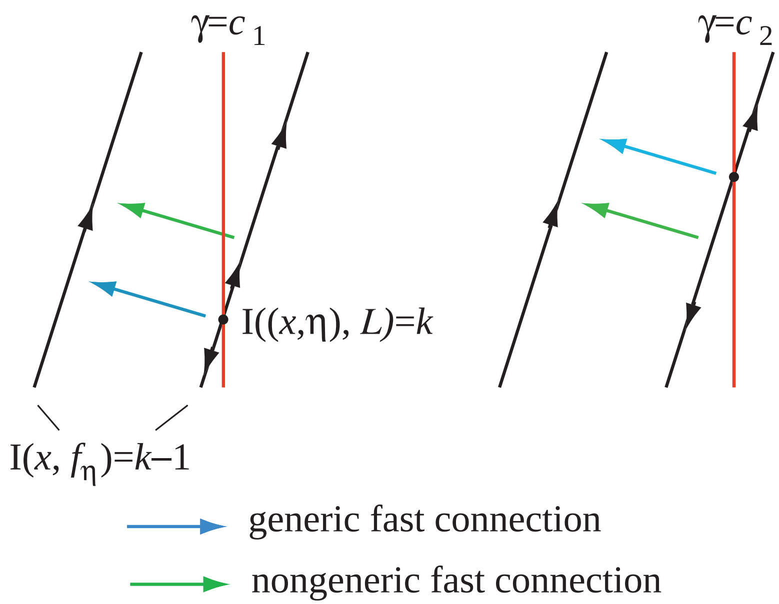
Rescaled ODE for the negative gradient flow of $\mathcal{L}_c$

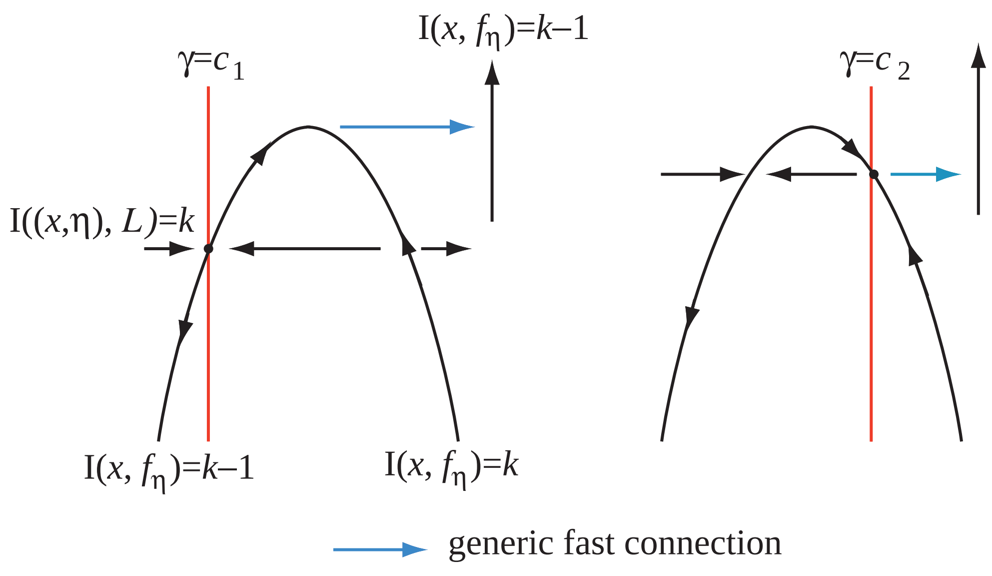
$$\begin{aligned} \dot{x} &= -\left(\nabla f(x) + \eta \nabla\gamma(x)\right), \\ \dot{\eta} &= -\lambda^2(\gamma(x) - c). \end{aligned}$$

- $\mathcal{S}_{\mathcal{L}} = \mathcal{S}_{\mathcal{L}_c}$.
- The fast flow does not change.
- The slow flow changes as the intersection of $\mathcal{S}_{\mathcal{L}}$ and $\gamma^{-1}(c) \times \mathbb{R}$ changes.

This should make it “easy” to check how slow-fast orbits appear and disappear as c varies.

If no critical value of γ is crossed, the homology of the chain complex should not change.





If a critical value of γ is crossed, the homology of the chain complex changes.

$$\dot{x} = -\left(\nabla f(x) + \eta \nabla \gamma(x)\right),$$
$$\dot{\eta} = -\lambda^2(\gamma(x) - c).$$

The slow flow changes at $\eta = \pm\infty$.