Journal of Dynamics and Differential Equations, Vol. 18, No. 1, January 2006 (© 2006)
DOI: 10.1007/s10884-005-9000-3

Eigenvalues of Self-Similar Solutions of the Dafermos Regularization of a System of Conservation Laws via Geometric Singular Perturbation Theory

Stephen Schecter[1]

Received March 29, 2005

The Dafermos regularization of a system of n conservation laws in one space dimension admits smooth self-similar solutions of the form $u = u(X/T)$. In particular, there are such solutions near a Riemann solution consisting of n possibly large Lax shocks. In Lin and Schecter (2004, *SIAM. J. Math. Anal.* **35**, 884–921), eigenvalues and eigenfunctions of the linearized Dafermos operator at such a solution were studied using asymptotic expansions. Here we show that the asymptotic expansions correspond to true eigenvalue–eigenfunction pairs. The proofs use geometric singular perturbation theory, in particular an extension of the Exchange Lemma.

KEY WORDS: Conservation law; Dafermos regularization; eigenfunction; eigenvalue; geometric singular perturbation theory; Riemann problem; stability.

Mathematics Subject Classification, 1991: 35L65, 35L67, 34E15.

1. INTRODUCTION

Consider a system of *viscous conservation laws* in one space dimension, i.e., a partial differential equation of the form

$$u_T + f(u)_X = (B(u)u_X)_X, \tag{1.1}$$

where $X \in \mathbb{R}, T \in [0,\infty), u \in \mathbb{R}^n, f\colon \mathbb{R}^n \to \mathbb{R}^n$, and $B(u)$ is an $n \times n$ matrix for which all eigenvalues have positive real part. We are interested in

[1]Mathematics Department, North Carolina State University, Box 8205, Raleigh, NC 27695, USA. E-mail: schecter@math.ncsu.edu

the behavior, as $T \to \infty$, of solutions of (1.1) that satisfy the constant boundary conditions

$$u(-\infty, T) = u^{\ell}, \qquad u(+\infty, T) = u^{r}, \quad 0 \leqslant T < \infty \tag{1.2}$$

and some initial condition $u(X, 0) = u^0(X)$. Our interest is not in the solution for any particular initial condition, but in the asymptotic behavior of solutions as $T \to \infty$.

It is believed [1] that as $T \to \infty$, solutions of such intitial-boundary-value problems typically approach Riemann solutions for the system of conservation laws

$$u_T + f(u)_X = 0. \tag{1.3}$$

These are solutions of (1.3) that depend only on $x = \frac{X}{T}$, and that satisfy the boundary conditions

$$u(-\infty) = u^{\ell}, \qquad u(+\infty) = u^{r}. \tag{1.4}$$

In numerical simulations, the convergence is seen when the solution is viewed in the rescaled spatial variable $x = \frac{X}{T}$; the rescaling counteracts the spreading of the solution as time increases. Discontinuities (shock waves) in the limiting Riemann solution satisfy the viscous profile criterion for the viscosity $B(u)$, i.e., they correspond to traveling waves of (1.1). Speaking very roughly, Riemann solutions are believed to play the same role for (1.1) and (1.2) that equilibria play for ordinary differential equations: they are the simplest asymptotic states. An important difference, however, is that Riemann solutions are not solutions of (1.1) but only of the related Eq. (1.3).

If the Riemann solution is a single shock wave, then it corresponds to a traveling wave solution of (1.1). Stability of such solutions has been studied using energy methods ([9, 11, 12, 14, 29, 30, 35]) and careful analysis of the linearization of (1.1) and (1.2) at the traveling wave ([3, 18, 20, 25–27, 32]). Beginning with [13], the Evans function has been used to study the spectrum of the linearization ([2, 10, 15, 16, 31, 41]).

Since Riemann solutions other than a single shock wave do not correspond to explicit solutions of (1.1) and (1.2), the study of their stability is less advanced. In some situations one can construct an approximate solutions of (1.1) and (1.2) near the Riemann solution and show that solutions of (1.1) and (1.2) that start near it approach it. See [36] for Riemann solutions consisting of a single rarefaction, and [24] for Riemann solutions consisting of weak Lax shock waves. The last paper is the only one we know of that deals with stability of Riemann solutions containing more than one wave.

Since it is in the variables (x, T) with $x = \frac{X}{T}$ that the convergence of solutions of (1.1) and (1.2) to Riemann solutions is observed, Lin and I proposed in [23] to make the following change of variables in (1.1) and (1.2):

$$x = \frac{X}{T}, \qquad t = \ln T. \tag{1.5}$$

(The substitution $t = \ln T$ is simply for convenience. Decay that is algebraic in T becomes exponential in t.) We obtain

$$u_t + (Df(u) - xI)u_x = e^{-t}(B(u)u_x)_x, \tag{1.6}$$

$$u(-\infty, t) = u^{\ell}, \qquad u(+\infty, t) = u^r, \quad 0 \leqslant t < \infty. \tag{1.7}$$

Of course the interval $0 \leqslant t < \infty$ corresponds to $1 \leqslant T < \infty$, but this is not important since we are interested in asymptotic behavior. The fact that (1.6) is nonautonomous implies that solutions can easily approach limits that are not themselves solutions.

In studying nonautonomous systems such as (1.6), it is natural to first freeze the time varying coefficient and study the resulting autonomous system. In this case one sets $\epsilon = e^{-t}$; for large t, ϵ is small. One obtains

$$u_t + (Df(u) - xI)u_x = \epsilon(B(u)u_x)_x \tag{1.8}$$

with the boundary conditions (1.7). Returning to (X, T) variables (1.8) becomes

$$u_T + f(u)_X = \epsilon T(B(u)u_X)_X. \tag{1.9}$$

Equation (1.9) is the *Dafermos regularization* of the system of conservation laws (1.3) associated with the viscosity $B(u)$ ([5]; see also [38, 39]), It is usually regarded as an artificial, nonphysical equation because of the factor T in the viscous term. As we have seen, however, in the variables (1.5), the Dafermos regularization is actually a natural simplification of the physical equations.

Tzavaras [40] first proposed that the Dafermos regularization might have a role to play in the study of asymptotic stability of solutions of viscous conservation laws.

Stationary solutions of (1.8) and (1.7) satisfy the ODE

$$(Df(u) - xI)u_x = \epsilon(B(u)u_x)_x, \tag{1.10}$$

with boundary conditions (1.4). We shall refer to a solution $u_\epsilon(x)$ of (1.10), (1.4) as a *Riemann–Dafermos solution* of (1.8). It is known in many cases that near a Riemann solution of (1.3), with shock waves that satisfy the viscous profile criterion for $B(u)$, there is a Riemann–Dafermos solution $u_\epsilon(x)$ of (1.8) ([28, 33, 34, 37, 40]). It is reasonable to expect that information about the stability of $u_\epsilon(x)$ as a solution of (1.8) will be helpful in the study of the stability of the corresponding Riemann solution as an asymptotic state of (1.1).

In [23], Lin and I began the study of the linearized stability of Riemann–Dafermos solutions of (1.8). We considered only the case in which $B(u) \equiv I$, the underlying Riemann solution consists of exactly n Lax shock waves, and the Riemann solution satisfies various nondegeneracy conditions. The shock waves are allowed to be large. It is shown in [23] that in an appropriate Banach space of exponentially decaying functions,

(1) the initial value problem for (1.8), (1.7) is well-posed in a neighborhood of a Riemann–Dafermos solution $u_\epsilon(x)$; and
(2) the linearization of (1.8) at $u_\epsilon(x)$ has its essential spectrum in $Re\ \lambda \leqslant -\delta < 0$.

In addition, eigenvalues and eigenfunctions of the linearization are constructed as asymptotic expansions. The constructed eigenvalues have expansions of the form $\lambda = \sum_{j=-1}^{\infty} \epsilon^j \lambda_j$.

(1) *Fast eigenvalues*, with $\lambda_{-1} \neq 0$, occur when λ_{-1} is a nonzero eigenvalue for the linearization of the PDE (1.1) at one of the viscous shock profiles. As described above, these eigenvalues have been much studied using Evans function methods.
(2) *Slow eigenvalues* have $\lambda_{-1} = 0$. It turns out that $\lambda_0 = -1$ is always among the slow eigenvalues. Its multiplicity is n. To lowest order, a basis for the eigenspace is given by the derivatives of the individual traveling waves in the n singular layers. These eigenfunctions correspond to shifts of the traveling waves.
(3) Other slow eigenvalues have λ_{-1} equal to an eigenvalue of a first-order hyperbolic system that arises in the study of inviscid stability of the underlying Riemann solution. The relationship of these eigenvalues to inviscid stability has recently been clarified by Lewicka [21].

The paper [23] did not resolve the question of whether these eigenvalue–eigenfunction expansions correspond to true eigenvalue–eigenfunction pairs; indeed, in the case of slow eigenvalues other than -1, it did not address the question of whether the expansions can be continued beyond the low order that was calculated. In this paper, we show that for both fast eigenvalues and slow eigenvalues other than -1, if the conditions required to start the expansions hold, and if appropriate nondegeneracy conditions are satisfied, then there are true eigenvalue–eigenfunction pairs nearby. In addition, we show that if the conditions required to start the expansions do not hold at λ_{-1} (resp. λ_0), then $u_\epsilon(x)$ has no fast eigenvalue near $\frac{\lambda_{-1}}{\epsilon}$ (resp. no slow eigenvalue near λ_0).

The nondegeneracy conditions used in this paper force the eigenvalues to be simple. We do not address multiple eigenvalues.

Our proofs use geometric singular perturbation theory rather than asymptotic expansions. The proofs are based on Szmolyan's construction of Riemann–Dafermos solutions using geometric singular perturbation theory [37]. Following Szmolyan, we write (1.10) as an ODE system; we then add additional variables to represent eigenvalues and linearized state variables. We look for solutions of an extended system that represent triples (Riemann–Dafermos solution, eigenvalue, and eigenfunction).

The related paper [22] gives a more analytic treatment of slow eigenvalues and eigenfunctions. There it is shown how to continue the slow eigenvalue–eigenfunction expansions to arbitrary order, and an analytic proof of existence of slow eigenvalues and eigenfunctions is given. While the geometric singular perturbation approach yields more geometric insight, the analytic approach promises to be more useful in the study of asymptotic behavior of solutions of (1.1).

For background on normally hyperbolic invariant manifolds, geometric singular perturbation theory, and the Exchange Lemma, all of which are heavily used in this paper, see [4, 17, 19]. Section 4 of this paper can also be used as an introduction to the Exchange Lemma.

The remainder of the paper is organized as follows. In Section 2, we recall the assumptions on the underlying Riemann solution that were used in [23] and review the construction of Riemann–Dafermos solutions. In Section 3, we prove the results about fast eigenvalues. In Section 4, we prove an extension of the Exchange Lemma that is needed to treat slow eigenvalues, and in Section 5, we prove the results about slow eigenvalues.

The heart of the paper is Sections 4 and 5. The treatment of fast eigenvalues uses only fairly standard geometric singular perturbation theory.

2. RIEMANN–DAFERMOS SOLUTIONS

In the remainder of the paper, we consider (1.8) with $B(u) \equiv I$:

$$u_t + (Df(u) - xI)u_x = \epsilon u_{xx} \tag{2.1}$$

with f sufficiently differentiable for the proofs.

Let $-\infty = \bar{x}^0 < \bar{x}^1 < \cdots < \bar{x}^{n+1} = \infty$. We consider a structurally stable Riemann solution $u_0(x), x = \frac{X}{T}$, of (1.3) that consists of exactly n Lax shock waves, each of which satisfies the viscous profile criterion for $B(u) \equiv I$:

$$u_0(x) = \bar{u}^i \quad \text{for } \bar{x}^i < x < \bar{x}^{i+1}, \ \ i = 0, \dots, n. \tag{2.2}$$

The jumps in $u_0(x)$ at the $\bar{x}^i$ may be large.

More precisely, we assume:

(R0) For all $u \in \mathbb{R}^n$, $Df(u)$ has n distinct real eigenvalues.

(R1) If we set $u^0 = \bar{u}^0$ and $u^n = \bar{u}^n$ in the system of equations

$$(f(u^1) - f(u^0) - x^1(u^1 - u^0), \dots, f(u^n) - f(u^{n-1}) \\ - x^n(u^n - u^{n-1})) = (0, \dots, 0) \tag{2.3}$$

then the resulting system of n^2 equations in the n^2 variables $(x^1, u^1, x^2, u^2, \dots, x^{n-1}, u^{n-1}, x^n)$ has $(\bar{x}^1, \bar{u}^1, \bar{x}^2, \bar{u}^2, \dots, \bar{x}^{n-1}, \bar{u}^{n-1}, \bar{x}^n)$ as a regular solution.

(R2) For each $i = 1, \dots, n, Df(\bar{u}^{i-1}) - \bar{x}^i I$ has $n - i + 1$ positive eigenvalues and $i - 1$ negative eigenvalues, and $Df(\bar{u}^i) - \bar{x}^i I$ has $n - i$ positive eigenvalues and i negative eigenvalues. (In particular, $\bar{u}^{i-1} \neq \bar{u}^i$ for $i = 1, \dots, n$.)

(R3) For each $i = 1, \dots, n$, the traveling wave ODE for $u_T + f(u)_x = u_{xx}$,

$$\dot{u} = f(u) - f(\bar{u}^{i-1}) - \bar{x}^i(u - \bar{u}^{i-1})$$

has a solution $q^i(\xi)$ in $W^u(\bar{u}^{i-1}) \cap W^s(\bar{u}^i)$.

(R4) $W^u(\bar{u}^{i-1})$ and $W^s(\bar{u}^i)$ meet transversally along $q^i(\xi)$. Equivalently, the linear differential equation

$$(Df(q^i(\xi)) - \bar{x}^i I)U_\xi = U_{\xi\xi}$$

has, up to scalar multiplication, a unique solution that approaches zero as $\xi \to \pm\infty$. It is $\dot{q}^i(\xi)$.

Stationary solutions $u_\epsilon(x)$ of (2.1), also called Riemann–Dafermos solutions, satisfy

$$(Df(u) - xI)u_x = \epsilon u_{xx}. \tag{2.4}$$

For small $\epsilon > 0$, there is a Riemann–Dafermos solution $u_\epsilon(x)$ near $u_0(x)$ with $u_\epsilon(-\infty) = \bar{u}^0$ and $u_\epsilon(\infty) = \bar{u}^n$. Let us review the construction by geometric singular perturbation theory of this solution

Let $x = x_0 + \epsilon\xi$, let a dot denote derivative with respect to ξ, and rewrite (2.4) as the system

$$\dot{u} = v, \tag{2.5}$$

$$\dot{v} = (Df(u) - xI)v, \tag{2.6}$$

$$\dot{x} = \epsilon. \tag{2.7}$$

We first consider the system (2.5)–(2.7) with $\epsilon = 0$:

$$\dot{u} = v, \tag{2.8}$$
$$\dot{v} = (Df(u) - xI)v, \tag{2.9}$$
$$\dot{x} = 0. \tag{2.10}$$

Let $\nu^1(u) < \cdots < \nu^n(u)$ be the eigenvalues of $Df(u)$, and let $r^1(u), \ldots, r^n(u)$ be corresponding eigenvectors. Let $\nu^0(u) = -\infty$ and $\nu^{n+1}(u) = \infty$.

For each $i = 0, \ldots, n$, let

$$S^i = \{(u, v, x) : v = 0 \quad \text{and} \quad \nu^i(u) < x < \nu^{i+1}(u)\}.$$

See Figure 1. Each S^i is an $(n+1)$-dimensional manifold of equilibria of (2.8)–(2.10). At $(u^0, 0, x^0)$ in S^i, the linearization of (2.8)–(2.10) has the semisimple eigenvalue 0 with multiplicity $n+1$, and n nonzero eigenvalues $\nu^k(u) - x, k = 1, \ldots, n$, of which the last $n - i$ are positive and the first i are negative.

Remark. Systems (2.5)–(2.7) is a singular perturbation problem expressed in the fast time ξ; the slow time is x. However, it is not in the standard form for such problems. Typically, for $\epsilon = 0$, the dimension of the set of equilibria in a singular perturbation problem written in the fast time equals the number of slow variables, which is not the case here. The system can be put into the standard form by setting $v = f(u) - xu + w$ and using

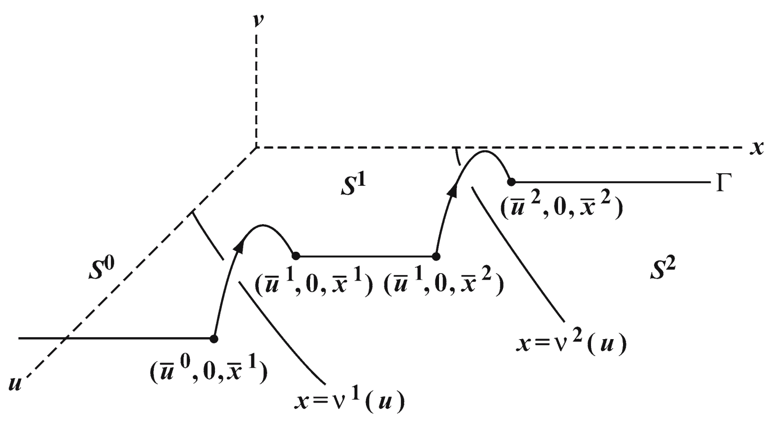

Figure 1. uvx-space and Γ.

the variables (u, w, x):

$$\dot{u} = f(u) - xu + w, \tag{2.11}$$

$$\dot{w} = -\epsilon u, \tag{2.12}$$

$$\dot{x} = \epsilon. \tag{2.13}$$

In this form, w and x are the slow variables, and, for $\epsilon = 0$, the set of equilibria $f(u) - xu + w = 0$ has dimension $n+1$ as expected. Nevertheless we prefer to retain the form (2.5)–(2.7), largely to take advantage of the invariance of the plane $v = 0$ for all ϵ. A secondary reason is to make it easy to generalize to systems not in conservation form, in which $Df(u)$ is replaced by $A(u)$ and the form (2.11)–(2.13) is not available. Because of this choice, in Section 4, we treat the Exchange Lemma for singular perturbation problems that are not in the standard form.

Let Γ denote the union of the $n+1$ line segments $\{(\bar{u}^i, 0, x) : \bar{x}^i \leqslant x \leqslant \bar{x}^{i+1}\}$ and the n connecting orbits $(q^i(\xi), \dot{q}^i(\xi), \bar{x}^i)$. (We use the conventions that $\bar{x}^0 \leqslant x$ means $-\infty < x$ and $x \leqslant \bar{x}^n$ means $x < \infty$.) see Figure 1. For small $\epsilon > 0$, we shall find a Riemann–Dafermos solution near Γ.

We denote the unstable and stable manifolds of $(u^0, 0, x^0)$ for (2.8)–(2.10) by $W_0^u(u^0, 0, x^0)$ and $W_0^s(u^0, 0, x^0)$; each is contained in the subspace $x = x^0$ of uvx-space, and they have dimensions $n - i$ and i, respectively. The subscript 0 stands for $\epsilon = 0$. For any submanifold N of S^i, $W_0^u(N)$ (resp. $W_0^s(N)$) denotes the union of the unstable (resp. stable) manifolds of points of N.

For $i = 0, \ldots, n$, let O^i be a small neighborhood of $\bar{u}^i$ in $\mathbb{R}^n$ and for $i = 1, \ldots, n$, let I^i be a small neighborhood of $\bar{x}^i$ in $\mathbb{R}$.

For $i = 0, \ldots, n$, we inductively define i-dimensional submanifolds M^i of O^i as follows:

(1) $M^0 = \{\bar{u}^0\}$.
(2) $u^i \in O^i$ is in M^i provided there exist $u^{i-1} \in M^{i-1}$ and $x^i \in I^i$ such that the triple (u^{i-1}, x^i, u^i) satisfies the Rankine–Hugoniot condition

$$f(u^i) - f(u^{i-1}) - x^i(u^i - u^{i-1}) = 0. \tag{2.14}$$

In fact, a consequence of (R1) is that for $i = 1, \ldots, n$, the mapping from $M^{i-1} \times I^i$ to $\mathbb{R}^n$ defined by solving (2.14) for u^i is a diffeomorphism onto an i-dimensional manifold M^i (see Figure 2).

It follows that there is a smooth inverse mapping from M^i to $M^{i-1} \times I^i, u^i \to (u^{i-1}(u^i), x^i(u^i))$. Let $P^i = \{(u^i, 0, x^i(u^i)) : u^i \in M^i\}$, an

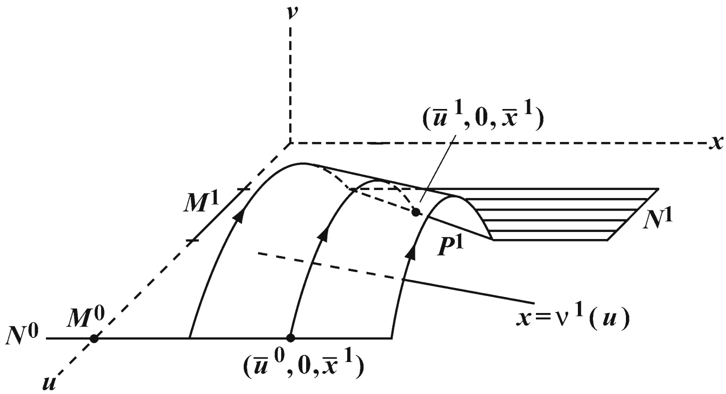

Figure 2. M^0, P^1, M^1, and N^1. The dimensions are, respectively, 0, 1, 1, and 2, regardless of the value of n. In the case $n=1$, which is shown accurately in the figure, $W^u(\bar{u}^0, 0, \bar{x}^1)$ has dimension 1 and $W^s(S^1)$ is open, so they meet transversally. This is described in Proposition 2.1.

i-dimensional submanifold of S^i. Also, note that by (R4), for $(u^{i-1}, x^i) \in M^{i-1} \times I^i$, the traveling wave equation

$$\dot{u} = f(u) - f(u^{i-1}) - x^i(u - u^{i-1}) \tag{2.15}$$

has a connecting orbit $u(\xi)$ from u^{i-1} to $u^i \in M^i$ near $q^i(\xi)$; moreover, the $(n-i+1)$-dimensional unstable manifold of u^{i-1} and the i-dimensional stable manifold of u^i meet transversally along this orbit.

Similarly, for $i = 0, \ldots, n$, we define $(n-i)$-dimensional submanifolds $\hat{M}^i$ of O^i by backwards induction as follows:

(1) $\hat{M}^n = \{\bar{u}^n\}$.
(2) $u^i \in O^i$ is in $\hat{M}^i$ provided there exist $x^{i+1} \in I^{i+1}$ and $u^{i+1} \in \hat{M}^{i+1}$ such that the triple (u^i, x^{i+1}, u^{i+1}) satisfies the Rankine–Hugoniot condition

$$f(u^i) - f(u^{i+1}) - x^{i+1}(u^i - u^{i+1}) = 0.$$

Each $u^i \in \hat{M}^i$ is associated with a unique point $(x^{i+1}(u^i), u^{i+1}(u^i)) \in I^{i+1} \times \hat{M}^{i+1}$. Let $\hat{P}^i = \{(u^i, 0, x^{i+1}(u^i)) : u^i \in \hat{M}^i\}$, an $(n-i)$-dimensional submanifold of S^i. Note that for $(x^{i+1}, u^{i+1}) \in I^{i+1} \times \hat{M}^{i+1}$, the traveling wave equation

$$\dot{u} = f(u) - f(u^{i+1}) - x^{i+1}(u - u^{i+1}) \tag{2.16}$$

has a connecting orbit $u(\xi)$ from $u^i \in \hat{M}^i$ to u^{i+1} that is near $q^{i+1}(\xi)$; moreover, the $(n-i)$-dimensional unstable manifold of u^i and the

$(i+1)$-dimensional stable manifold of u^{i+1} meet transversally along this orbit.

Proposition 2.1. *For $i=1,\dots,n$, let $(u^{i-1},0,x^i)\in M^{i-1}\times\{0\}\times I^i\subset S^{i-1}$. Then $W_0^u(u^{i-1},0,x^i)$ and $W_0^s(S^i)$ meet transversally along the curve $(u(\xi),\dot u(\xi),x^i)$, where $u(\xi)$ is a solution of (2.15) from u^{i-1} to the corresponding point $u^i\in M^i$.*

For $i=0,\dots,n-1$, let $(u^{i+1},0,x^{i+1})\in \hat M^{i+1}\times\{0\}\times I^{i+1}\subset S^{i+1}$. Then $W_0^u(S^i)$ and $W_0^s(u^{i+1},0,x^{i+1})$ meet transversally along the curve $(u(\xi),\dot u(\xi),x^{i+1})$, where $u(\xi)$ is a solution of (2.16) to u^{i+1} from the corresponding point $u^i\in\hat M^i$ (see Figure 2).

Proof. We prove only the first part. Let

$$H=\{(u,v,x): v=f(u)-f(u^{i-1})-x(u-u^{i-1}) \text{ and } x=x^i\}.$$

H is an n-dimensional manifold that is invariant under (2.8)–(2.10), contains $(u^{i-1},0,x^i)$, and meets S^i transversally at $(u^i,0,x^i)$. In fact, H meets $W_0^s(S^i)$ transversally in $W_0^s(u^i,0,x^i)$. Using u as the coordinate on H, we see that the system (2.8)–(2.10) reduces on H to (2.15). By (R4), $W_0^u(u^{i-1},0,x^i)$ and $W_0^s(u^i,0,x^i)$ meet transversally within H, The result follows. □

We remark that in the system (2.11)–(2.13), H has the equation $w=$ constant.

Note that $\dim W_0^u(u^{i-1},0,x^i)=n-i+1, \dim W_0^s(S^i)=n+1+i$, and the sum of these numbers is $2n+2$, which is one more than the dimension of uvx-space. Thus a transverse intersection of these two manifolds is a curve. Similarly, $\dim W_0^u(S^i)=n+1+n-i=2n+1-i$ and $\dim W_0^s(u^{i+1},0,x^{i+1})=i+1$.

Let $N^i=\{(u,0,x)\in S^i : u\in M^i\}$ and $\hat N^i=\{(u,0,x)\in S^i : u\in\hat M^i\}$. Note that $\dim N^i=i+1, \dim W^u(N^i)=n+1$ independent of $i, \dim \hat N^i=n-i+1$, and $\dim W^s(\hat N^i)=n+1$ independent of i (see Figure 2).

Proposition 2.2. *For $i=1,\dots,n, W_0^u(N^{i-1})$ meets $W_0^s(S^i)$ transversally along an i-parameter family of connecting orbits from $M^{i-1}\times\{0\}\times I^i$ to S^i. For each point $(u^{i-1},0,x^i)\in M^{i-1}\times\{0\}\times I^i$ there is a unique u^i such that one orbit of the family connects $(u^{i-1},0,x^i)$ to $(u^i,0,x^i)$. The set of such u^i is M^i, and the set of such $(u^i,0,x^i)$ is P^i. If $(u^i,0,x^i)\in P^i$ and $(\tilde u,\tilde v,x^i)\in W_0^u(N^{i-1})\cap W_0^s(u^i,0,x^i)$, then the tangent spaces to $W_0^u(N^{i-1})$ and $W_0^s(u^i,0,x^i)$ at $(\tilde u,\tilde v,x^i)$ have one-dimensional intersection.*

For $i=0,\dots,n-1, W_0^u(S^i)$ meets $W_0^s(\hat N^{i+1})$ transversally along an $(n-i)$-parameter family of connecting orbits from S^i to $\hat M^{i+1}\times\{0\}\times I^{i+1}$. For each point $(u^{i+1},0,x^{i+1})\in\hat M^{i+1}\times\{0\}\times I^{i+1}$ there is a unique

u^i such that one orbit of the family connects $(u^i, 0, x^{i+1})$ to $(u^{i+1}, 0, x^{i+1})$. The set of such u^i is $\hat{M}^i$, and the set of such $(u^i, 0, x^{i+1})$ is $\hat{P}^i$. If $(u^i, 0, x^{i+1}) \in \hat{P}^i$ and $(\tilde{u}, \tilde{v}, x^{i+1}) \in W_0^u(u^i, 0, x^{i+1}) \cap W_0^s(\hat{N}^{i+1})$, then the tangent spaces to $W_0^u(u^i, 0, x^{i+1})$ and $W_0^s(\hat{N}^{i+1})$ at $(\tilde{u}, \tilde{v}, x^{i+1})$ have one-dimensional intersection.

See Figure 2. This proposition is an immediate consequence of the previous one, except for the last sentence of each paragraph, which is a consequence of (R4).

For $\delta > 0$, let

$$S_\delta^0 = \left\{(u, 0, x) : \| u \| \leqslant \frac{1}{\delta} \text{ and } x \leqslant \nu_1(u) - \delta\right\},$$

$$S_\delta^i = \left\{(u, 0, x) : \| u \| \leqslant \frac{1}{\delta} \text{ and } \nu_i(u) + \delta \leqslant x \leqslant \nu_{i+1}(u) - \delta\right\}, \quad i = 1, \ldots, n-1,$$

$$S_\delta^n = \left\{(u, 0, x) : \| u \| \leqslant \frac{1}{\delta} \text{ and } \nu_n(u) + \delta \leqslant x\right\}.$$

Proposition 2.3. *For any $\delta > 0$, all $S_\delta^i, i = 0, \ldots, n$, are normally hyperbolic invariant manifolds of equilibria.*

Proof. At any $(u, 0, x) \in S_\delta^i$, there are $n-i$ positive eigenvalues and i negative eigenvalues. For $S_\delta^1, \ldots, S_\delta^{n-1}$, which are compact, the conclusion follows immediately. S_δ^0 and S_δ^n are not compact, but the conclusion follows from an easy compactification argument (see Appendix A). □

For small $\epsilon > 0$, each S_δ^i remains a normally hyperbolic, locally invariant manifold [8]. It no longer consists of equilibria, since the systems (2.5)–(2.7) on the invariant manifold $v = 0$ is

$$\dot{u} = 0, \tag{2.17}$$

$$\dot{x} = \epsilon. \tag{2.18}$$

The orbits are the lines $u = \text{constant}$. Suppose M is a submanifold of u-space, and let $N = \{(u, 0, x) \in S_\delta^i : u \in M\}$. Then for each ϵ, N is invariant under the flow of (2.17) and (2.18). From normal hyperbolicity it follows that for $\epsilon > 0$ sufficiently small, N has unstable and stable manifolds $W_\epsilon^u(N)$ and $W_\epsilon^s(N)$ that are close to $W_0^u(N)$ and $W_0^s(N)$, respectively.

A Riemann–Dafermos solution with left state $\bar{u}^0$ and right state $\bar{u}^n$ corresponds to an intersection of $W_\epsilon^u(N^0)$ and $W_\epsilon^s(\hat{N}^n)$.

Rewriting (2.17) and (2.18) in the slow time gives

$$u' = 0, \tag{2.19}$$

$$x' = 1. \tag{2.20}$$

Proposition 2.4. *The vector field defined by* (2.19) *and* (2.20) *is nowhere tangent to any* P^i *or* $\hat{P}^i$.

The following result of Szmolyan [37, unpublished] says that for small $\epsilon > 0$, there is a Riemann–Dafermos solution near Γ, with the same left and right states.

Theorem 2.1. *If* f *is sufficiently differentiable, then for small* $\epsilon > 0$, $W_\epsilon^u(N^0)$ *and* $W_\epsilon^s(\hat{N}^n)$ *meet transversally along a single orbit* $u_\epsilon(x)$ *near* Γ.

Proof. The Exchange Lemma implies that for small $\epsilon > 0, W_\epsilon^u(N^0)$ is C^1-close to $W_0^u(N^1)$ near $(\bar{u}^1, 0, \bar{x}^2)$. See Figure 3; the hypotheses of the Exchange Lemma are verified by Proposition 2.2 with $i = 1$ and Proposition 2.4 for P^1.

(Of course, N^0 and S^1 are not normally hyperbolic invariant manifolds, so before using the Exchange Lemma they should be replaced by $N_\delta^0 = N^0 \cap S_\delta^0$ and S_δ^1, respectively, for some sufficiently small $\delta > 0$. The latter are normally hyperbolic invariant manifolds by Proposition 2.3. We shall ignore this sort of detail. Also, strictly speaking, we are using the Exchange Lemma for systems not in the standard slow–fast form, as presented in Section 4.)

For consistency with the treatment of eigenvalues in Sections 3 and 5, the remainder of the proof is somewhat more complicated than necessary.

Let ℓ be a number between 1 and n. It is enough to show that $W_\epsilon^u(N^0)$ and $W_\epsilon^s(\hat{N}^n)$ meet transversally near $(q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell)$.

Proceeding inductively, we see that $W_\epsilon^u(N^0)$ is C^1-close to $W_0^u(N^{\ell-1})$ near $(\bar{u}^{\ell-1}, 0, \bar{x}^\ell)$. Using Proposition 2.2. $W_\epsilon^u(N^0)$ meets $W_0^s(S^\ell)$ in an ℓ-parameter family of connecting orbits to

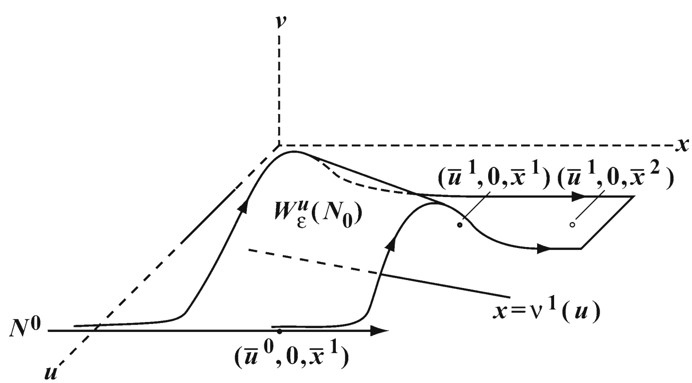

Figure 3. For small $\epsilon > 0, W_\epsilon^u(N^0)$ is C^1-close to $W_0^u(N^1)$ near $(\bar{u}^1, 0, \bar{x}^2)$. Compare Figure 2.

$$P^\ell = \{(u^\ell, 0, x^\ell) \in O^\ell \times \{0\} \times I^\ell : \\ f(u^1) - f(\bar{u}^0) - x^1(u^1 - \bar{u}^0) = 0, \dots, f(u^\ell) - f(u^{\ell-1}) \\ -x^\ell(u^\ell - u^{\ell-1}) = 0 \\ \text{for some } (x^1, u^1, \dots, x^{\ell-1}, u^{\ell-1}) \in I^1 \times O^1 \times \cdots \times I^{\ell-1} \times O^{\ell-1}\}.$$

Similarly,

$$\hat{P}^\ell = \{(u^\ell, 0, x^{\ell+1}) \in O^\ell \times \{0\} \times I^{\ell+1} : \\ f(u^{\ell+1}) - f(u^\ell) - x^{\ell+1}(u^{\ell+1} - u^\ell) \\ = 0, \dots, f(\bar{u}^n) - f(u^{n-1}) - x^n(\bar{u}^n - u^{n-1}) = 0 \\ \text{for some } (u^{\ell+1}, x^{\ell+2}, \dots, u^{n-1}, x^n) \in O^{\ell+1} \times I^{\ell+2} \times \cdots \times O^{n-1} \times I^n\},$$

and, for small $\epsilon > 0$, $W^s_\epsilon(\hat{N}^n)$ is C^1-close to $W^s_0(\hat{N}^\ell)$ near $(\bar{u}^\ell, 0, \bar{x}^\ell)$.

From (R1), P^ℓ, of dimension ℓ, and $\hat{N}^\ell$, of dimension $n - \ell + 1$, meet transversally within S^ℓ in the point $(\bar{u}^\ell, 0, \bar{x}^\ell)$. It follows from Proposition 2.2 with $i = \ell$ that $W^u_0(N^{\ell-1})$ and $W^s_0(\hat{N}^\ell)$ meet transversally along the connecting orbit $(q^\ell(\xi), \dot{q}^\ell(\xi), \bar{x}^\ell)$. Hence, $W^u_\epsilon(N^0)$ and $W^s_\epsilon(N^n)$ meet transversally in a curve that passes near $(q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell)$. □

Remark. The Exchange Lemma has been used $\ell - 1$ times in following $W^u_\epsilon(N^0)$ and $n - \ell$ times in following $W^s_\epsilon(\hat{N}^n)$. Let $L = \max(\ell - 1, n - \ell)$. Thus, using the Exchange Lemma as stated in Section 4, f must be at least C^{6L+1}. A similar remark applies to the theorems in Sections 3 and 5.

3. FAST EIGENVALUES

Eigenvalues λ and corresponding eigenfunctions $U(x)$ of the linearized Dafermos operator at $u_\epsilon(x)$ satisfy

$$\lambda U + (Df(u) - xI)U_x + D^2 f(u) u_x U = \epsilon U_{xx} \quad \text{with} \quad u = u_\epsilon(x) \quad \text{and} \quad U(\pm\infty) = 0. \tag{3.1}$$

The Eqs. (2.4) and (3.1) may be combined into a first-order autonomous system. Letting $x = x_0 + \epsilon\xi$, using a dot to denote derivative with respect to ξ, and letting $\rho = \epsilon\lambda$, we convert this system to

$$\begin{aligned} \dot{u} &= v, & (3.2)\\ \dot{v} &= (Df(u) - xI)v, & (3.3)\\ \dot{x} &= \epsilon, & (3.4)\\ \dot{\rho} &= 0, & (3.5)\\ \dot{U} &= V, & (3.6)\\ \dot{V} &= \rho U + (Df(u) - xI)V + D^2 f(u)vU. & (3.7) \end{aligned}$$

(We have added ρ as a state variable.) The system (3.2)–(3.7) is a linear skew-product flow on the trivial vector bundle $uvx\rho$-space $\times UV$-space. We should take take $\rho \in \mathbb{C}$ and $(U, V) \in \mathbb{C} \times \mathbb{C}$.

We consider the systems (3.2)–(3.7) with $\epsilon = 0$:

$$\begin{aligned} \dot{u} &= v, & (3.8)\\ \dot{v} &= (Df(u) - xI)v, & (3.9)\\ \dot{x} &= 0, & (3.10)\\ \dot{\rho} &= 0, & (3.11)\\ \dot{U} &= V, & (3.12)\\ \dot{V} &= \rho U + (Df(u) - xI)V + D^2 f(u)vU. & (3.13) \end{aligned}$$

For each $i = 0, \dots, n$, let

$$\mathcal{S}^i = \{(u, v, x, \rho, U, V) : (u, v, x) \in S^i \text{ and } U = V = 0\}.$$

Each $\mathcal{S}^i$ is a manifold of equilibria of (3.8)–(3.13).

The linear system (3.12) and (3.13), with $v = 0$ and (u, x, ρ) fixed, has $2n$ eigenvalues

$$\mu^j_{\pm}(u, v, \rho) = \frac{1}{2}\left(\nu^j(u) - x \pm ((\nu^j(u) - x)^2 + 4\rho)^{\frac{1}{2}}\right), \quad j = 1, \dots, n.$$

For $(u, 0, x)$ in one of the sets S^i, let $r(u, x) = \min_j |\nu^j(u) - x|$, and let $G(u, x) = \{\rho = \sigma + i\omega \in \mathbb{C} : \sigma > -\frac{\omega^2}{r(u,x)^2}\}$. For $\rho \in G(u, x)$, each $\mu^j_+(u, x, \rho)$ has positive real part and each $\mu^j_-(u, x, \rho)$ has negative real part. Thus, at $(u, 0, x, \rho, 0, 0)$ in $\mathcal{S}^i$ with $\rho \in G(u, x)$, the linearization of (3.8)–(3.13) has the semisimple eigenvalue 0 with multiplicity $n + 2$; n nonzero "spatial" eigenvalues $\nu^k(u) - x, k = 1, \dots, n$, of which $n - i$ are positive and i are negative; and $2n$ "linear" eigenvalues $\mu^k_{\pm}(u, x, \rho), k = 1, \dots, n$, of which n have positive real part and n have negative real part.

Let $G = \cap_{i=1}^n G(\bar{u}^{i-1}, \bar{x}^i) \cap \cap_{i=1}^n G(\bar{u}^i, \bar{x}^i)$, which includes the real interval $(0, \infty)$. The theorems of this section apply to any $\bar{\rho} \in G$. For simplicity, however, we will only consider $\bar{\rho} \in (0, \infty)$. Therefore, in defining the state space, we shall take $\rho \in \mathbb{R}$ and $(U,V) \in \mathbb{R}^n \times \mathbb{R}^n$. Note that for $(u,0,x)$ in any S^i and any $\rho > 0$, all $\mu_+^j(u,x,\rho)$ are positive and all $\mu_-^j(u,x,\rho)$ are negative.

Let $(\tilde{u},\tilde{v},x^0) \in W_0^u(u^0,0,x^0)$, and let $(u,v)(\xi)$ be the solution of (3.8) and (3.9) with $(u,v)(0)=(\tilde{u},\tilde{v})$. Then $\lim_{\xi\to-\infty}(u,v)(\xi)=(u^0,0)$. The linear system (3.12) and (3.13) with $(u,v)=(u,v)(\xi), x=x^0$, and $\rho=\rho^0>0$ has n linearly independent solutions that approach 0 exponentially as $\xi\to-\infty$. Let $E_0^u(\tilde{u},\tilde{v},x^0,\rho^0)$ denote their span. Then for the system (3.8)–(3.13),

$$W_0^u(u^0,0,x^0,\rho^0,0,0)=\{(u,v,x^0,\rho^0,U,V):\\ (u,v,x^0)\in W_0^u(u^0,0,x^0) \text{ and } (U,V)\in E_0^u(u,v,x^0,\rho^0)\}.$$

Similarly, let $(\tilde{u},\tilde{v},x^0) \in W_0^s(u^0,0,x^0)$, and let $(u,v)(\xi)$ be the solution of (3.8) and (3.9) with $(u,v)(0)=(\tilde{u},\tilde{v})$. Then $\lim_{\xi\to\infty}(u,v)(\xi)=(u^0,0)$. The linear system (3.12) and (3.13) with $(u,v)=(u,v)(\xi), x=x^0$, and $\rho=\rho^0>0$ has n linearly independent solutions that approach 0 exponentially as $\xi\to\infty$. Let $E_0^s(\tilde{u},\tilde{v},x^0,\rho^0)$ denote their span. Then for the system (3.8)–(3.13),

$$W_0^s(u^0,0,x^0,\rho^0,0,0)=\{(u,v,x^0,\rho^0,U,V):\\ (u,v,x^0)\in W_0^s(u^0,0,x^0) \text{ and } (U,V)\in E_0^s(u,v,x^0,\rho^0)\}.$$

Recall that each $q^i(\xi)$, with $\xi = X-\bar{x}^iT$, is a traveling wave solution of $u_T+f(u)_x=u_{xx}$, so an Evans function [7] for $q^i(\xi)$ for the PDE (1.1) can be defined. For a number $\bar{\rho}>0$, consider the following assumptions:

- (F1) There is a number $\ell, 1\leqslant \ell \leqslant n$, such that $\bar{\rho}$ is a simple zero of the Evans function for $q^\ell(\xi)$.
- (F2) For $i\neq\ell, \bar{\rho}$ is not a zero of the Evans function for $q^i(\xi)$.

In geometric language, the second assumption just says that for $i\neq \ell$, $E_0^u(q^i(0),\dot{q}^i(0),\bar{x}^i,\bar{\rho})$ and $E_0^s(q^i(0)),\dot{q}^i(0),\bar{x}^i,\bar{\rho})$ are transverse, so their intersection is the origin. The first assumption says that $E_0^u(q^\ell(0),\dot{q}^\ell(0),\bar{x}^\ell,\bar{\rho})$ and $E_0^s(q^\ell(0),\dot{q}^\ell i(0),\bar{x}^\ell,\bar{\rho})$ have one-dimensional intersection, and this intersection breaks in a nondegenerate manner as ρ varies. The Evans function is described in more detail in the proof of Proposition 3.3.

Let J be an interval around $\bar{\rho}$ in $\mathbb{R}$.

Proposition 3.1. *Suppose $\bar{\rho}$ satisfies assumption* (F2).

For $i=1,\dots,\ell-1$, let $(u^{i-1},0,x^i,\rho^0,0,0)\in\mathcal{S}^{i-1}$ with $u^{i-1}\in M^{i-1},x^i\in I^i$, and $\rho^0\in J$. Then $W_0^u(u^{i-1},0,x^i,\rho^0,0,0)$ and $W_0^s(\mathcal{S}^i)$ meet transversally along the curve $(u(\xi),\dot{u}(\xi),x^i,\rho^0,0,0)$, where $u(\xi)$ is a solution of (2.15) *from u^{i-1} to the corresponding point $u^i\in M^i$.*

For $i = \ell, \ldots, n-1$, *let*$(u^{i+l}, 0, x^{i+1}, \rho^0, 0, 0) \in \mathcal{S}^{i+1}$ *with* $u^{i+1} \in \hat{M}^{i+1}$, $x^{i+1} \in I^{i+1}$, *and* $\rho^0 \in J$. *Then* $W_0^u(\mathcal{S}^i)$ *and* $W_0^s(u^{i+1}, 0, x^{i+1}, \rho^0, 0, 0)$ *meet transversally along the curve* $(u(\xi), \dot{u}(\xi), x^{i+1}, \rho^0, 0, 0)$ *where* $u(\xi)$ *is a solution of* (2.16) *to* u^{i+l} *from the corresponding point* $u^i \in M^i$.

Note that dim $W_0^u(u^{i-1}, 0, x^i, \rho^0, 0, 0) = 2n - (i-1)$, dim $W_0^s(\mathcal{S}^i) = 2n + i + 2$, and the sum of these numbers is $4n+3$, which is one more than the dimension of $uvxUV\ \rho$-space. Similarly, dim $W_0^u(\mathcal{S}^i) = 3n - i + 2$ and dim $W_0^s(u^{i+l}, 0, x^{i+1}, \rho^0, 0, 0) = n + i + 1$.

Let

$$
\begin{aligned}
\mathcal{Q}^i &= \{(u,0,x,\rho,0,0) \in \mathcal{S}^i : (u,0,x) \in P^i\} \quad \text{and} \\
\mathcal{R}^i &= \{(u,0,x,\rho,0,0) \in \mathcal{S}^i : u \in M^i\}, \\
\hat{\mathcal{Q}}^i &= \{(u,0,x,\rho,0,0) \in \mathcal{S}^i : (u,0,x) \in \hat{P}^i\} \quad \text{and} \\
\hat{\mathcal{R}}^i &= \{(u,0,x,\rho,0,0) \in \mathcal{S}^i : u \in \hat{M}^i\}.
\end{aligned}
$$

Note that dim $\mathcal{Q}^i = i+1$, dim $\mathcal{R}^i = i+2$, dim $W_0^u(\mathcal{R}^i) = 2n+2$ independent of i, dim $\hat{\mathcal{Q}}^i = n - i + 1$, dim $\hat{\mathcal{R}}^i = n - i + 2$, and dim $W_0^s(\hat{\mathcal{R}}^i) = 2n+2$ independent of i.

Note that for $\epsilon > 0$, an intersection of $W_\epsilon^u(\mathcal{R}^0)$ and $W_\epsilon^s(\hat{\mathcal{R}}^n)$ for which $(U, V) \neq (0,0)$ gives simultaneously a Riemann–Dafermos solution $(u_\epsilon(\xi), v_\epsilon(\xi), \epsilon\xi)$ as in Section 2, an eigenvalue $\lambda(\epsilon) = \frac{\rho(\epsilon)}{\epsilon}$ and a corresponding eigenfunction $(U_\in(\xi), V_\epsilon(\xi))$.

Proposition 3.2. *Suppose* $\bar{\rho}$ *satisfies assumption* (F2).

For $i = 1, \ldots, \ell - 1$, $W_0^u(\mathcal{R}^{i-1})$ *meets* $W_0^s(\mathcal{S}^i)$ *transversally along an* $(i+1)$*-dimensional family of connecting orbits. In particular, for each point* $(u^{i-1}, 0, x^i, \rho^0, 0, 0) \in \mathcal{R}^{i-1}$ *with* $x^i \in I^i$, *there is a unique* u^i *such that one orbit* $(u(\xi), \dot{u}(\xi), x^i, \rho^0, 0, 0)$ *of the family connects it to* $(u^i, 0, x^i, \rho^0, 0, 0)$. *The set of such* u^i *is* M^i, *and the set of such* $(u^i, 0, x^i, \rho^0, 0, 0)$ *is* $\mathcal{Q}^i$. *If* $(u^i, 0, x^i, \rho^0, 0, 0) \in \mathcal{Q}^i$ *and* $(\tilde{u}, \tilde{v}, x^i, \rho^0, 0, 0) \in W_0^u(\mathcal{R}^{i-1}) \cap W_0^s(u^i, 0, x^i, \rho^0, 0, 0)$, *then the tangent spaces to* $W_0^u(\mathcal{R}^{i-1})$ *and* $W_0^s(u^i, 0, x^i, \rho^0, 0, 0)$ *at* $(\tilde{u}, \tilde{v}, x^i, \rho^0, 0, 0)$ *have one-dimensional intersection.*

For $i = \ell, \ldots, n-1$, $W_0^u(\mathcal{S}^i)$ *meets* $W_0^s(\hat{\mathcal{R}}^{i+1})$ *transversally along an* $n - i + 1$*-dimensional family of connecting orbits. In particular, for each point* $(u^{i+1}, 0, x^{i+l}, \rho^0, 0, 0) \in \hat{\mathcal{R}}^{i+1}$ *with* $x^{i+1} \in I^{i+1}$, *there is a unique* u^i *such that one orbit* $(u(\xi), \dot{u}(\xi), x^{i+1}, \rho^0, 0, 0)$ *of the family connects* $(u^i, 0, x^{i+1}, \rho^0, 0, 0)$ *to it. The set of such* u^i *is* M^i, *and the set of such* $(u^i, 0, x^i, \rho^0, 0, 0)$ *is* $\hat{\mathcal{Q}}^i$. *If* $(u^i, 0, x^i, \rho^0, 0, 0) \in \hat{\mathcal{Q}}^i$ *and* $(\tilde{u}, \tilde{v}, x^i, \rho^0, 0, 0) \in W_0^u(u_i, 0, x_i, \rho^0, 0, 0) \cap W_0^S(\mathcal{R}^{i+1})$, *then the tangent spaces to* $W_0^u(u_i, 0, x^i, \rho^0, 0, 0)$ *and* $W_0^s(\mathcal{R}^{i+1})$ *at* $(\tilde{u}, \tilde{v}, x^i, \rho^0, 0, 0)$ *have one-dimensional intersection.*

Suppose $\bar{\rho}$ satisfies assumption (F1). Let (U_1, V_1) span $E_0^u(q^\ell(0), \dot{q}^\ell i(0), \bar{x}^\ell, \bar{\rho} \cap E_0^s(q^\ell(0), \bar{x}^\ell, \bar{\rho})$ and let $(U_1(\xi), V_1(\xi))$ be the solution of the linear systems (3.12) and (3.13) with $(u, v, x, \rho) = (q^\ell(\xi), \dot{q}^\ell(\xi), \bar{x}^\ell \bar{\rho})$ and $(U(0), V(0)) = (U_1, V_1)$.

Proposition 3.3. *Suppose $\bar{\rho}$ satisfies assumption* (F1). *Then $W_0^u(\mathcal{R}^{\ell-1})$ and $W_0^s(\hat{\mathcal{R}}^\ell)$, both of which have dimension $2n + 2$ in $(4n + 2)$-dimensional $uvx\rho UV$ -space, meet transversally along*

$$(q^\ell(\xi), \dot{q}^\ell(\xi), \bar{x}^\ell, \bar{\rho}, U_1(\xi), V_1(\xi)). \tag{3.14}$$

Their intersection includes the two-dimensional manifold $(q^\ell(\xi), \dot{q}^\ell(\xi), \bar{x}^\ell, \bar{\rho}, aU_1(\xi), aV_1(\xi))$, and includes no other points near $(q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell, \bar{\rho}, U_1(0), V_1(0))$.

Proof. The intersection clearly includes the given manifold. Assuming transversality, the intersection is two-dimensional and therefore cannot include other points near $(q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell, \bar{\rho}, aU_1(0), aV_1(0))$. Thus we only need to show that the intersection is transverse, i.e., that the tangent spaces to $W_0^u(\mathcal{R}^{\ell-1})$ and $W_0^s(\hat{\mathcal{R}}^\ell)$ at $(q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell, \bar{\rho}, U_1(0), V_1(0))$ span $\mathbb{R}^{4n+2}$.

Choose $(U_2, V_2), \ldots, (U_n, V_n)$ such that

(1) $\{(U_1, V_1), (U_2, V_2), \ldots, (U_n, V_n)\}$ is a basis for $E_0^u(q^\ell(0)), \dot{q}^\ell(0), \bar{x}^\ell, \bar{\rho})$;

(2) $\{(U_1, V_1), (U_{n+1}, V_{n+1}), \ldots, (U_{2n-1}, V_{2n-1})\}$ is a basis for E_0^s $(q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell, \bar{\rho})$.

On UV-space we use the coordinate system

$$(U, V) = a(U_1, V_1) + \sum_{k=2}^{n} b_k(U_k, V_k) + \sum_{k=n+1}^{2n-1} c_k(U_k, V_k) + z(U_n, V_n).$$

Let $\tilde{p} = (q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell, \bar{\rho}, U_1, V_1)$.

A neighborhood of $\tilde{p}$ in $W_0^u(\mathcal{R}^{\ell-1})$ can be parameterized as follows. $W_0^u(N^{\ell-1})$ is locally parameterized by a map $\phi\colon \mathbb{R}^{n+1} \to uvx$-space with $\phi(0) = (q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell)$ and $D\phi$ injective. $W_0^u(\mathcal{R}^{\ell-1})$ is locally parameterized by a map

$$(\alpha, \rho, a, b) \in \mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \to uvx\rho abcz - \text{space},$$
$$(\alpha, \rho, a, b) \to (\phi(\alpha), \rho, a, b, C(\alpha, \rho, a, b), Z(\alpha, \rho, a, b)),$$

with $C(\alpha, \rho, a, b) = C_0(\alpha, \rho)a + C_1(\alpha, \rho)b$, $Z(\alpha, \rho, a, b) = Z_0(\alpha, \rho)a + Z_1(\alpha, \rho)b$, and $C_i(0, \bar{\rho})$ and $Z_i(0, \bar{\rho})$ all equal to 0. Note that $(\alpha, \rho, a, b) = (0, \bar{\rho}, 1, 0)$ corresponds to $\tilde{p}$.

Similarly, a neighborhood of $\tilde{p}$ in $W_0^s(\hat{\mathcal{R}}^\ell)$ can be parameterized as follows. $W_0^s(\hat{N}^\ell)$ is locally parameterized by a map $\hat{\phi}: \mathbb{R}^{n+1} \to uvx$-space with $\hat{\phi} = (q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell)$ and $D\hat{\phi}$ injective. $W_0^s(\hat{\mathcal{R}}^\ell)$ is locally parameterized by a map

$$(\beta, \rho, a, c) \in \mathbb{R}^{(n+1)+1+(n-1)+1} \to uvx\rho abcz\text{-space},$$
$$(\beta, \rho, a, c) \to (\hat{\phi}(\beta, \rho, a, \hat{B}(\beta, \rho, a, c), c, \hat{Z}(\beta, \rho, a, c))$$

with $\hat{B}(\beta, \rho, a, c) = \hat{B}_0(\beta, \rho)a + \hat{B}_1(\beta, \rho)c$, $\hat{Z}(\beta, \rho, a, c) = \hat{Z}_0(\beta, \rho)a, + \hat{Z}_1(\beta, \rho)c$, and $\hat{B}_i(0, \bar{\rho})$ and $\hat{Z}_i(0, \bar{\rho})$ all equal to 0. Note that $(\beta, \rho, a, c) = (0, \bar{\rho}, 1, 0)$ corresponds to $\tilde{p}$.

The tangent spaces to $W_0^u(\mathcal{R}^{\ell-1})$ and $W_0^s(\hat{\mathcal{R}}^\ell)$ at $\tilde{p}$ are spanned, respectively, by the column vectors in the matrices

$$\begin{pmatrix} D\phi(0) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \\ D_\alpha C(0, \bar{\rho}, 1, 0) & D_\rho C(0, \bar{\rho}, 1, 0) & 0 & 0 \\ D_\alpha Z(0, \bar{\rho}, 1, 0) & D_\rho Z(0, \bar{\rho}, 1, 0) & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} D\hat{\phi}(0) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ D_\beta \hat{B}(0, \bar{\rho}, 1, 0) & D_\rho \hat{B}(0, \bar{\rho}, 1, 0) & 0 & 0 \\ 0 & 0 & 0 & I \\ D_\rho \hat{Z}(0, \bar{\rho}, 1, 0) & D_\rho \hat{Z}(0, \bar{\rho}, 1, 0) & 0 & 0 \end{pmatrix}.$$

The span of all these column vectors equals the span of the column vectors in the matrix

$$\begin{pmatrix} D\phi(0) & 0 & D\hat{\phi}(0) & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ D_\alpha Z(0, \bar{\rho}, 1, 0) & D_\rho Z(0, \bar{\rho}, 1, 0) & D_\beta \hat{Z}(0, \bar{\rho}, 1, 0) & D_\rho \hat{Z}(0, \bar{\rho}, 1, 0) & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

Now the $n+1$ column vectors in $D\phi(0)$ and the $n+1$ column vectors in $D\hat{\phi}(0)$ together span $(2n+1)$-dimensional uvx-space by [33], so the span of the $2n+2$ column vectors in the matrix

$$\begin{pmatrix} D\phi(0) & D\hat{\phi}(0) \\ D_\alpha Z(0,\bar{\rho},1,0) & D_\beta \hat{Z}(0,\bar{\rho},1,0) \end{pmatrix} \tag{3.16}$$

has dimension at least 2n+1. In fact the dimension is exactly $2n+1$. The reason is that the known intersection (3.14) of $W_0^u(\mathcal{R}^{\ell-1})$ and $W_0^s(\hat{\mathcal{R}}^\ell)$ implies the existence of functions $(\alpha(\xi), b(\xi))$ and $(\beta(\xi), c(\xi))$, with $(\alpha(0), b(0)) = (0,0)$, $(\beta(0), c(0)) = (0,0)$, and $\alpha'(0)$ and $\beta'(0)$ nonzero, such that

$$\begin{aligned} &(\phi(\alpha(\xi)), \hat{\rho}, 1, b(\xi), C(\alpha(\xi), \bar{\rho}, 1, b(\xi)), Z(\alpha(\xi), \bar{\rho}, 1, b(\xi))) \\ &= (\hat{\phi}(\beta(\xi)), \bar{\rho}, 1, \hat{B}(\beta(\xi), \bar{\rho}, 1, c(\xi)), c(\xi), \hat{Z}(\beta(\xi), \bar{\rho}, 1, c(\xi))). \end{aligned}$$

Therefore,

$$\begin{pmatrix} D\phi(0)\alpha'(0) \\ D_\alpha Z(0,\bar{\rho},1,0)\alpha'(0) \end{pmatrix} = \begin{pmatrix} D\hat{\phi}(0)\beta'(0) \\ D_\beta \hat{Z}(0,\bar{\rho},1,0)\beta'(0) \end{pmatrix}.$$

It follows easily that the span of the columns of (3.16) has dimension $2n+1$, not $2n+2$. An easy consequence is that the columns of (3.15) span $\mathbb{R}^{4n+2}$ if and only if the matrix

$$\begin{pmatrix} 1 & 1 \\ D_\rho Z(0,\bar{\rho},1,0) & D_\rho \hat{Z}(0,\bar{\rho},1,0) \end{pmatrix}$$

is invertible, which occurs if and only if

$$D_\rho Z(0,\bar{\rho},1,0) - D_\rho \hat{Z}(0,\bar{\rho},1,0) \neq 0. \tag{3.17}$$

The latter is equivalent to the assumption that $\bar{\rho}$ is a simple 0 for the Evans function of q^ℓ. This may be seen as follows: $E_0^u(q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell, \bar{\rho})$ and $E_0^s(q^\ell(0), \dot{q}^\ell(0), \bar{x}^\ell, \bar{\rho})$ are spanned, respectively, by the column vectors in the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & I \\ C_0(0,\rho) & C_1(0,\rho) \\ Z_0(0,\rho) & Z_1(0,\rho) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \hat{B}_0(0,\rho) & \hat{B}_1(0,\rho) \\ 0 & I \\ \hat{Z}_0(0,\rho) & \hat{Z}_1(0,\rho) \end{pmatrix}$$

Therefore, by definition, the Evans function is

$$E(\rho) = \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & I & \hat{B}_0(0,\rho) & \hat{B}_1(0,\rho) \\ C_0(0,\rho) & C_1(0,\rho) & 0 & I \\ Z_0(0,\rho) & Z_1(0,\rho) & \hat{Z}_0(0,\rho) & \hat{Z}_1(0,\rho) \end{pmatrix}$$

$$= \det \begin{pmatrix} I & \hat{B}_0(0,\rho) & \hat{B}_1(0,\rho) \\ C_1(0,\rho) & 0 & I \\ Z_1(0,\rho) & \hat{Z}_0(0,\rho) & \hat{Z}_1(0,\rho) \end{pmatrix} + (-1)^n \det \begin{pmatrix} 0 & I & \hat{B}_1(0,\rho) \\ C_0(0,\rho) & C_1(0,\rho) & I \\ Z_0(0,\rho) & Z_1(0,\rho) & \hat{Z}_0(0,\rho) \end{pmatrix}.$$

Since all the functions are 0 at $\rho=\bar{\rho}$, $E(\bar{\rho})=0$ and

$$E'(\bar{\rho})=\det\begin{pmatrix} I & D_\rho\hat{B}_0(0,\bar{\rho}) & 0\\ 0 & 0 & I\\ 0 & D_\rho\hat{Z}_0(0,\bar{\rho}) & 0\end{pmatrix}+(-1)^n\det\begin{pmatrix} 0 & I & 0\\ D_\rho C_0(0,\bar{\rho}) & 0 & I\\ D_\rho Z_0(0,\bar{\rho}) & 0 & 0\end{pmatrix}$$
$$=(-1)^n(D_\rho Z_0(0,\bar{\rho})-D_\rho\hat{Z}_0(0,\bar{\rho})).$$

Thus $E'(\bar{\rho})\neq 0$ if and only if (3.17) holds. □

Let $\tilde{\Gamma}$ denote the union of the $n+1$ line segments $\{(\bar{u}^i,0,x,\bar{\rho},0,0): \bar{x}^i\leqslant x\leqslant\bar{x}^{i+1}\}$; the $n-1$ connecting orbits $(q^i(\xi),\dot{q}^i(\xi),\bar{x}^i,\bar{\rho},0,0)$, $i\neq\ell$; and the connecting orbit $(q^\ell(\xi),\dot{q}^\ell(\xi),\bar{x}^\ell,\bar{\rho},U_1(\xi),V_1(\xi))$.

Analogous to the situation in Section 2, the $\mathcal{R}^i$ and $\mathcal{S}^i$, which remain invariant for $\epsilon>0$, are not normally hyperbolic invariant manifolds, but subsets $\mathcal{R}^i_\delta$ and $\mathcal{S}^i_\delta$ are. We shall ignore this detail and simply speak of $W^u_\epsilon(\mathcal{R}^i)$, etc.

The following theorem says that if $\bar{\rho}$ satisfies assumptions (F1) and (F2), then for small $\epsilon>0$, there is an eigenvalue near $\frac{\bar{\rho}}{\epsilon}$ with eigenfunction near $\tilde{\Gamma}$. In particular, consistent with the asymptotic expansion of [23], the eigenfunction is approximately 0 except near $\bar{x}^\ell$.

Theorem 3.1. *Suppose f is sufficiently differentiable and $\bar{\rho}$ satisfies assumptions* (F1) *and* (F2)*. Then for small $\epsilon>0$, $W^u_\epsilon(\mathcal{R}^0)$ and $W^s_\epsilon(\hat{\mathcal{R}}^n)$, both of which have dimension $2n+2$ in $(4n+2)$-dimensional $uvx\rho UV$-space, meet transversally near $\tilde{\Gamma}$. Their intersection includes a two-dimensional manifold $(u_\epsilon(\xi),\dot{u}_\epsilon(\xi),\epsilon\xi,\rho(\epsilon),aU_\epsilon(\xi),aV_\epsilon(\xi))$, with $u_\epsilon(\xi)$ the Riemann–Dafermos solution and $\rho(\epsilon)=\bar{\rho}+O(\epsilon)$.*

Remark. The theorem holds for any $\bar{\rho}\in G$ that satisfies (F1) and (F2), except that the intersection has one real dimension (ξ) plus one complex dimension (a).

Proof. Following $W^u_\epsilon(\mathcal{R}^0)$ by the Exchange Lemma, whose hypotheses are verified by Proposition 3.2 and an analogue of Proposition 2.4, we see that $W^u_\epsilon(\mathcal{R}^0)$ is eventually C^1-close to $W^u_\epsilon(\mathcal{R}^{\ell-1})$. Following $W^s_\epsilon(\hat{\mathcal{R}}^n)$ backwards by the Exchange Lemma, we see that $W^s_\epsilon(\hat{\mathcal{R}}^n)$ is eventually C^1-close to $W^s_0(\hat{\mathcal{R}}^\ell)$. By Proposition 3.3, $W^u_\epsilon(\mathcal{R}^0)$ and $W^s_\epsilon(\hat{\mathcal{R}}^n)$ meet transversally near $(q^\ell(0),\dot{q}^\ell(0),\bar{x}^\ell,\bar{\rho},U_1(0),V_1(0))$. The intersection therefore contains points with nonzero (U,V)-component. Since the manifolds both have dimension $2n+2$, the intersection has dimension 2, and must have the given form. □

Remark. Since the intersection of $W^u_0(\mathcal{R}^{\ell-1})$ and $W^s_0(\bar{u}^\ell,0,x^\ell,0,0,\bar{\rho})$ is two-dimensional (it is the manifold described in Proposition 3.3), we cannot use the Exchange Lemma at this stage.

Theorem 3.2. *Suppose that f is sufficiently differentiable and for all $i=1,\dots,n$, $\bar\rho\in G$ is not a zero of the Evans function for $q^i(\xi)$. Then there are numbers $\delta_0>0$ and $\epsilon_0>0$ such that for $|\rho-\bar\rho|<\delta_0$ and $0<\epsilon<\epsilon_0$, $\lambda=\frac{\rho}{\epsilon}$ is not an eigenvalue of the linearized Dafermos operator at the Riemann–Dafermos solution u_ϵ.*

Proof. We give the proof for $\bar\rho>0$. In this case, $W_0^u(\mathcal{R}^{\ell-1})$ and $W_0^s(\hat{\mathcal{R}}^\ell)$ meet transversally along the two-dimensional manifold $(q^\ell(\xi),\dot q^\ell(\xi),\bar x^\ell,\rho,0,0)$ at $(q^\ell(0),\dot q^\ell(0),\bar x^\ell,\bar\rho,0,0)$. It follows that for small $\epsilon>0$, $W_\epsilon^u(\mathcal{R}^0)$ and $W_\epsilon^s(\hat{\mathcal{R}}^n)$ meet transversally near $(q^\ell(0),\dot q^\ell(0),\bar x^\ell,\bar\rho,0,0)$ along their known two-dimensional intersection $(u_\epsilon(\xi),\dot u_\epsilon(\xi),\epsilon\xi,\rho,0,0)$. The result is just a restatement of this fact. □

4. EXCHANGE LEMMA

The treatment of slow eigenvalues will require an extension of the Exchange Lemma to a certain degenerate situation involving a skew-product flow on a vector bundle. In this section, we state and prove this extension. Our proof includes a proof of the usual Exchange Lemma, with the context changed a little to accommodate singular perturbation problems that are not in the standard slow-fast form. It is based on the methods of [4] and [6].

On the trivial vector bundle $\mathbb{R}^{p+1}\times\mathbb{R}^p$, with coordinates (θ,Θ), consider a differential equation

$$\dot\theta=f(\theta,\epsilon), \tag{4.1}$$

$$\dot\Theta=A(\theta,\epsilon)\Theta \tag{4.2}$$

with f and $A(\theta,\epsilon)\Theta$ $C^{r+6}, r\geqslant 1$. Note that (4.2) is linear in Θ, and that $A(\theta,\epsilon)\Theta$ is C^{r+6} if and only if A is C^{r+6}.

We assume that (4.1) satisfies the usual hypotheses of the Exchange Lemma at a point $(\theta_0,0)$, modified slightly since (4.1) is not in the standard slow-fast form. In particular, we assume that there are integers $k\geqslant 0,\ell\geqslant 1,m\geqslant 1$, and $n\geqslant 1$ such that $k+\ell+m+n=p$ and

- (E1) $\dot\theta=f(\theta,0)$ has a $(k+\ell+1)$-dimensional normally hyperbolic manifold of equilibria S_0.
- (E2) For each $\theta_0\in S_0$, $D_\theta f(\theta_0,0)$ has m eigenvalues with negative real part and n eigenvalues with positive real part.

(Actually, the Exchange Lemma permits $\ell=0$, but the extension discussed in this section requires $\ell\geqslant 1$.)

For the differential equation $\dot\theta=f(\theta,0)$, each point of S_0 has a stable manifold $W_0^s(\theta_0)$ of dimension m and an unstable manifold $W_0^u(\theta_0)$ of dimension n. The union of the stable manifolds of points of S_0 is $W_0^s(S_0)$, which has dimension $k+\ell+1+m$; the union of the unstable manifolds of points of S_0 is $W_0^u(S_0)$, which has dimension $k+\ell+1+n$ (see Figure 4).

By normal hyperbolicity, for small $\epsilon>0$ there is a normally hyperbolic, locally invariant manifold S_ϵ near S_0 with stable and unstable manifolds $W_\epsilon^s(S_\epsilon)$ and $W_\epsilon^u(S_\epsilon)$, respectively.

For small $\epsilon\geqslant 0$, let H_ϵ be a C^{r+6} submanifold of θ-space of dimension $k+n$. We assume that the sets $H_\epsilon\times\{\epsilon\}$ fit together to form a C^{r+6} submanifold of $\theta\epsilon$-space.

We assume (see Figure 4):

(E3) H_0 is transverse to $W_0^s(S_0)$ at a point p in $W_0^s(\theta_0)\backslash\{\theta_0\}$, and $T_pH_0\cap T_pW_0^s(\theta_0)=\{0\}$.

Then for small $\epsilon\geqslant 0$, the intersection of H_ϵ and $W_\epsilon^s(S_\epsilon)$ is a manifold of dimension k that projects along the invariant foliation of $W_\epsilon^s(S_\epsilon)$ to a k-dimensional submanifold P_ϵ of S_ϵ.

The sets $S_\epsilon\times\{\epsilon\}$ fit together to form a C^{r+6} submanifold S of $\theta\epsilon$-space. S can be parameterized by (w,ϵ) with $w\in\mathbb{R}^{k+\ell+1}$ and $(w,\epsilon)=(0,0)$ corresponding to $(\theta,\epsilon)=(\theta_0,0)$. Since S_0 consists of equilibria, the differential equation $\dot\theta=f(\theta,\epsilon)$ restricted to S has the form

$$\dot w=\epsilon g(w,\epsilon).$$

We assume

(E4) $g(0,0)$ is not tangent to the submanifold of w-space that corresponds to P_0.

Assumptions (E1)–(E4) are the usual hypotheses for the Exchange Lemma.

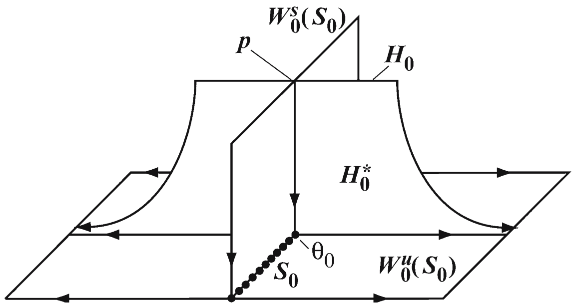

Figure 4. Assumptions (E1)–(E3) with $k=\ell=0$ and $m=n=1$. H_0 is one-dimensional. It is transverse to the two-dimensional set $W_0^s(S_0)$, and meets the one-dimensional set $W_0^s(\theta_0)$ in the point p. Under the flow, H_0 becomes the two-dimensional set H_0^*.

Next we discuss (4.2). We assume:

(E5) For each $\theta_0 \in S_0, A(\theta_0, 0)$ has the semisimple eigenvalue 0 with multipicity $k + \ell, m$ eigenvalues with negative real part, and n eigenvalues with positive real part.

The numbers k, ℓ, m, and n could be different from those used previously, but they are the same in the application to slow eigenvalues, so we assume them the same.

For each $\theta_0 \in S_0$, let $E_0^c(\theta_0), E_0^s(\theta_0)$, and $E_0^u(\theta_0)$ denote the center, stable, and unstable subspaces of $A(\theta_0, 0)$. Note that the restriction of $A(\theta_0, 0)$ to $E_0^c(\theta_0)$ is 0 (see Figure 5 (a)).

Let $\mathcal{S}_0 = \{(\theta_0, \Theta_0) : \theta_0 \in S_0 \text{ and } \Theta_0 \in E_0^c(\theta_0)\}$, a C^{r+6} vector bundle over S_0. For the differential equation

$$\dot{\theta} = f(\theta, 0), \qquad \dot{\Theta} = A(\theta, 0)\Theta, \tag{4.3}$$

$\mathcal{S}_0$ is a manifold of normally hyperbolic equilibria. Its stable manifold

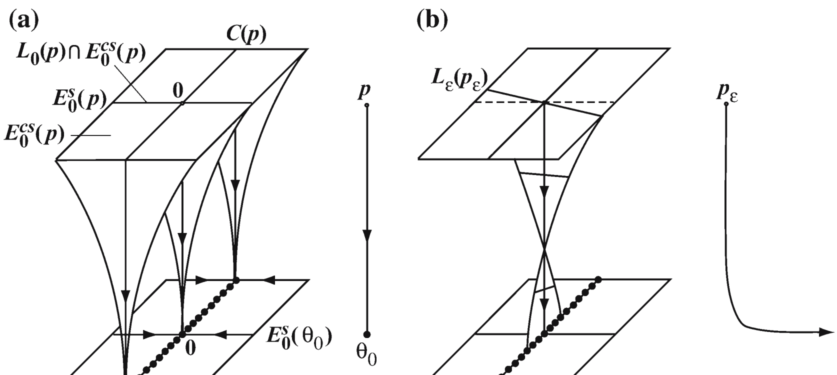

Figure 5. (a) The orbit of $\dot{\theta} = f(\theta, 0)$ through a point p in $H_0 \cap W_0^s(S_0)$, the corresponding point $\theta_0 \in S_0$, and the portion of the linear flow $\dot{\Theta} = A(\theta, 0)\Theta$ that lies above this set. In the picture, the intersection of $L_0(p)$ and $E_0^{cs}(p)$ is one-dimensional and coincides with the one-dimensional space $E_0^s(p)$. (b) The perturbed flow. The orbit of a point p_ϵ in $H_\epsilon \cap W_\epsilon^s(S_\epsilon)$ is shown at the right. $L_\epsilon(p_\epsilon) \cap E_\epsilon^{cs}(p_\epsilon)$ no longer coincides with $E_\epsilon^s(p_\epsilon)$, so the linear flow $\dot{\Theta} = A(\theta, \epsilon)\Theta$ takes it toward a one-dimensional space. To lowest order the direction of this space is $\Theta^c(\theta_0)$. This one-dimensional space will be transported to the right by the linear flow. This process is described by Theorem 4.1.

$W_0^s(\mathcal{S}_0)$ is a vector bundle over $W_0^s(S_0)$ with fiber dimension $k+\ell+m$. We denote the fibers $E_0^{cs}(\theta), \theta \in W_0^s(S_0)$.

To describe the foliation of $W_0^s(\mathcal{S}_0)$ by stable manifolds of points (θ_0, Θ_0) in $\mathcal{S}_0$, we first note that there are m-dimensional subspaces $E_0^s(\theta) \subset E_0^{cs}(\theta)$ such that for each $\theta_0 \in S_0$, the stable manifold of $(\theta_0, 0)$ for the system (4.3) is $W_0^s(\theta_0, 0) = \{(\theta, \Theta) : \theta \in W_0^s(\theta_0)$ and $\Theta \in E_0^s(\theta)\}$ (see Figure 5(a)).

Choose a complementary subspace $C(\theta)$ to $E_0^s(\theta)$ in $E_0^{cs}(\theta)$ such that $C(\theta)$ depends smoothly on $\theta \in W_0^s(S_0)$ and, for $\theta_0 \in S_0, C(\theta_0) = E_0^c(\theta_0)$; $C(\theta)$ has dimension $k+\ell$ (see Figure 5(a)). There is a projection Π from $W_0^s(\mathcal{S}_0)$ to $\mathcal{S}_0$ defined by $\Pi(\theta, \Theta) = (\theta_0, \Theta_0)$, where $\theta \in W_0^s(\theta_0)$ and $(\theta, \Theta) \in W_0^s(\theta_0, \Theta_0)$. For fixed $\theta \in W_0^s(\theta_0), \Pi|\{\theta\} \times C(\theta)$ is an isomorphism onto $\{\theta_0\} \times E_0^c(\theta_0)$. Let $\Theta(\theta, \Theta_0)$ denote the point of $C(\theta)$ such that $\Pi(\theta, \Theta(\theta, \Theta_0)) = (\theta_0, \Theta_0)$; $\Theta(\theta, \cdot)$ is an isomorphism from $E_0^c(\theta_0)$ to $C(\theta)$. Then the stable manifold of (θ_0, Θ_0) for the system (4.3) is $W_0^s(\theta_0, \Theta_0) = \{(\theta, \Theta) : \theta \in W_0^s(\theta_0)$ and $\Theta \in \Theta(\theta, \Theta_0) + E_0^s(\theta)\}$.

The unstable manifold $W_0^u(\mathcal{S}_0)$ has a similar description.

By normal hyperbolicity, for small $\epsilon > 0$ there is a normally hyperbolic, locally invariant C^{r+6} vector bundle $\mathcal{S}_\epsilon$ near $\mathcal{S}_0$, whose stable and unstable manifolds $W_\epsilon^s(\mathcal{S}_\epsilon)$ and $W_\epsilon^u(\mathcal{S}_\epsilon)$ are vector bundles over $W_\epsilon^s(S_\epsilon)$ and $W_\epsilon^u(S_\epsilon)$, respectively. We denote the fibers of $W_\epsilon^s(\mathcal{S}_\epsilon)$ by $E_\epsilon^{cs}(\theta), \theta \in W_\epsilon^s(S_\epsilon)$.

Note that $S_\epsilon \times \{0\}$ is invariant; its stable manifold is a C^{r+6} vector bundle over $W_\epsilon^s(S_\epsilon)$ with fibers $E_\epsilon^s(\theta) \subset E_\epsilon^{cs}(\theta)$.

For small $\epsilon \geqslant 0$, let $\mathcal{H}_\epsilon$ be a C^{r+6} vector bundle over H_ϵ,

$$\mathcal{H}_\epsilon = (\theta, \Theta) : \theta \in H_\epsilon \quad \text{and} \quad \Theta \in L_\epsilon(\theta),$$

where $L_\epsilon(\theta)$ is a subspace of Θ-space of dimension $k+1+n$. We assume that the sets $\mathcal{H}_\epsilon \times \{\epsilon\}$ fit together to form a C^{r+6} vector bundle in $\theta\Theta\epsilon$-space.

We assume

- (E6) $L_0(p)$ is transverse to $E_0^{cs}(p)$.
- (E7) For all $\theta \in H_0 \cap W^s(S_0)$, $L_0(\theta)$ intersects $E_0^s(\theta)$ in a space of dimension one.

See Figure 5(a). Assumption (E7) is highly nongeneric, but it occurs in the treatment of slow eigenvalues in Section 5. It is this degeneracy that requires an extension of the Exchange Lemma.

The vector bundle over $H_0 \cap W_0^s(S_0)$ whose fibers are $L_0(\theta) \cap E_0^{cs}(\theta)$, with fiber dimension $k+1$, projects by Π to a vector subbundle $\mathcal{P}_0$ of $\mathcal{S}_0$; the base is P_0, and the fibers $F_0(\theta_0)$ have dimension k because of (E7).

Our final assumption is that the intersection of $L_\epsilon(\theta)$ and $E^s_\epsilon(\theta)$ breaks in a nondegenerate manner as ϵ varies. In order to state this assumption more precisely, let $(\theta(\epsilon), \Theta(\epsilon)), \epsilon \geqslant 0$, be a smooth curve such that

$$\text{for } \epsilon \geqslant 0, (\theta(\epsilon), \Theta(\epsilon)) \text{ lies in } \mathcal{H}_\epsilon \cap W^s_\epsilon(\mathcal{S}_\epsilon), \tag{4.4}$$

$$\theta(0) = p \text{ and } \Theta(0) \in E^s(p) \backslash \{0\}. \tag{4.5}$$

Each point $(\theta(\epsilon), \Theta(\epsilon))$ lies in the stable fiber of a point of a point $(\theta_0(\epsilon), \Theta_0(\epsilon))$ in $\mathcal{S}_\epsilon$. We assume

(E8) There exists a curve $(\theta(\epsilon), \Theta(\epsilon))$ satisfying (4.4) and (4.5) for which $\Theta_0'(0) \notin F_0(p_0)$.

If we extend $(\theta(\epsilon), \Theta(\epsilon))$ to a smooth family of curves in $\mathcal{H}_\epsilon \cap W^s_\epsilon(\mathcal{S}_\epsilon)$, with $\theta(0)$ a parametrization of H_0, and $\Theta(0) \in E^s_0(\theta(0))$, then the above construction yields a smooth vector field $\Theta^c(\theta_0)$ on P_0, with $\Theta^c(\theta_0) \notin F_0(\theta_0)$ (see Figure 5(b)).

For $\theta_0 \in P_0$, let $G_0(\theta_0) = F_0(\theta_0) \oplus$ span $\Theta^c(\theta_0)$, of dimension $k+1$. We shall see that $G_0(\theta_0)$ is a subspace of $E^{cs}_0(\theta_0)$. Let $\mathcal{R}_0$ be the vector bundle over P_0 whose fibers are $G_0(\theta_0)$.

The sets $\mathcal{S}_\epsilon \times \{\epsilon\}$ fit together to form a C^{r+6} vector bundle $\mathcal{S}$ in $\theta\Theta\epsilon$-space. $\mathcal{S}$ can be parameterized by (w, W, ϵ) with $w \in \mathbb{R}^{k+\ell+1}$, $W \in \mathbb{R}^{k+\ell}$, and $(w, \epsilon) = (0, 0)$ corresponding to $(\theta, \epsilon) = (\theta_0, 0)$. Since $\mathcal{S}_0$ consists of equilibria, the differential equation (4.1) and (4.2) restricted to $\mathcal{S}$ has the form

$$\dot{w} = \epsilon g(w, \epsilon), \qquad \dot{W} = \epsilon B(w, \epsilon) W.$$

Let $w_0 \neq 0$ be a point on the positive semiorbit of $\dot{w} = g(w, 0)$ through 0, and let $q_0 \neq \theta_0$ be the corresponding point in θ-space. For small $\epsilon > 0$, under the flow of (4.1) and (4.2), H_ϵ becomes a manifold H^*_ϵ of dimension $k+n+1$ that passes near q_0 (see Figure 6). Also, $\mathcal{H}_\epsilon$ becomes a vector bundle $\mathcal{H}^*_\epsilon$ over H^*_ϵ.

Under the flow of

$$w' = g(w, 0)$$

the manifold corresponding to P_0 becomes a submanifold of w-space of dimension $k+1$. Let P^*_0 be the corresponding submanifold of S_0, which passes through q_0.

Under the flow of

$$w' = g(w, 0), \qquad W' = B(w, 0) W$$

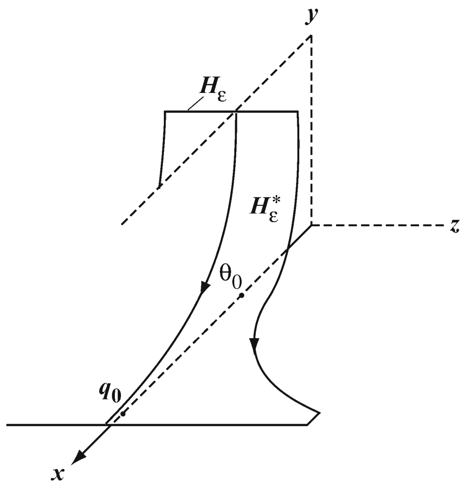

Figure 6. The perturbed flow $\dot{\theta} = f(\theta, \epsilon)$. The coordinates are consistent with Lemma 4.2, but the extra center manifold dimensions corresponding to u and v cannot be shown. Near q^0, the pictured H^*_ϵ is very close to xz-space. In general it is close to xuz-space.

the vector bundle corresponding to $\mathcal{R}_0$ becomes a vector bundle in wW-space. Let $\mathcal{R}^*_0$ be the corresponding vector subbundle of $\mathcal{S}_0$. $\mathcal{R}^*_0$ is a vector bundle over P^*_0 with fiber dimension $k+1$.

Let O be a small neighborhood of q_0 in θ-space.

Theorem 4.1. *As $\epsilon \to 0$, $H^*_\epsilon \cap O$ approaches $W^u(P^*_0) \cap O$ in the C^r topology, and $\mathcal{H}^*_\epsilon \cap (O \times \mathbb{R}^P)$ approaches $W^u(\mathcal{R}^*_0) \cap (O \times \mathbb{R}^P)$ in the C^r topology. The sets $(H^*_\epsilon \cap O) \times \{\epsilon\}$ and $(W^u(P^*_0) \cap O) \times \{0\}$ fit together to form a C^r submanifold of $\theta\epsilon$-space, and the sets $(\mathcal{H}^*_\epsilon \cap (O \times \mathbb{R}^P)) \times \{\epsilon\}$ and $(W^u(\mathcal{R}^*_0) \cap (O \times \mathbb{R}^p)) \times \{0\}$ fit together to form a C^r vector bundle in $\theta\Theta\epsilon$-space*

The first conclusion is just the usual Exchange Lemma. It only requires assumptions (E1)–(E4).

To prove the theorem, we first construct a convenient coordinate system. Let $\Gamma_0 \subset S_0$ be the curve that corresponds to the portion of the orbit of $w' = g(w, 0)$ from 0 to w_0. Γ_0 is a curve with endpoints p_0 and q_0.

Lemma 4.2. *Near Γ_0 there are coordinates $\theta = (x, u, v, y, z) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^\ell \times \mathbb{R}^m \times \mathbb{R}^n$ such that the point p_0 corresponds to the origin; the point q_0 corresponds to $(\bar{x}, 0, 0, 0, 0)$ with $\bar{x} > 0$; and the differential equation* (4.1) *takes the form*

$$\dot{x} = \epsilon + y^\top Az, \tag{4.6}$$
$$\dot{u} = y^\top Bz, \tag{4.7}$$
$$\dot{v} = y^\top Cz, \tag{4.8}$$
$$\dot{y} = Dy, \tag{4.9}$$
$$\dot{z} = Ez, \tag{4.10}$$

where A is an $m \times n$ matrix, B is a k-tuple of $m \times n$ matrices, C is an ℓ-tuple of $m \times n$ matrices, D is $m \times m$, and E is $n \times n$. The entries of these matrices are functions of $(x, u, v, y, z, \epsilon)$. The eigenvalues of D have negative real part, and those of E have positive real part.

Moreover, there are coordinates $\Theta = (U, V, Y, Z) \in \mathbb{R}^k \times \mathbb{R}^\ell \times \mathbb{R}^m \times \mathbb{R}^n$ such that the differential equation (4.2) takes the form

$$\dot{U} = y^\top Fz\Theta, \tag{4.11}$$
$$\dot{V} = y^\top Gz\Theta, \tag{4.12}$$
$$\dot{Y} = HY + y^\top I\Theta, \tag{4.13}$$
$$\dot{Z} = JZ + z^\top K\Theta, \tag{4.14}$$

where F is a $k \times p$ array of $m \times n$ matrices, G is an $\ell \times p$ array of $m \times n$ matrices, H is $m \times m$, I is an m-tuple of $m \times p$ matrices, J is $n \times n$, and K is an n-tuple of $n \times p$ matrices. The entries of these matrices are functions of $(x, u, v, y, z, \epsilon)$. The eigenvalues of H have negative real part, and those of J have positive real part.

The coordinates can be chosen so that $\mathcal{H}_\epsilon$ (or one of its forward iterates under the flow) is parameterized by $(u, z, U, a, Z), a \in \mathbb{R}$, as follows:

$$x = L(u, z, \epsilon)z, \tag{4.15}$$
$$v = M(u, z, \epsilon)z, \tag{4.16}$$
$$y = y(u, \epsilon) + N(u, z, \epsilon)z, \tag{4.17}$$
$$V = a(\epsilon P(u, \epsilon) + zQ(u, z, \epsilon)) + zR(u, z, \epsilon)(U, Z), \tag{4.18}$$
$$Y = a(Y(u, \epsilon) + zS(u, z, \epsilon)) + zT(u, z, \epsilon)(U, Z) \tag{4.19}$$

with $P(0, 0) \neq 0$. Equations (4.15) and (4.16) give a parametrization of H_ϵ by (u, z).

The coordinate change can be chosen to be a C^{r+2} vector bundle map, so the skew-product system (4.6)–(4.14) is C^{r+2}, and the vector bundles $\mathcal{H}_\epsilon$ are now C^{r+2}.

In the new coordinates, for each $\epsilon \geqslant 0, S_\epsilon$ is xuv-space, $W^s_\epsilon(S_\epsilon)$ is $xuvy$-space, and $W^u_\epsilon(S_\epsilon)$ is $xuvz$-space. On both $xuvy$-space and $xuvz$-space, $(\dot{x}, \dot{u}, \dot{v})$ depends only on (x, u, v), the coordinates on S_ϵ. The stable foliation of $xuvy$-space is by planes $(x, u, v) =$ constant; similarly, the unstable foliation of $xuvz$-space is by planes $(x,u,v)=$ constant. P_0 is u-space; in fact, we have arranged that for each $\epsilon\geqslant 0$, $H_\epsilon \cap W^s_\epsilon(S_\epsilon)$ projects along the stable foliation of $xuvy$-space to u-space. P_0^* is xu-space.

In addition, for each $\epsilon\geqslant 0, \mathcal{S}_\epsilon$ is $xuvUV$-space, $W^s_\epsilon(\mathcal{S}_\epsilon)$ is $xuvyUVY$-space, and $W^u_\epsilon(\mathcal{S}_\epsilon)$ is $xuvzUVZ$-space. We have arranged that on both these spaces, $\dot U=\dot V=0$. The stable foliation of $xuvyUVY$-space is by planes $(x,u,v,U,V)=$ constant; similarly, the unstable foliation of $xuvzUVZ$-space is by planes $(x,u,v,U,V)=$ constant. $\mathcal{P}_0$ is uU-space and $\mathcal{P}_0^*$ is xuU-space. In fact, we have arranged that for each $\epsilon\geqslant 0$, the image of the projection of $\mathcal{H}^\epsilon \cap W^s_\epsilon(\mathcal{S}_\epsilon)$ to $xuvUV$-space along the stable foliation of $xuvyUVY$-space includes uU-space. For $\epsilon=0$, it is exactly uU-space; for $\epsilon>0$, to lowest order, the image has one more dimension: U space is augmented by the span of $P(u,0)$ in V-space.

Proof. Using [6], we first choose C^{r+4} skew-product Fenichel coordinates $(\theta,\Theta)=(w,y,z,W,Y,Z)$ such that

$$\dot w = \epsilon h(w,\epsilon)+y^\top Az, \tag{4.20}$$
$$\dot y = Dy, \tag{4.21}$$
$$\dot z = Ez, \tag{4.22}$$
$$\dot W = \epsilon F(w,\epsilon)W+y^\top Gz\Theta, \tag{4.23}$$
$$\dot Y = HY+y^\top I\Theta, \tag{4.24}$$
$$\dot Z = JZ+z^\top K\Theta \tag{4.25}$$

with $h(0,0)\neq 0$ by (E4). The skew-product system is now C^{r+4}. By a further change of coordinates $w=(x,\tilde w)$ that depends on (w,ϵ) only, we can convert (4.20) to

$$\dot x = \epsilon + y^\top A_1 z, \tag{4.26}$$
$$\dot{\tilde w} = y^\top \tilde A z. \tag{4.27}$$

Equation (4.23) now reads $\dot W=\epsilon F(x,\tilde w,\epsilon)W + y^\top Gz\Theta$. Dividing $\dot W = \epsilon F(x,\tilde w,\epsilon)W$ by (4.26), we obtain $W_x = F(x,\tilde w,\epsilon)W$. By a change of coordinates in W that depends only on $(x,\tilde w, W,\epsilon)$, we can convert this differential equation to $W_x=0$. Then (4.23) becomes

$$\dot W = y^\top Gz\Theta.$$

The skew-product system is now C^{r+3}.

Next we make a change of coordinates $\tilde w=(u,v)$ that depends only on $(x,\tilde w,\epsilon)$ such that for all small ϵ, P_ϵ is u-space. We can arrange this change of coordinates so that the curve $\theta(\epsilon)$ of (E8) becomes $(x,u,v,y,z)(\epsilon)=(0,0,0,y(\epsilon),0)$. In the new coordinates, (4.1) is given by

(4.6)–(4.10), and the parameterization of H_ϵ is given by (4.15) and (4.16). The skew-product system is still C^{r+3}, and the parameterization of H_ϵ is also C^{r+3}.

For each $0 \in H_\epsilon, L_\epsilon(0) \cap E_\epsilon^{cs}(\theta)$ is spanned by $k+1$ vectors that depend C^{r+3} on (θ, ϵ), with the first k not in $E_\epsilon^s(\theta)$ for all $\epsilon \geqslant 0$, and the last vector, for $\epsilon = 0$, in $E_\epsilon^s(\theta)$. The span of the first k of these vectors defines a vector bundle over H_ϵ of fiber dimension k that projects by Π to a vector bundle $\mathcal{P}_\epsilon$ over $\mathcal{P}_\epsilon$, of fiber dimension k, with $\mathcal{P}_0$ as previously defined. By a change of coordinates in W that depends only on (u, W), we arrange that the vector bundle $\mathcal{P}_\epsilon$ corresponds to uU-space. The skew-product system is still C^{r+3}.

In these coordinates, $\mathcal{H}_\epsilon$ is given by (4.15)–(4.19). $P(0,0) \neq 0$ by (E8). □

Since $P(u, 0) \neq 0$, by a change of coordinates $V = (V_1, \tilde{V}), (V_1, \tilde{V}) \in \mathbb{R} \times \mathbb{R}^{\ell-1}$, that depends only on (u, V, ϵ), we can replace (4.18) by

$$V_1 = \alpha(\epsilon + zQ_1(u, z, \epsilon)) + zR_1(u, z, \epsilon)(U, Z), \tag{4.28}$$

$$\tilde{V} = az\tilde{Q}(u, z, \epsilon) + z\tilde{R}(u, z, \epsilon)(U, Z). \tag{4.29}$$

Since $P(u, \epsilon)$ is only C^{r+2}, the differential equation is now C^{r+2}.

To prove Theorem 4.1, we must study, in $xuvyzUV_1\tilde{V}Z$-coordinates the convergence, as $\epsilon \to 0$, of $H_\epsilon^* \cap O$ to xuz-space, and of $\mathcal{H}_\epsilon^* \cap (O \times \mathbb{R}^p)$ to $xuzUV_1Z$-space.

Let $\tau > 0$. The solution of (4.6)–(4.14) on the interval $0 \leqslant t \leqslant \tau$ with boundary conditions

$$\begin{aligned} x(\tau) &= x^1, \\ u(0) &= u^0, \\ v(0) &= v^0, \\ y(0) &= y^0, \\ z(\tau) &= z^1, \\ U(\tau) &= U^1, \\ V_1(\tau) &= V_1^1, \\ \tilde{V}(0) &= \tilde{V}^0, \\ Y(0) &= Y^0, \\ Z(\tau) &= Z^1 \end{aligned}$$

is

$$(x, u, v, y, z, U, V_1, \tilde{V}, Y, Z)(t, \tau, x^1, u^0, v^0, y^0, z^1, U^1, V_1^1, \tilde{V}^0, Y^0, Z^1, \epsilon).$$

(Of course, (x, u, v, y, z) depends only on $(t, \tau, x^1, u^0, v^0, y^0, z^1, \epsilon)$.) From [4] and [6], this function is C^r, and there exist ρ with $0 < \rho < \frac{\bar{x}}{3}$, $\lambda < 0 < \mu$, and $K > 0$ such that for $-\rho \leqslant x^1 \leqslant \bar{x} + \rho$ and $\max(\|u^0\|, \|v^0\|, \|y^0\|, \|z^1\|, \|U^1\|, \|V_1^1\|, \|\tilde{V}^0\|, \|Y^0\|, \|Z^1\|) \leqslant \rho$, and for any multi-index i with $|i| \leqslant r$,

$$\|D^i(x - (x^1 + \epsilon(t - \tau)))\| \leqslant Ke^{\lambda t + \mu(t-\tau)}, \tag{4.30}$$
$$\|D^i(u - u^0)\| \leqslant Ke^{\lambda t + \mu(t-\tau)}, \tag{4.31}$$
$$\|D^i(v - v^0)\| \leqslant Ke^{\lambda t + \mu(t-\tau)}, \tag{4.32}$$
$$\|D^i y\| \leqslant Ke^{\lambda t}, \tag{4.33}$$
$$\|D^i z\| \leqslant Ke^{\mu(t-\tau)}, \tag{4.34}$$
$$\|D^i(U - U^1)\| \leqslant Ke^{\lambda t + \mu(t-\tau)}, \tag{4.35}$$
$$\|D^i(V_1 - V_1^1)\| \leqslant Ke^{\lambda t + \mu(t-\tau)}, \tag{4.36}$$
$$\|D^i(\tilde{V} - \tilde{V}^0)\| \leqslant Ke^{\lambda t + \mu(t-\tau)}, \tag{4.37}$$
$$\|D^i Y\| \leqslant Ke^{\lambda t}, \tag{4.38}$$
$$\|D^i Z\| \leqslant Ke^{\mu(t-\tau)}. \tag{4.39}$$

Moreover, we can write

$$\begin{aligned}(U, V_1, \tilde{V}, Y, Z)(t, \tau, x^1, u^0, v^0, y^0, z^1, U^1, V_1^1, \tilde{V}^0, Y^0, Z^1, \epsilon)\\ = \mathcal{L}_{(t,\tau,x^1,u^0,v^0,y^0,z^1,\epsilon)}(U^1, V_1, \tilde{V}^0, Y^0, Z^1),\end{aligned}$$

where $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_5)$ is linear for fixed $(t, \tau, x^1, u^0, v^0, y^0, z^1)$. Let us define

$$\mathcal{I}_1(U^1, V_1^1, \tilde{V}^0, Y^0, Z^1) = U^1, \tag{4.40}$$
$$\mathcal{I}_2(U^1, V_1^1, \tilde{V}^0, Y^0, Z^1) = V_1^1, \tag{4.41}$$
$$\mathcal{I}_3(U^1, V_1^1, \tilde{V}^0, Y^0, Z^1) = \tilde{V}^0. \tag{4.42}$$

For any multi-index i with $|i| \leqslant r$, we have

$$\|D^i(\mathcal{L}_1 - \mathcal{I}_1)\| \leqslant Ke^{\lambda t + \mu(t-\tau)}, \tag{4.43}$$
$$\|D^i(\mathcal{L}_2 - \mathcal{I}_2)\| \leqslant Ke^{\lambda t + \mu(t-\tau)}, \tag{4.44}$$
$$\|D^i(\mathcal{L}_3 - \mathcal{I}_3)\| \leqslant Ke^{\lambda t + \mu(t-\tau)}, \tag{4.45}$$
$$\|D^i \mathcal{L}_4\| \leqslant Ke^{\lambda t}, \tag{4.46}$$
$$\|D^i \mathcal{L}_5\| \leqslant Ke^{\mu(t-\tau)}, \tag{4.47}$$

Let

$$\mathcal{A} = \{(x^1, u^1, v^1, y^1, z^1) : \max(|x^1 - \bar{x}|, \|u^1\|, \|v^1\|, \|y^1\|, \|z^1\|) \leqslant \rho\},$$
$$\mathcal{B} = \{(x^1, u^0, z^1, \epsilon) : \max(|x^1 - \bar{x}|, \|u^0\|, \|z^1\|) \leqslant \rho \text{ and } 0 < \epsilon < \epsilon_0\},$$
$$\mathcal{C} = \{(x^1, u^1, z^1, \epsilon) : \max(|x^1 - \bar{x}|, \|u^1\|, \|z^1\|) \leqslant \rho \text{ and } 0 < \epsilon < \epsilon_0\},$$
$$\mathcal{D} = \{(x^0, v^0, \bar{y}^0) : \max(|x^0|, \|v^0\|, 2\|\bar{y}^0\|) \leqslant \rho\}.$$

Then $q_0 = (\bar{x}, 0, 0, 0, 0) \in \mathcal{A}$. We may assume that $U \subset \mathcal{A}$ and, after replacing $\mathcal{H}_\epsilon$ by one of its forward iterates under the flow, that $\|y(u, \epsilon)\| \leqslant \frac{\rho}{2}$, where $y(u, \epsilon)$ is given by (4.17).

Write $y_0 = y(u_0, \epsilon) + \bar{y}_0$ and $Y^0 = aY(u^0, \epsilon) + \bar{Y}^0$, where $Y(u^0, \epsilon)$ is given by (4.19). For $\epsilon > 0$ we wish to solve the following system of equations in the variables $(x^0, x^1, u^0, v^0, \bar{y}^0, z^1, U^1, V_1^1, \tilde{V}^0, a, \bar{Y}^0, Z^1, \epsilon)$:

$$x^0 = L(u^0, z(0), \epsilon) z(0), \tag{4.48}$$
$$v^0 = M(u^0, z(0), \epsilon) z(0), \tag{4.49}$$
$$\bar{y}^0 = N(u^0, z(0), \epsilon) z(0), \tag{4.50}$$
$$\tilde{V}^0 = az(0)\tilde{Q}(u^0, z(0), \epsilon) + z(0)\tilde{R}(u^0, z(0), \epsilon), (U(0), Z(0)), \tag{4.51}$$
$$a = \frac{1}{\epsilon}(V_1(0) - az(0)Q(u^0, z(0), \epsilon) - z(0)R_1(u^0, z(0), \epsilon)(U(0), Z(0)), \tag{4.52}$$
$$\bar{Y}^0 = az(0)S(u^0, z(0), \epsilon) + z(0)T(u^0, z(0), \epsilon)(U(0), Z(0))), \tag{4.53}$$

where $\tau = \frac{1}{\epsilon}(x^1 - x^0)$, $z(0)$ means $z(0, \tau, x^1, u^0, v^0, y(u^0, \epsilon) + \bar{y}^0, z^1, \epsilon)$,

$$U(0) = \mathcal{L}_{1(0,\tau,x^1,u^0,v^0,y(u^0,\epsilon)+\bar{y}^0,z^1,\epsilon)}(U^1, V_1^1, \tilde{V}^0, aY(u^0, \epsilon) + \bar{Y}^0, Z^1),$$

etc. Each solution yields a point $(x^0, u^0, v^0, y^0, z(0), U(0), V_1(0), \tilde{V}^0, aY(u(0), \epsilon) + \bar{Y}^0, Z(0))$ in $\mathcal{H}_\epsilon$ and a point $(x^1, u(\tau), v(\tau), y(\tau), z^1, U^1, V_1^1, \tilde{V}(\tau), Y(\tau), Z^1)$ in $\mathcal{H}_\epsilon^*$; the latter is the point reached by the former after time $\tau = \frac{1}{\epsilon}(x^1 - x^0)$. To prove the theorem we must describe the second set of points in the form

$$(v(\tau), y(\tau), \tilde{V}(\tau), Y(\tau)) = \text{function of } (x^1, u(\tau), z^1, U^1, V_1^1, Z^1, \epsilon).$$

To solve (4.48)–(4.50), we reorder the first six variables and define $\mathcal{F}(x^0, v^0, \bar{y}^0, x^1, u^0, z^1, \epsilon)$ to be the right-hand side of (4.48)–(4.50). For $\epsilon > 0, \mathcal{F}$ is C^r. Let $0 < \nu < \min(-\lambda, \mu)$.

Lemma 4.3. *For $\epsilon_0 > 0$ sufficiently small, $\mathcal{F}$ maps $\mathcal{D} \times \mathcal{B}$ into $\mathcal{D}$ and is a contraction of $\mathcal{D}$ for fixed $(x^1, u^0, z^1, \epsilon) \in \mathcal{B}$. There is a constant M independent of ϵ such that all partial derivatives of $\mathcal{F}$ are bounded by $Me^{-\frac{\nu\rho}{\epsilon}}$ on*

$\mathcal{D}\times\mathcal{B}$. Let the fixed point be

$$(x^0, v^0, \bar{y}^0) = \mathcal{G}(x^1, u^0, z^1, \epsilon). \tag{4.54}$$

Then $\mathcal{G}$ is C^r on $\mathcal{B}$. The components of $\mathcal{G}$, along with their partial derivatives through order r, are bounded by $Me^{-\frac{\nu\rho}{\epsilon}}$ on $\mathcal{B}$.

Proof. Note that $\tau = \frac{1}{\epsilon}(x^1 - x^0) \geqslant \frac{\rho}{\epsilon}$., By (4.34),

$$\|\mathcal{F}_1(x^0, v^0, \bar{y}^0, x^1, u^0, z^1, \epsilon)\| \leqslant Ke^{-\mu\tau} \leqslant Ke^{-\nu\rho/\epsilon}. \tag{4.55}$$

The same estimate applies to $\mathcal{F}_2$ and $\mathcal{F}_3$. Therefore, for ϵ sufficiently small, $\mathcal{F}(\cdot,\cdot,\cdot,x^1,u^0,z^1,\epsilon)$ maps $\mathcal{D}$ into itself. The same estimates apply to the partial derivatives with respect to all variables except x^0, x^1, u^0, and ϵ. To estimate the partial derivative of $\mathcal{F}_i$ with respect x^0, note that for small $\epsilon > 0$,

$$\left|\left|\frac{\partial z(0)}{\partial x^0}\right|\right| = \frac{1}{\epsilon}\left|\left|\frac{\partial z(0)}{\partial \tau}\right|\right| \leqslant \frac{K}{\epsilon}e^{-\mu\tau} \leqslant Me^{-\nu\rho/\epsilon}.$$

The partial derivatives with respect to x^1 and ϵ include a similar term. To estimate the partial derivative with respect u^0, note that $\frac{\partial z(0)}{\partial u^0}$ includes the term $\frac{\partial z(0)}{\partial y^0}\frac{\partial y}{\partial u}$. To estimate it, note that in replacing $\mathcal{H}_\epsilon$ by a forward iterate for which $|y(u,\epsilon)| \leqslant \frac{\rho}{2}$, partial derivatives of y_i with respect to u_j shrink by a factor of $L\rho^\omega$, where ω is at worst approximately the ratio of the least to most negative eigenvalues of $E(0)$. Then

$$\left|\left|\frac{\partial z(0)}{\partial y^0}\frac{\partial y}{\partial u}\right|\right| \leqslant Ke^{-\mu\tau}L\rho^\omega \leqslant Me^{-\nu\rho/\epsilon}.$$

The estimates for the components of $\mathcal{G}$ follow from the estimates for the components of $\mathcal{F}$. The estimates for the partial derivatives of $\mathcal{G}$ follow from differentiation with respect to (x^1, u^0, z^1) of the formula

$$(x^0, v^0, \bar{y}^0) = \mathcal{F}(x^0, v^0, \bar{y}^0, x^1, u^0, z^1, \epsilon).$$

□

To describe the solution set of (4.48)–(4.50) in the desired way, we first define

$$u^1(x^1, u^0, z^1, \epsilon) = u(\tau, \tau, x^1, u^0, v^0, y(u^0, \epsilon) + \bar{y}^0, z^1, \epsilon),$$

where $(x^0, v^0, \bar{y}^0)$ is given by (4.54) and $\tau = \frac{1}{\epsilon}(x^1 - x^0)$. Now $u^1 - u^0$ and its partial derivatives with respect to all variables are exponentially small, so we can solve the equation $u^1 = u^1(x^1, u^0, z^1, \epsilon)$ for u^0. We obtain

$$u^0 = u^0(x^1, u^1, z^1, \epsilon), \tag{4.56}$$

where $u^0 - u^1$ and its partial derivatives through order r are bounded by $Me^{-\nu\rho/\epsilon}$ on $\mathcal{C}$.

Define $\mathcal{K}: \mathcal{C} \to vy$-space by

$$(v^1, y^1) = \mathcal{K}(x^1, u^1, z^1, \epsilon) = (v, y)(\tau, \tau, x^1, u^0, v^0, y(u^0, \epsilon) + \bar{y}^0, z^1, \epsilon), \tag{4.57}$$

where u^0 is given by (4.56), $(x^0, v^0, \bar{y}^0)$ is given by (4.54), and $\tau = \frac{1}{\epsilon}(x^1 - x^0)$. $\mathcal{K}$ is C^r.

Lemma 4.4. *$\mathcal{K}$ and its partial derivatives through order r are bounded by $Me^{-\nu\rho/\epsilon}$ on $\mathcal{C}$.*

Proof. Note that $||v^1|| \leqslant ||v^0|| + Ke^{\lambda\tau} \leqslant Me^{-\nu\rho/\epsilon}$. A similar (easier) estimate holds for y^1. The estimates on the partial derivatives follow from differentiation of (4.57) with respect to $(x^1, u^1, z^1, \epsilon)$. □

To solve (4.51)–(4.53) we note that the right-hand side of (4.51)–(4.53) defines a C^r family of linear mappings

$$\mathcal{M}_{(x^0, x^1, u^0, v^0, \bar{y}^0, z^1, \epsilon)}(U^1, V_1^1, \tilde{V}^0, a, \bar{Y}^0, Z^1).$$

We easily check that $\mathcal{M}_{(x^0, x^1, u^0, v^0, \bar{y}^0, z^1, \epsilon)}$ has the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\epsilon} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

plus a remainder whose terms, along with their partial derivatives through order r, are of order $e^{-\frac{\nu\rho}{\epsilon}}$. Therefore,

Lemma 4.5. *The solution of (4.51)–(4.53) is*

$$(\tilde{V}^0, a, \bar{Y}^0) = \mathcal{N}_{(x^0, x^1, u^1, v^0, \bar{y}^0, z^1, \epsilon)}(U^1, V_1^1, Z^1), \tag{4.58}$$

where $\mathcal{N}_{(x^0, x^1, u^1, v^0, \bar{y}^0, z^1, \epsilon)}$ is a C^r family of linear mappings with the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\epsilon} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

plus a remainder whose terms, along with their partial derivatives through order r, are of order $e^{-\nu\rho/\epsilon}$.

To describe the solution sets of (4.51)–(4.53) in the desired way, let $(x^0, v^0, \bar{y}^0) = \mathcal{G}(x^1, u^1, z^1, \epsilon)$ and let $\tau = \frac{1}{\epsilon}(x^1 - x^0)$. Let (v^1, y^1) be given by (4.57), and let $(\tilde{V}^0, a, \bar{Y}^0)$ be given by (4.58).

Define

$$\begin{aligned}(\tilde{V}^1, Y^1 &= \mathcal{P}_{(x^1,u^1,z^1,\epsilon)}(U^1, V_1^1, Z^1) = (\tilde{V}, Y)(\tau, \tau, x^1, u^1, v^0, y(u^0, \epsilon) \\ &\quad + \bar{y}^0, z^1, U^1, V_1^1, \tilde{V}^0, aY(u^0, \epsilon) + \bar{Y}^0, Z^1, \epsilon) \\ &= \mathcal{L}34_{(\tau,\tau,x^1,u^1,u^0,y(u^0,\epsilon)+\bar{y}^0,z^1,\epsilon)}(U^1, V_1^1, \tilde{V}^0, aY(u^0, \epsilon) + \bar{Y}^0, Z^1).\end{aligned}$$

Lemma 4.6. *The entries of $\mathcal{P}$, along with their partial derivatives through order r, are bounded by $Me^{-\nu\rho/\epsilon}$ on $\mathcal{C}$.*

Proof. We have $\mathcal{P} = \mathcal{L}_{34}\mathcal{Q}$ with

$$\mathcal{L}_{34} = \begin{pmatrix} 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Q} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\epsilon} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

The rows of $\mathcal{L}_{34}$ represent $(\tilde{V}^1, Y^1)$; the columns of $\mathcal{L}_{34}$ and the rows of $\mathcal{Q}$ represent $(U^1, V_1^1, \tilde{V}^0, a, \bar{Y}^0, Z^1)$; the columns of $\mathcal{Q}$ represent (U^1, V_1^1, Z^1). The third through fifth rows of $\mathcal{Q}$ are the matrix $\mathcal{N}$. These rows of $\mathcal{Q}$, and the matrix $\mathcal{L}_{34}$, are given modulo terms that, along with their partial derivatives through order r, are bounded by $Me^{-\nu\rho/\epsilon}$ on $\mathcal{C}$. The lemma follows. □

Lemmas 4.4 and 4.6 prove the result.

5. SLOW EIGENVALUES

In order to study the system (3.2)–(3.5) for ρ near 0, we let $\rho = \epsilon\lambda$, i.e., we undo the rescaling of λ used in Section 3 to study fast eigenvalues. We obtain:

$$\dot{u} = v, \tag{5.1}$$
$$\dot{v} = (Df(u) - xI)v, \tag{5.2}$$
$$\dot{x} = \epsilon, \tag{5.3}$$
$$\dot{\lambda} = 0, \tag{5.4}$$
$$\dot{U} = V, \tag{5.5}$$
$$\dot{V} = \epsilon\lambda U + (Df(u) - xI)V + D^2 f(u)vU. \tag{5.6}$$

The system (5.1)–(5.6) is a skew-product flow on the trivial vector bundle $uvx\lambda$-space $\times UV$-space.

Setting $\epsilon = 0$, we obtain

$$\dot{u} = v, \tag{5.7}$$
$$\dot{v} = (Df(u) - xI)v, \tag{5.8}$$
$$\dot{x} = 0, \tag{5.9}$$
$$\dot{\lambda} = 0, \tag{5.10}$$
$$\dot{U} = V, \tag{5.11}$$
$$\dot{V} = (Df(u) - xI)V + D^2 f(u)vU. \tag{5.12}$$

Notice that (5.7)–(5.12) is independent of λ. Also, suppose $(u(\xi), \dot{u}(\xi), x^0)$ is a solution of (5.7)–(5.9). The linearization of (5.7) and (5.8) along this solution is

$$\dot{U} = V, \tag{5.13}$$
$$\dot{V} = (Df(u(\xi)) - x^0 I)V + D^2 f(u(\xi))\dot{u}(\xi)U, \tag{5.14}$$

which is just (5.11) and (5.12) with $(u, v, x) = (u(\xi), \dot{u}(\xi), x^0)$.

In this section, we could consider any $\lambda \in \mathbb{C}$. For simplicity, however, we will only consider $\lambda \in \mathbb{R}$. Hence we take $(U, V) \in \mathbb{R}^n \times \mathbb{R}^n$ as in Section 3.

For each $i = 0, \dots, n$, let

$$T^i = \{(u, v, x, \lambda) : (u, v, x) \in S^i\} = \{(u, v, x, \lambda) : v = 0 \text{ and } \nu^i(u) < x < \nu^{i+1}(u)\}.$$

Each T^i is a normally hyperbolic $(2n+2)$-dimensional manifold of equilibria of (5.7)–(5.12). At each point $(u^0, 0, x^0, \lambda^0)$ of T^i, the linearization of (5.7)–(5.12) has the semisimple eigenvalue 0 with multiplicity $n+2$, and n nonzero eigenvalues $\nu^k(u) - x, k = 1, \dots, n$, of which the last $n-i$ are positive and the first i are negative.

Recall $W_0^u(u^0, 0, x^0) \subset uvx$-space defined in Section 2. Then in $uvx\lambda$-space, $W_0^u(u^0, 0, x^0, \lambda^0) = W_0^u(u^0, 0, x^0) \times \{\lambda^0\}$ is an $(n-i)$-dimensional manifold that is contained in the subspace $(x, \lambda) = (x^0, \lambda^0)$. The union of these manifolds is $W_0^u(T^i)$, which has dimension $2n+2-i$. There are analogous descriptions of $W_0^s(u^0, 0, x^0, \lambda^0)$ and $W_0^s(T^i)$.

Given $(u^0, 0, x^0, \lambda^0) \in T^i$, consider the constant-coefficient linear differential equation (5.11) and (5.12) with $(u, v, x) = (u^0, 0, x^0)$. It has the semisimple eigenvalue 0 with multiplicity n, and n nonzero eigenvalues $\nu^k(u) - x, k = 1, \dots, n$, of which the first $n-i$ are positive and the last i are negative. Its center subspace is $\{(U, V) : V = 0\}$, which consists of equilibria.

For each $i = 0, \dots, n$, let

$$\mathcal{T}_0^i = \{(u, v, x, \lambda, U, V) : (u, v, x, \lambda) \in T^i \text{ and } V = 0\}.$$

Each $\mathcal{T}_0^i$ is a vector bundle over T^i that is a normally hyperbolic $(2n+2)$-dimensional manifold of equilibria of (5.7)–(5.10).

$W_0^u(\mathcal{T}_0^i)$ is a vector bundle over $W_0^u(T^i)$ with fiber dimension $2n-i$. We denote the fibers, which are subspaces of UV-space, by $E^{cu}(\tilde{u}, \tilde{v}, x^0, \lambda^0)$, $(\tilde{u}, \tilde{v}, x^0, \lambda^0) \in W_0^u(T^i)$. They are independent of λ_0.

More precisely, let $(u^0, 0, x^0, \lambda^0) \in T^i$, let $(\tilde{u}, \tilde{v}, x^0, \lambda^0) \in W_0^u(u^0, 0, x^0)$, and let $(u(\xi), v(\xi))$ be the solutions of (5.7) and (5.8) with $(u(0), v(0)) = (\tilde{u}, \tilde{v})$. Then there is an $(n-i)$-dimensional subspace $E^u(\tilde{u}, \tilde{v}, x^0, \lambda^0) \subset E^{cu}(\tilde{u}, \tilde{v}, x^0, \lambda^0)$ such that if $(U_0, V_0) \in E^u(\tilde{u}, \tilde{v}, x^0, \lambda^0)$ and $(U(\xi), V(\xi))$ is the solution of (5.13) and (5.14) with $(u, v, x) = (u(\xi), v(\xi), x^0)$ and $(U(0), V(0)) = (U_0, V_0)$, then $(U(\xi), V(\xi))$ approaches 0 exponentially as $\xi \to -\infty$. The spaces $E^u(\tilde{u}, \tilde{v}, x^0, \lambda^0)$ are independent of λ^0.

For each $(\tilde{u}, \tilde{v}, x^0, \lambda^0) \in W_0^u(T^i)$ choose a complementary subspace $C^u(\tilde{u}, \tilde{v}, x^0, \lambda^0)$ to $E^u(\tilde{u}, \tilde{v}, x^0, \lambda^0)$ in $E^{cu}(\tilde{u}, \tilde{v}, x^0, \lambda^0)$, independent of λ^0, such that C^u depends smoothly on $(\tilde{u}, \tilde{v}, x^0)$ and $C^u(u^0, 0, x^0, \lambda^0)$ is U-space.

There is a projection Π^u from $W_0^u(\mathcal{T}_0^i)$ to $\mathcal{T}_0^i$ defined by $\Pi^u(\tilde{u}, \tilde{v}, x, \lambda, U, V) = (u^0, 0, x^0, \lambda^0, U^0, 0)$ where $(\tilde{u}, \tilde{v}, x, \lambda, U, V) \in W_0^u(u^0, 0, x^0, \lambda^0, U^0, 0)$ (so in particular $x = x^0, \lambda = \lambda^0$, and $(\tilde{u}, \tilde{v}, x^0) \in W_0^u(u^0, 0, x^0)$). For fixed $(\tilde{u}, \tilde{v}, x^0, \lambda^0) \in W_0^u(u^0, 0, x^0, \lambda^0)$, $\Pi^u|\{(\tilde{u}, \tilde{v}, x^0, \lambda^0)\} \times C^u(\tilde{u}, \tilde{v}, x^0, \lambda^0)$ is an isomorphism onto $\{(u^0, 0, x^0, \lambda^0)\} \times U$-space. If we regard this isomorphism as a map from $C^u(\tilde{u}, \tilde{v}, x^0, \lambda^0)$ to U-space, then it has an inverse, which we denote $B^u_{(\tilde{u}, \tilde{v}, x^0, \lambda^0)}$; it is independent of λ^0. Then

$$\begin{aligned} W_0^u(u^0, 0, x^0, \lambda^0, U^0, 0) = \{&(u, v, x^0, \lambda^0, U, V): \\ &(u, v, x^0, \lambda^0) \in W_0^u(u^0, 0, x^0, \lambda^0) \text{ and } (U, V) \in B^u_{(u, v, x^0, \lambda^0)} U^0 \\ &+ E^u(u, v, x^0, \lambda^0)\}. \end{aligned}$$

There is an analogous description of $W_0^s(\mathcal{T}_0^i)$.

Proposition 5.1. *Let $(u^{i-1}, 0, x^i) \in M^{i-1} \times I^i$, and let $(u(\xi), \dot{u}(\xi), x^i)$ be the connection from $(u^{i-1}, 0, x^i)$ to some $(u^i, 0, x^i) \in M^i \times I^i$. Then for each fixed ξ and λ^0,*

1. *$E^u(u(\xi), v(\xi), x^i, \lambda^0)$ and $E^{cs}(u(\xi), v(\xi), x^i, \lambda^0)$ are transverse. Their intersection is spanned by $(\dot{u}(\xi), \ddot{u}(\xi))$.*
2. *$E^{cu}(u(\xi), v(\xi), x^i, \lambda^0)$ and $E^s(u(\xi), v(\xi), x^i, \lambda^0)$ are transverse. Their intersection is spanned by $(\dot{u}(\xi), \ddot{u}(\xi))$.*

Proof. To prove (1), note that $\dim E^u(u(\xi), v(\xi), x^i, \lambda^0) + \dim E^{cs}(u(\xi), v(\xi), x^i, \lambda^0) = n - (i-1) + (n+i) = 2n+1$. Therefore, the two spaces are transverse if and only if their intersection has dimension one.

Let $(U(\xi), V(\xi))$ belong to the intersection. Then $\lim_{\xi\to-\infty}(U(\xi), V(\xi)) = (0,0)$, and there is a vector U^i such that $\lim_{\xi\to\infty}(U(\xi), V(\xi)) = (U^i, 0)$. Now the system (5.13) and (5.14) can be written

$$\dot{V} = \ddot{U} = \frac{d}{d\xi}((Df(u(\xi))-x^0 I)U). \tag{5.15}$$

Integrating from $\xi = -\infty$ to $\xi = \infty$, we obtain

$$0 = (Df(u^i)-x^i I)U^i - (Df(u^{i-1})-x^i I)0 = (Df(u^i)-x^i I)U^i.$$

Therefore, $U^i = 0$, so $(U(\xi), V(\xi))$ belongs to the intersection of $E^u(u(\xi), v(\xi), x^i, \lambda^0)$ and $E^s(u(\xi), v(\xi), x^i, \lambda^0)$. By assumption (R4), this intersection is spanned by $(\dot{u}(\xi), \ddot{u}(\xi))$.

The proof of (2) is similar. □

The normally hyperbolic manifold of equilibria $\mathcal{T}_0^i$ perturbs, for small $\epsilon > 0$, to a normally hyperbolic, locally invariant manifold $\mathcal{T}_\epsilon^i$, a fiber bundle over T^i:

$$\mathcal{T}_\epsilon^i = \{(u, v, x, \lambda, U, V) : (u, v, x, \lambda) \in T^i \text{ and } V = \epsilon A(u, x, \lambda, \epsilon)U\}. \tag{5.16}$$

We determine $A(u, x, \lambda, 0)$ as follows: From (5.16), we have

$$\dot{V} = \epsilon\left(\frac{\partial A}{\partial u}\dot{u}U + \frac{\partial A}{\partial x}\dot{x}U + A\dot{U}\right) = \epsilon^2\left(\frac{\partial A}{\partial x} + A^2\right)U. \tag{5.17}$$

Substituting (5.6) for $\dot{V}$ in (5.17), then setting $v = 0$ and $V = \epsilon AU$, and retaining only terms of order ϵ, we obtain

$$\lambda U + (Df(u) - xI)A(u, x, \lambda, 0)U = 0.$$

Therefore,

$$A(u, x, \lambda, 0) = -\lambda(Df(u) - xI)^{-1}.$$

The differential equation (5.1)–(5.6) on the invariant manifold $\mathcal{T}_\epsilon^i$, with coordinates (u, x, U), is therefore

$$\dot{u} = 0, \tag{5.18}$$

$$\dot{x} = \epsilon, \tag{5.19}$$

$$\dot{U} = V = \epsilon A(u, x, \lambda, \epsilon)U = \epsilon(-\lambda(Df(u) - xI)^{-1} + O(\epsilon))U. \tag{5.20}$$

Dividing (5.20) by (5.19), we obtain

$$U_x = A(u, x, \lambda, \epsilon)U = (-\lambda(Df(u) - xI)^{-1} + O(\epsilon))U. \tag{5.21}$$

To lowest order, (5.21) can be written

$$\lambda U + (Df(u) - xI)U_x = 0. \tag{5.22}$$

With some abuse of notation, we think of $\mathcal{T}_\epsilon^i$ as $T^i \times U$-space. Then to lowest order in ϵ, the differential equation (5.1)–(5.6) on $\mathcal{T}_\epsilon^i$ is simply the nonautonomous system (5.22) with parameters (u, λ). We denote the solution operator by

$$U(x) = \Phi(x, y; u, \lambda)U(y), \qquad \nu^i(u) < x, y < \nu^{i+1}(u).$$

Let

$$Q^i = \{(u, 0, x, \lambda) \in T^i : (u, 0, x) \in P^i\} \quad \text{and} \quad R^i = \{(u, 0, x, \lambda) \in T^i : u \in M^i\},$$
$$\hat{Q}^i = \{(u, 0, x, \lambda) \in T^i : (u, 0, x) \in \hat{P}^i\} \quad \text{and} \quad \hat{R}^i = \{(u, 0, x, \lambda) \in T^i : u \in \hat{M}^i\}.$$

We inductively define subspaces $\Sigma^i(u, 0, x, \lambda)$ of U-space on R^i as follows:

(1) On R^0, $\Sigma^0(\bar{u}^0, 0, x, \lambda) = \{0\}$.
(2) Assuming Σ^{i-1} is defined on R^{i-1}, let $(u^i, 0, x^i, \lambda) \in Q^i$ and let $(u^{i-1}, 0, x^i, \lambda)$ be the corresponding point of R^{i-1}, so that $f(u^i) - f(u^{i-1}) - x^i(u^i - u^{i-1}) = 0$. Then $U^i \in \Sigma^i(u^i, 0, x^i, \lambda)$ if and only if there exist $U^{i-1} \in \Sigma^{i-1}(u^{i-1}, 0, x^i, \lambda)$ and $\rho^i \in \mathbb{R}$ such that

$$(Df(u^i) - x^i I)U^i - (Df(u^{i-1}) - x^i I)U^{i-1} - \rho^i(u^i - u^{i-1}) = 0.$$

In other words,

$$\Sigma^i(u^i, 0, x^i, \lambda) = (Df(u^i) - x^i I)^{-1}\{(Df(u^{i-1}) - x^i I)\Sigma^{i-1}(u^{i-1}, 0, x^i, \lambda) + \text{span }(u^i - u^{i-1})\}.$$

(3) For other $(u^i, 0, x, \lambda) \in R^i$,

$$\Sigma^i(u^i, 0, x, \lambda) = \Phi(x, x^i; u^i, \lambda)\Sigma^i(u^i, 0, x^i, \lambda).$$

Notice that $\dim \Sigma^{i-1} \leqslant \dim \Sigma^i \leqslant \dim \Sigma^{i-1} + 1$.

For $1 \leqslant i \leqslant n$, another description of the spaces $\Sigma^i(u, 0, x, \lambda)$ is as follows: Fix i, $1 \leqslant i \leqslant n$. For $i \leqslant k \leqslant n$ we inductively defined vector fields $w^{k,i}(u, 0, x, \lambda)$ on R^k as follows:

(1) Let $(u^i, 0, x^i, \lambda) \in Q^i$, and let $(u^{i-1}, 0, x^i, \lambda)$ be the corresponding point of R^{i-1}, so that $f(u^i) - f(u^{i-1}) - x^i(u^i - u^{i-1}) = 0$. Then

$$w^{i,i}(u^i, 0, x^i, \lambda) = (Df(u^i) - x^i I)^{-1}(u^i - u^{i-1}).$$

(2) If $w^{k,i}(u^k, 0, x^k, \lambda)$ has been defined on Q^k and $(u^k, 0, x, \lambda)$ is another point of R^k, let $w^{k,i}(u^k, 0, x, \lambda) = \Phi(x, x^k; u^k, \lambda) w^{k,i}(u^k, 0, x^k, \lambda)$.

(3) If $w^{k,i}(u^k, 0, x, \lambda)$ has been defined on R^k and $(u^{k+1}, 0, x^{k+1}, \lambda)$ is a point of Q^{k+1} with corresponding point $(u^k, 0, x^{k+1}, \lambda) \in R^k$, then

$$w^{k+1,i}(u^{k+1}, 0, x^{k+1}, \lambda) = (Df(u^{k+1}) - x^{k+1} I)^{-1} (Df(u^k) - x^{k+1} I) w^{k,i}(u^k, 0, x^{k+1}, \lambda).$$

Then for $k \geqslant 1$, if $(u, 0, x, \lambda) \in R^k$, $\Sigma^k(u, 0, x, \lambda) =$ span $(w^{k,1}(u, 0, x, \lambda), \dots, w^{k,k}(u, 0, x, \lambda))$.

For $i = 0, \dots, n$, let

$$\mathcal{R}_0^i = \{(u, v, x, \lambda, U, V) : (u, v, x, \lambda) \in R^i, U \in \Sigma^i(u, v, x, \lambda), V = 0\}.$$

Proposition 5.2. *For some i, $1 \leqslant i \leqslant n$, and all $0 < \epsilon < \epsilon_0$, let $\mathcal{R}_\epsilon^{i-1}$ be a* vector bundle *over R^{i-1} that is contained in $\mathcal{T}_\epsilon^{i-1}$ and is invariant under (5.1)–(5.6). Assume that the sets $\mathcal{R}_\epsilon^{i-1} \times \{\epsilon\}$ fit together with the set $\mathcal{R}_0^{i-1} \times \{0\}$ defined above to form a C^{r+6} submanifold of $uvx\lambda UV\epsilon$-space, $r \geqslant 2$. For $0 \leqslant \epsilon < \epsilon_0$, let H_ϵ be a cross-section of $W_\epsilon^u(R^{i-1})$ such that H_0 contains a point $(q^i(\xi), \dot{q}^i(\xi), \bar{x}^i, \bar{\lambda})$ with $\bar{\lambda} \neq -1$, and the sets $H_\epsilon \times \{\epsilon\}$ fit together to form a C^{r+6} submanifold of $uvx\lambda\epsilon$-space. Let $\mathcal{H}_\epsilon$ be the restriction to H_ϵ of the vector bundle $W_\epsilon^u(\mathcal{R}_\epsilon^{i-1})$. Then, with $\mathcal{S}_0 = \mathcal{S}_0^i$, $\theta_0 = (\bar{u}^i, 0, \bar{x}^i, \bar{\lambda})$, and $p = (q^i(\xi), \dot{q}^i(\xi), \bar{x}^i, \bar{\lambda}, \dot{q}^i(\xi), \ddot{q}^i(\xi))$, assumptions* (E1)–(E7) *of Section* 4 *are satisfied. Assumption* (E8) *is satisfied if and only if $dim\,\Sigma^i = dim\,\Sigma^{i-1} + 1$.*

Proof. (E1)–(E4) were shown in Section 2, and (E5) has been explained. For (E6), note that the fiber of $\mathcal{H}_0$ at (u, v, x, λ) contains $E^u(u, v, x, \lambda)$, so this fiber is transverse to $E^{cs}(u, v, x, \lambda)$ by Proposition 5.1 (1). For (E7), note that the fiber of $\mathcal{H}_0$ at (u, v, x, λ) is contained in $E^{cu}(u, v, x, \lambda)$, so from Proposition 5.1 (2) its intersection with $E^s(u, v, x, \lambda)$ is at most one-dimensional. It is exactly one-dimensional since this fiber contains $E^u(u, v, x, \lambda)$, whose intersection with $E^s(u, v, x, \lambda)$ is one-dimensional (the span of $(\dot{q}^i(\xi), \ddot{q}^i(\xi))$).

Suppose $(q^i(\xi), \dot{q}^i(\xi), \bar{x}^i, \bar{\lambda}, U_0(\xi), V_0(\xi))$ lies in $W_0^u(\mathcal{R}_0^{i-1}) \cap W_0^s(\mathcal{T}_0^i)$ and approaches, as $\xi \to -\infty$, $(\bar{u}^{i-1}, 0, \bar{x}^i, \bar{\lambda}, U_0^{i-1}, 0) \in \mathcal{R}_0^{i-1}$. Then as $\xi \to \infty$, the solution approaches $(\bar{u}^i, 0, \bar{x}^i, \bar{\lambda}, U_0^i, 0) \in \mathcal{T}_0^i$, where, from (5.15),

$$(Df(\bar{u}^i) - \bar{x}^i I) U^i - (Df(\bar{u}^{i-1}) - \bar{x}^i I) U^{i-1} = 0.$$

The set of such U^i is a subspace of $\Sigma^i(\bar{u}^i, 0, \bar{x}^i, \bar{\lambda})$ with dimension equal to that of $\Sigma^{i-1}(\bar{u}^{i-1}, 0, \bar{x}^i, \bar{\lambda})$.

To check whether (E8) is satisfied, suppose

$$\begin{aligned}(u^i_\epsilon(\xi), v^i_\epsilon(\xi), x^i_\epsilon(\xi), \lambda(\epsilon), U^i_\epsilon(\xi), V^i_\epsilon(\xi))\\ = (q^i(\xi), \dot q^i(\xi), \bar x^i, \bar\lambda, \dot q^i(\xi), \ddot q^i(\xi))\\ +\epsilon(u^i_1(\xi), v^i_1(\xi), x^i_1(\xi), \lambda^i_1, U^i_1(\xi), V^i_1(\xi)) + O(\epsilon^2)\end{aligned} \tag{5.23}$$

lies in $W^u_\epsilon(\mathcal{R}^{i-1}_\epsilon)\cap W^s_\epsilon(\mathcal{T}^i_\epsilon)$. Then for $k=i-1, i$. there are solutions

$$\begin{aligned}(\tilde u^k_\epsilon(\xi), 0, \tilde x^k_\epsilon(\xi), \tilde\lambda^k(\epsilon), \tilde U^k_\epsilon(\xi), \tilde V^k_\epsilon(\xi))\\ = (\tilde u^k_0(\xi), 0, \tilde x^k_0(\xi), \tilde\lambda^k_0, \tilde U^k_0(\xi), \tilde V^k_0(\xi))\\ +\epsilon(\tilde u^k_1(\xi), 0, \tilde x^k_1(\xi), \tilde\lambda^k_1, \tilde U^k_1(\xi), \tilde V^k_1(\xi)) + O(\epsilon^2)\end{aligned} \tag{5.24}$$

in $\mathcal{T}^k_\epsilon$ that (5.23) approaches exponentially as $\xi\to-\infty$ and $\xi\to\infty$, respectively. Clearly

$$\lambda(\epsilon)=\tilde\lambda^{i-1}(\epsilon)=\tilde\lambda^i(\epsilon)=\bar\lambda+\epsilon\lambda_1+O(\epsilon^2).$$

Also, since $\dot x=\epsilon$, we must have

$$x^i_\epsilon(\xi)=\tilde x^{i-1}_\epsilon(\xi)=\tilde x^i_\epsilon(\xi)=x^i(\epsilon)+\epsilon\xi=(\bar x^i+\epsilon x^i_1+O(\epsilon^2))+\epsilon\xi.$$

Since $\dot u=0$ when $v=0$, we have

$$\tilde u^k_\epsilon(\xi)=\tilde u^k(\epsilon)=\bar u^k+\epsilon\tilde u^k_1+O(\epsilon^2),\quad k=i-1, i.$$

From invariant manifold theory,

$$\begin{aligned}0=\lim_{\xi\to-\infty} u^i_1(\xi)-\tilde u^{i-1}_1(\xi)=\lim_{\xi\to-\infty} u^i_1(\xi)-\tilde u^{i-1}_1,\\ 0=\lim_{\xi\to\infty} u^i_1(\xi)-\tilde u^i_1(\xi)=\lim_{\xi\to\infty} u^i_1(\xi)-\tilde u^i_1.\end{aligned} \tag{5.25}$$

For $k=i-1, i$, $\tilde U^k_\epsilon(\xi)$ satisfies (5.20) with $(u, x, \lambda)=(\tilde u^k_\epsilon(\xi), x^i_\epsilon(\xi), \lambda(\epsilon))$. It follows easily that $\tilde U^k_0(\xi)$ is constant. Since

$$\lim_{\xi\to-\infty}(\dot q^i(\xi)-\tilde U^{i-1}_0(\xi))=0 \quad\text{and}\quad \lim_{\xi\to\infty}(\dot q^i(\xi)-\tilde U^i_0(\xi))=0,$$

we see that $\tilde U^k_0(\xi)=0$ for $k=i-1, i$. Substituting this into the expansion of $\tilde U^k_\epsilon(\xi)$ in (5.24), then substituting the expansion into (5.20), we find that for $k=i-1, i$, $\tilde U^k_1(\xi)$ is constant, so there is a vector $\tilde U^k_1$ in $\mathbb{R}^n$ such that $\tilde U^k_1(\xi)=\tilde U^k_1$. Also, since $(\tilde u^{i-1}_\epsilon(\xi), 0, x^i_\epsilon(\xi), \lambda(\epsilon), \tilde U^{i-1}_\epsilon(\xi), \tilde V^{i-1}_\epsilon(\xi))\in\mathcal{R}^{i-1}_\epsilon$, and $\tilde U^{i-1}_0=0$, it follows easily that $\tilde U^{i-1}_1\in\Sigma^{i-1}(\bar u^{i-1}, 0, \bar x^i, \bar\lambda)$.

If we substitute (5.23) into

$$\ddot U=\epsilon\lambda U+(Df(u)-xI)V+D^2f(u)vU,$$

which is equivalent to (5.5) and (5.6), we find that $U_1(\xi)$ satisfies

$$U^i_{1\xi\xi} = (\bar{\lambda}+1)\dot{q}^i + ((D^2 f(q^i)u^i_1 - (x^i_1+\xi)I)\dot{q}^i)_\xi + ((Df(q^i) - x^i_0 I)U^i_1)_\xi. \tag{5.26}$$

Integrating (5.26) from $\xi = -\infty$ to $\xi = \infty$, we obtain

$$(\bar{\lambda}+1)(\bar{u}^i_0 - \bar{u}^{i-1}_0) + (Df(\bar{u}^i_0) - x^i_0 I)\tilde{U}^i_1 - (Df(\bar{u}^{i-1}_0) - x^i_0 I)\tilde{U}^{i-1}_1 = 0. \tag{5.27}$$

Since $\bar{\lambda} \neq -1$, the curve (5.23) shows that (E8) is satisfied provided

$$\tilde{U}^i_1 \notin (Df(\bar{u}^i) - \bar{x}^i I)^{-1}(Df(\bar{u}^{i-1}) - \bar{x}^i I)\Sigma^{i-1}(\bar{u}^{i-1}, 0, \bar{x}^i, \bar{\lambda}).$$

Equivalently,

$$\bar{u}^i_0 - \bar{u}^{i-1}_0 \notin (Df(\bar{u}^{i-1}) - \bar{x}^i I)\Sigma^{i-1}(\bar{u}^{i-1}, 0, \bar{x}^i, \bar{\lambda}).$$

This is equivalent to the given condition. □

Similarly, we define subspaces $\hat{\Sigma}^i(u,0,x,\lambda)$ of U-space on R^i by backwards induction as follows:

(1) On R^n, $\hat{\Sigma}^n(\bar{u}^n, 0, x, \lambda) = \{0\}$.
(2) Assuming $\hat{\Sigma}^{i+1}$ is defined on R^{i+1}, let $(u^i, 0, x^{i+1}, \lambda) \in \hat{Q}^i$ and let $(u^{i+1}, 0, x^{i+1}, \lambda)$ be the corresponding point of R^{i+1}, so that $f(u^i) - f(u^{i+1}) - x^{i+1}(u^i - u^{i+1}) = 0$. Then $U^i \in \hat{\Sigma}^i(u^i, 0, x^{i+1}, \lambda)$ if and only if there exist $U^{i+1} \in \hat{\Sigma}^{i+1}(u^{i+1}, 0, x^{i+1}, \lambda)$ and $\rho^i \in \mathbb{R}$ such that

$$(Df(u^i) - x^{i+1}I)U^i - (Df(u^{i+1}) - x^{i+1}I)U^{i+1} - \rho^i(u^i - u^{i+1}) = 0.$$

In other words,

$$\hat{\sum}^i(u^i, 0, x^{i+1}, \lambda) = (Df(u^i) - x^{i+1}I)^{-1}\{(Df(u^{i+1}) - x^{i+1}I)\hat{\Sigma}^{i+1}(u^{i+1}, 0, x^{i+1}, \lambda) + \text{ span } (u^i - u^{i+1})\}.$$

(3) For other $(u^i, 0, x, \lambda) \in R^i$,

$$\hat{\Sigma}^i(u^i, 0, x, \lambda) = \Phi(x, x^{i+1}; u^i, \lambda)\hat{\Sigma}^i(u^i, 0, x^{i+1}, \lambda).$$

Notice that dim $\hat{\Sigma}^{i+1} \leqslant$ dim $\hat{\Sigma}^i \leqslant$ dim $\hat{\Sigma}^{i+1} + 1$.

A somewhat more precise description of these spaces is as follows: Fix i, $1 \leqslant i \leqslant n$. For $0 \leqslant k \leqslant i-1$ we define vector fields $\hat{w}^{k,i}(u,0,x,\lambda)$ on $\hat{R}^k$ by reverse induction as follows

(1) Let $(u^{i-1},0,x^i,\lambda)\in\hat{Q}^{i-1}$, and let $(u^i,0,x^i,\lambda)$ be the corresponding point of R^i, so that $f(u^{i-1})-f(u^i)-x^i(u^{i-1}-u^i)=0$. Then

$$\hat{w}^{i-1,i}(u^{i-1},0,x^i,\lambda)=(Df(u^{i-1})-x^iI)^{-1}(u^{i-1}-u^i).$$

(2) If $\hat{w}^{k-1,i}(u^{k-1},0,x^k,\lambda)$ has been defined on $\hat{Q}^{k-1}$ and $(u^{k-1},0,x,\lambda)$ is another point of $\hat{R}^{k-1}$, let $\hat{w}^{k-1,i}(u^{k-1},0,x,\lambda)=\Phi(x,x^k;u^{k-1,i},\lambda)\hat{w}^{k-1,i}(u^{k-1},0,x^k,\lambda)$.

(3) If $\hat{w}^{k,i}(u^k,0,x,\lambda)$ has been defined on $\hat{R}^k$ and $(u^{k-1},0,x^k,\lambda)$ is a point of $\hat{Q}^{k-1}$, with corresponding point $(u^k,0,x^k,\lambda)\in\hat{R}^k$, then

$$\hat{w}^{k-1,i}(u^{k-1},0,x^k,\lambda)=(Df(u^{k-1})\\ -x^kI)^{-1}(Df(u^k)-x^kI)\hat{w}^{k,i}(u^k,0,x^k,\lambda).$$

Then for $k\leqslant n-1$, if $(u,0,x,\lambda)\in\hat{R}^k$,

$$\hat{\Sigma}^k(u,0,x,\lambda)=\text{ span }(\hat{w}^{k,k+1}(u,0,x,\lambda),\dots,\hat{w}^{k,n}(u,0,x,\lambda)).$$

For $i=1,\dots,n$, let

$$\hat{\mathcal{R}}_0^i=\{(u,v,x,\lambda,U,V):(u,v,x,\lambda)\in\hat{R}^i,\,U\in\hat{\Sigma}^i(u,v,x,\lambda),\,V=0\}.$$

Proposition 5.3. *For some $i, 0\leqslant i\leqslant n-1$, and all $0<\epsilon<\epsilon_0$, let $\hat{\mathcal{R}}_\epsilon^{i+1}$ be a* vector bundle *over $\hat{\mathcal{R}}^{i+1}$ that is contained in $\mathcal{T}_\epsilon^{i+1}$ and is invariant under (5.22). Assume that the sets $\hat{\mathcal{R}}_\epsilon^{i+1}\times\{\epsilon\}$ fit together with the set $\hat{\mathcal{R}}_0^{i+1}\times\{0\}$ defined above to form a C^{r+6} submanifold of $uvx\lambda UV\epsilon$-space, $r\geqslant 2$. For $0\leqslant\epsilon\leqslant\epsilon_0$, let $\hat{H}_\epsilon$ be a cross-section of $W_\epsilon^s(\hat{R}^{i+1})$ such that $\hat{H}_0$ contains the point $(q^{i+1}(\xi),\dot{q}^{i+1}(\xi),\bar{x}^{i+1},\bar{\lambda})$ with $\bar{\lambda}\neq-1$, and the sets $\hat{H}_\epsilon\times\{\epsilon\}$ fit together to form a C^{r+6} submanifold of $uvx\lambda\epsilon$-space. Let $\hat{\mathcal{H}}_\epsilon$ be the restriction to $\hat{H}_\epsilon$ of the vector bundle $W_\epsilon^u(\hat{\mathcal{R}}_\epsilon^{i+1})$. Then, with $\mathcal{S}_0=\mathcal{S}_0^i,\theta_0=(\bar{u}^i,0,\bar{x}^{i+1},\bar{\lambda})$, and $p=(q^{i+1}(\xi),\dot{q}^{i+1}(\xi),\bar{x}^{i+1},\bar{\lambda},\dot{q}^{i+1}(\xi),\ddot{q}^{i+1}(\xi))$, assumptions* (E1)–(E7) *of Section* 4 *are satisfied for the backwards flow. Assumption* (E8) *is satisfied if and only if dim $\hat{\Sigma}^i=$ dim $\hat{\Sigma}^{i+1}+1$.*

Let $M(\lambda)$ denote the $n\times n$ matrix whose columns are $w^{n,i}(\bar{u}^n,\bar{x}^n,\lambda), i=1,\dots,n$, and let $E(\lambda)=\det M(\lambda)$.

Assume:

(S1) $\bar{\lambda}$ is a simple zero of $M(\lambda)$.

Let $\ell, 1\leqslant\ell\leqslant n-1$, denote the last integer such that the ℓ vectors $w^{n,1}(\bar{u}^n,\bar{x}^n,\lambda),\dots,w^{n,\ell}(\bar{u}^n,\bar{x}^n,\lambda)$ are linearly independent. Then the vectors $w^{n,\ell+1}(\bar{u}^n,\bar{x}^n,\lambda),\dots,w^{n,n}(\bar{u}^n,\bar{x}^n,\lambda)$ are also linearly independent.

Theorem 5.4 says that if $\bar{\lambda}\neq-1$ satisfies (S1), then there is an eigenvalue nearby.

Theorem 5.4. *Suppose f is sufficiently differentiable and $\bar{\lambda} \neq -1$ satisfies* (S1). *Then for small $\epsilon > 0$, $W^u_\epsilon(\mathcal{R}^0_0)$ and $W^s_\epsilon(\hat{\mathcal{R}}^n_0)$ meet transversally along a one-parameter family of orbits $(u_\epsilon(\xi), \dot{u}_\epsilon(\xi), \epsilon\xi, aU_\epsilon(\xi), aV_\epsilon(\xi), \lambda(\epsilon))$, with $u_\epsilon(\xi)$ the Riemann–Dafermos solution and $\lambda(\epsilon) = \bar{\lambda} + O(\epsilon)$.*

Proof. We follow $W^u_\epsilon(\mathcal{R}^0_0), \epsilon > 0$, forward past $x = \bar{x}^1$ using Theorem 4.1, whose hypotheses are verified by Proposition 5.2. We see that near $x = \bar{x}^2$, the $W^u_\epsilon(\mathcal{R}^0_0), \epsilon > 0$, fit together with $W^u_0(\mathcal{R}^1_0)$ to form a smooth manifold. The U-part of $\mathcal{R}^1_0$ has dimension one greater than the U-part of $\mathcal{R}^1_0$; the process by which this occurs is pictured in Figure 5.

Proceding inductively, we see that near $x = \bar{x}^{\ell+1}$, the $W^u_\epsilon(\mathcal{R}^0_0), \epsilon > 0$, fit together with $W^u_0(\mathcal{R}^\ell_0)$ to form a smooth manifold.

Similarly, we follow $W^s_\epsilon(\mathcal{R}^n_0), \epsilon > 0$, backward past $x = \bar{x}^n$ using Theorem 4.1, whose hypotheses are verified by Proposition 5.3. We see that near $x = \bar{x}^{n-1}$, the $W^s_\epsilon(\mathcal{R}^n_0), \epsilon > 0$, fit together with $W^s_0(\mathcal{R}^{n-1}_0)$ to form a smooth manifold. Proceding inductively, we see that near $x = \bar{x}^\ell$, the $W^s_\epsilon(\mathcal{R}^n_0), \epsilon > 0$, fit together with $W^s_0(\mathcal{R}^\ell_0)$ to form a smooth manifold.

Let $\bar{x} = \frac{1}{2}(\bar{x}^\ell + \bar{x}^{\ell+1})$, and let $p = (\bar{u}^\ell, 0, \bar{x}, \bar{U}, 0)$ be a point in the intersection of $W^u_0(\mathcal{R}^\ell_0)$ and $W^s_0(\hat{\mathcal{R}}^\ell_0)$.

A neighborhood of p in $W^u_0(\mathcal{R}^\ell_0)$ is parameterized by a map

$$(\alpha, \beta, x^L, \lambda^L, a) \in \mathbb{R}^\ell \times \mathbb{R}^{n-\ell} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to (u, v, x, \lambda, U, V)$$

of the form

$$\begin{aligned}
u &= A(\alpha) + O(\beta),\\
v &= B(\alpha, x^L)\beta + O(\beta^2),\\
x &= x^L,\\
\lambda &= \lambda^L,\\
U &= \sum_{i=1}^{\ell} a^i w^{\ell,i}(A(\alpha), x^L, \lambda^L) + \sum_{i=\ell+1}^{n} a^i r^i(A(\alpha)) + O(\beta),\\
V &= \sum_{i=\ell}^{n} a^i(\nu^i(A(\alpha)) - x^L) r^i(A(\alpha)) + O(\beta)
\end{aligned}$$

with $A(0) = \bar{u}^\ell$, DA of rank ℓ, and B of rank $n - \ell$.

A neighborhood of p in $W^s_0(\hat{\mathcal{R}}^\ell_0)$ is parameterized by a map

$$(\gamma, \delta, x^R, \lambda^R, b) \in \mathbb{R}^{n-\ell} \times \mathbb{R}^\ell \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to (u, v, x, \lambda, U, V)$$

of the form

$$\begin{aligned}
u &= \hat{A}(\gamma) + O(\delta),\\
v &= \hat{B}(\gamma, x^R)\delta + O(\delta^2),\\
x &= x^R,\\
\lambda &= \lambda^R,\\
U &= \sum_{i=1}^{\ell} b^i r^i(\hat{A}(\gamma)) + \sum_{i=\ell+1}^{n} b^i \hat{w}^{\ell,i}(\hat{A}(\gamma), x^R, \lambda^R) + O(\delta),\\
V &= \sum_{i=1}^{\ell} b^i(\nu^i(\hat{A}(\gamma)) - x^R) r^i(\hat{A}(\gamma)) + O(\delta)
\end{aligned}$$

with $\hat{A}(0) = \bar{u}^\ell$, $D\hat{A}$ of rank $n-\ell$ and B of rank ℓ.

Define linear maps $\Psi(\lambda)\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\begin{aligned}
\Psi(\lambda) = {} & (Df(\bar{u}^n) - \bar{x}^n I)^{-1}(Df(\bar{u}^{n-1}) - \bar{x}^n I)\Phi(\bar{x}^n, \bar{x}^{n-1}; \bar{u}^{n-1}, \lambda)\\
& \cdots (Df(\bar{u}^{\ell+1}) - \bar{x}^{\ell+1} I)^{-1}(Df(\bar{u}^\ell) - \bar{x}^{\ell+1} I)\Phi(\bar{x}^{\ell+1}, \bar{x}; \bar{u}^\ell, \lambda).
\end{aligned}$$

Then

$$\begin{aligned}
w^{n,i}(\bar{u}^n, \bar{x}^n, \lambda) &= \Psi(\lambda) w^{\ell,i}(\bar{u}^\ell, \bar{x}, \lambda), \quad i = 1, \ldots, \ell,\\
w^{n,i}(\bar{u}^n, \bar{x}^n, \lambda) &= \Psi(\lambda) \hat{w}^{\ell,i}(\bar{u}^\ell, \bar{x}, \lambda), \quad i = \ell+1, \ldots, n.
\end{aligned}$$

Therefore, there exist $(\bar{a}^1, \ldots, \bar{a}^\ell)$ such that

$$\hat{w}^{\ell,\ell+1}(\bar{u}^\ell, \bar{x}, \bar{\lambda}) = \sum_{i=1}^{\ell} \bar{a}^i w^{\ell,i}(\bar{u}^\ell, \bar{x}, \bar{\lambda}) \tag{5.28}$$

and we may take $\bar{U} = \hat{w}^{\ell,\ell+1}(\bar{u}^\ell, \bar{x}, \lambda)$.

The tangent spaces to $W_0^u(\mathcal{R}_0^\ell)$ and $W_0^s(\hat{\mathcal{R}}_0^\ell)$ at p are spanned, respectively, by the column vectors in the matrices

$$\begin{pmatrix}
A'(0) & * & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0\\
0 & B(0,\bar{x}) & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0\\
0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0\\
0 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0\\
* & * & \sum_{i=1}^{\ell} \bar{a}^i \frac{\partial}{\partial x^L} w^{\ell,i} & \sum_{i=1}^{\ell} \bar{a}^i \frac{\partial}{\partial \lambda^L} w^{\ell,i} & w^{\ell,1} & \ldots & w^{\ell,\ell} & r^{\ell+1} & \ldots & r^n\\
* & * & * & 0 & 0 & \ldots & 0 & (\nu^{\ell+1} - \bar{x}) r^{\ell+1} & \ldots & (\nu^n - \bar{x}) r^n
\end{pmatrix}$$

and

$$\begin{pmatrix} \hat{A}'(0) & * & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \hat{B}(0,\bar{x}) & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ * & * & \frac{\partial}{\partial x^R}\hat{w}^{\ell,\ell+1} & \frac{\partial}{\partial \lambda^R}\hat{w}^{\ell,\ell+1} & r^1 & \dots & r^\ell & \hat{w}^{\ell,\ell+1} & \dots & \hat{w}^{\ell,n} \\ * & * & * & 0 & (\nu^1-\bar{x})r^1 & \dots & (\nu^\ell-\bar{x})r^\ell & 0 & \dots & 0 \end{pmatrix},$$

where the $w^{\ell,i}$ and $\hat{w}^{\ell,i}$ are evaluated at $(\bar{u}^\ell,\bar{x},\lambda)$, the ν^i and r^i are evaluated at $\bar{u}^\ell$, and the starred entries are not important. The span of all these column vectors is $uvx\lambda UV$-space if and only if the span of the $n+4$ vectors in the following matrix, in which the $w^{\ell,i}$ and $\hat{w}^{\ell,i}$ are considered to be functions of $(\bar{u}^\ell,x,\lambda)$ evaluated at $(\bar{u}^\ell,\bar{x},\bar{\lambda})$, is $\mathbb{R}^{n+2}$, i.e., $x\lambda U$-space:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \sum_{i=1}^{\ell}\bar{a}^i\frac{\partial}{\partial x}w^{\ell,i} & \frac{\partial}{\partial x}\hat{w}^{\ell,\ell+1} & \sum_{i=1}^{\ell}\bar{a}^i\frac{\partial}{\partial\lambda}w^{\ell,i} & \frac{\partial}{\partial\lambda}\hat{w}^{\ell,\ell+1} & w^{\ell,1} & \dots & w^{\ell,\ell} & \hat{w}^{\ell,\ell+1} & \dots & \hat{w}^{\ell,n} \end{pmatrix}.$$

Of course, the column containing $\hat{w}^{\ell,\ell+1}$ is a linear combination of the ℓ previous columns. Also, from (5.28) and the fact that the $w^{\ell,i}$ and $\hat{w}^{\ell,i}$ satisfy the same linear differential equation, we see that the first two columns are equal. Therefore, the following matrix has the same column span as the previous one:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \frac{\partial}{\partial x}\hat{w}^{\ell,\ell+1} & \sum_{i=1}^{\ell}\bar{a}^i\frac{\partial}{\partial\lambda}w^{\ell,i} & \frac{\partial}{\partial\lambda}\hat{w}^{\ell,\ell+1}-\sum_{i=1}^{\ell}\bar{a}^i\frac{\partial}{\partial\lambda}w^{\ell,i} & w^{\ell,1} & \dots & w^{\ell,\ell} & \hat{w}^{\ell,\ell+2} & \dots & \hat{w}^{\ell,n} \end{pmatrix}.$$

This matrix has full rank if and only if the vector $\frac{\partial}{\partial\lambda}\hat{w}^{\ell,\ell+1}-\sum_{i=1}^{\ell}\bar{a}^i\frac{\partial}{\partial\lambda}w^{\ell,i}$ is independent of the vectors $w^{\ell,1},\dots,w^{\ell,\ell},\hat{w}^{\ell,\ell+2},\dots,\hat{w}^{\ell,n}$.

We claim that this condition is equivalent to $E'(\bar{\lambda})\neq 0$. To see this, let $N(\lambda)$ denote the $n\times n$ matrix

$$\left(w^{\ell,1}(\bar{u}^\ell,\bar{x},\lambda)\dots w^{\ell,\ell}(\bar{u}^\ell,\bar{x},\lambda) \quad \hat{w}^{\ell,\ell+1}(\bar{u}^\ell,\bar{x},\lambda)\dots\hat{w}^{\ell,n}(\bar{u}^\ell,\bar{x},\lambda)\right)$$

and let $F(\lambda)=\det N(\lambda)$. Then $M(\lambda)=\Psi(\lambda)N(\lambda)$, so $E(\lambda)=\det\Psi(\lambda)F(\lambda)$ and $E'(\bar{\lambda})=\det\Psi(\bar{\lambda})F'(\lambda)$. Hence $E'(\bar{\lambda})$ is nonzero if and only if $F'(\bar{\lambda})$ is nonzero.

After some column operations, we see that

$$F(\lambda)=\det\left(w^{\ell,1}\ \dots\ w^{\ell,\ell}\ \hat{w}^{\ell,\ell+1}-\sum_{i=1}^{\ell}\bar{a}^i w^{\ell,i}\ \hat{w}^{\ell,\ell+2}\ \dots\ \hat{w}^{\ell,n}\right).$$

Since the $(\ell+1)$st column is 0 at $\lambda=\bar{\lambda}$,

$$F'(\bar{\lambda})=\det\left(w^{\ell,1}\ \ldots\ w^{\ell,\ell}\ \frac{\partial}{\partial\lambda}\hat{w}^{\ell,\ell+1}-\sum_{i=1}^{\ell}\bar{a}^i\frac{\partial}{\partial\lambda}w^{\ell,i}\ \hat{w}^{\ell,\ell+2}\ \ldots\ \hat{w}^{\ell,n}\right).$$

Therefore $F'(\bar{\lambda})$ is nonzero if and only if the desired linear independence condition holds. □

Theorem 5.5. *Suppose f is sufficiently differentiable, $\bar{\lambda}\neq -1$, and $M(\bar{\lambda})\neq 0$. Then there are numbers $\delta_0>0$ and $\epsilon_0>0$ such that for $|\lambda-\bar{\lambda}|<\delta_0$ and $0<\epsilon<\epsilon_0$, λ is not an eigenvalue of the linearized Dafermos operator at the Riemann–Dafermos solution u_ϵ.*

Proof. Choose an ℓ between 1 and n. Proceeding as in the proof of Theorem 5.4, we see that there do not exist $(\bar{a}^1,\ldots,\bar{a}^\ell)$ and $(\bar{b}^{\ell+1},\ldots,\bar{b}^n)$ such that (5.28) holds. In this case, $W_0^u(\mathcal{R}^{\ell_0})$ and $W_0^s(\hat{\mathcal{R}}_0^\ell)$ meet transversally along the two-dimensional manifold

$$\{(u,v,x,\lambda,U,V): u=\bar{u}^\ell, v=0, \mu^\ell(\bar{u}^\ell)<x<\mu^{\ell+1}(\bar{u}^\ell), |\lambda-\bar{\lambda}|<2\delta_0, U=V=0\}.$$

It follows that for small $\epsilon>0$, $W_\epsilon^u(\mathcal{R}_0^0)$ and $W_\epsilon^s(\hat{\mathcal{R}}_0^n)$ meet transversally near $(\bar{u}^\ell,0,\bar{x},\bar{\lambda},0,0)$ along their known two-dimensional intersection $(u_\epsilon(\xi),\dot{u}_\epsilon(\xi),\epsilon\xi,\lambda,0,0)$. The result is just a restatement of this fact.

APPENDIX A. NORMAL HYPERBOLICITY OF S_δ^0 AND S_δ^n

In Section 2, we defined manifolds S_δ^0 and S_δ^n, which are not compact. To treat their normal hyperbolicity, we compactify the x-component of uvx-space by adding points at $x=-\infty$ and $x=\infty$. We denote the compactified $\mathbb{R}$ with its usual topology by $\hat{\mathbb{R}}=[-\infty,\infty]$. We define $\hat{S}_\delta^0$ and $\hat{S}_\delta^n$ in $\hat{\mathbb{R}}$ by the same equations used in Section 2, so that $\hat{S}_\delta^0$ and $\hat{S}_\delta^n$ extend all the way to $x=-\infty$ and $x=\infty$, respectively.

On $\hat{\mathbb{R}}$ we use three coordinates: x, $-\infty<x<\infty$, on $\mathbb{R}\subset\hat{\mathbb{R}}$; y, $-\infty<y\leqslant 0$, on $[-\infty,0)\subset\hat{\mathbb{R}}$; and z, $0\leqslant z<\infty$, on $[0,\infty]\subset\hat{\mathbb{R}}$. The coordinate transformation between x and y (respectively, x and z) is $x=\frac{1}{y}$, $-\infty<y<0$ (resp. $x=\frac{1}{z}$, $0<z<\infty$). Of course, $y=0$ corresponds to $-\infty\in\hat{\mathbb{R}}$, and $z=0$ corresponds to $\infty\in\hat{\mathbb{R}}$.

In uvy-coordinates, the systems (2.5)–(2.7) becomes

$$\dot{u}=v, \tag{A.1}$$

$$\dot{v}=(Df(u)-\frac{1}{y}I)v, \tag{A.2}$$

$$\dot{x}=-y^2\epsilon. \tag{A.3}$$

Since (A.2) is undefined at $y=0$, corresponding to $-\infty \in \hat{\mathbb{R}}$, we multiply the system by $-y$, which is positive for $-\infty < y < 0$. We obtain

$$\dot{u} = -yv, \tag{A.4}$$

$$\dot{v} = (I - yDf(u))v, \tag{A.5}$$

$$\dot{x} = y^3\epsilon. \tag{A.6}$$

The system on $\mathbb{R}^n \times \mathbb{R}^n \times \hat{\mathbb{R}}$ that we consider is constructed from (A.4)–(A.6), the corresponding equation in uvz-coordinates, and (2.5)–(2.7) using a partition of unity, so that it coincides with (A.4)–(A.6) near $-\infty \in \hat{\mathbb{R}}$, with the corresponding equation in uvz-coordinates near $\infty \in \hat{\mathbb{R}}$, and with (2.5)–(2.7) near $0 \in \hat{\mathbb{R}}$. It has the same flow as (2.5)–(2.7), up to reparameterization of time, in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

For $\epsilon = 0$, (A.4)–(A.6) reduces to

$$\dot{u} = -yv, \tag{A.7}$$

$$\dot{v} = (I - yDf(u))v, \tag{A.8}$$

$$\dot{x} = 0, \tag{A.9}$$

which has the region $\|u\| \leqslant \frac{1}{\delta}, v = 0, -K \leqslant y \leqslant 0$ as a compact normally hyperbolic invariant manifold. It follows easily that $\hat{S}^0_\delta$ is a compact normally hyperbolic invariant manifold in $\mathbb{R}^n \times \mathbb{R}^n \times \hat{\mathbb{R}}$, and hence persists as a normally hyperbolic, locally invariant manifold for small $\epsilon > 0$.

A similar argument applies to S^n_δ.

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation under grant DMS-0406016.

REFERENCES

1. Azevedo, A., Marchesin, D., Plohr, B. J., and Zumbrun, K. (1996). Nonuniqueness of solutions of Riemann problems. *Zeit. angew. Math. Phys.* **47**, 977–998.
2. Benzoni-Gavage, S., Serre, D., and Zumbrun, K. (2001). Alternate Evans functions and viscous shock waves. *SIAM J. Math. Anal.* **32**, 929–962.
3. Beyn, W.-J., and Lorenz, J. (2005). Stability of viscous profiles: proofs via dichotomies, preprint, 2005.
4. Brunovsky, P. (1999). C^r-inclination theorems for singularly perturbed equations, *J. Diff. Eqs.* **155**, 133–152.
5. Dafermos, C. M. (1973). Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method. *Arch. Ration. Mech. Anal.* **52**, 1–9.
6. Deng, B. (1990). Homoclinic bifurcations with nonhyperbolic equilibria. *SIAM J. Math. Anal.* **21**, 693–720.

7. Evans, J. W. (1975). Nerve axon equations. IV. The stable and the unstable impulse, *Indiana Univ. Math. J.* **24**, 1169–1190.
8. Fenichel, N. (1979). Geometric singular perturbation theory for ordinary differential equations. *J. Diff. Eqs.* **31**, 53–98.
9. Freistühler, H., and Serre, D. (1998). L^1-stability of shock waves in scalar viscous conservation laws. *Comm. Pure Appl. Math.* **51**, 291–301.
10. Freistühler, H., and Szmolyan, P. (2002). Spectral stability of small shock waves. *Arch. Ration. Mech. Anal.* **164**, 287–309.
11. Fries, C. (1998). Nonlinear asymptotic stability of general small-amplitude viscous Laxian shock waves. *J. Diff. Eqs.* **146**, 185–202.
12. Fries, C. (2000). Stability of viscous shock waves associated with non-convex modes. *Arch. Ration. Mech. Anal.* **152**, 141–186.
13. Gardner, R., and Zumbrun, K. (1998). The gap lemma and geometric criteria for instability of viscous shock profiles. *Comm. Pure Appl. Math.* **51**, 789–847.
14. Goodman, J. (1986). Nonlinear asymptotic stability of viscous shock profiles for conservation laws. *Arch. Ration. Mech. Anal.* **95**, 325–344.
15. Howard, P. (2002). Pointwise estimates and stability for degenerate viscous shock waves. *J. Reine Angew. Math.* **545**, 19–65.
16. Howard, P., and Zumbrun, K. (2004). The Evans function and stability criteria for degenerate viscous shock waves. *Discrete Contin. Dyn. Syst.* **10**, 837–855.
17. Jones, C. K. R. T. (1995). *Geometric Singular Perturbation Theory. Dynamical Systems,* Montecatini Terme, 1994 Lecture Notes in Math. **1609**, Springer, Berlin, pp. 44–118.
18. Jones, C. K. R. T., Gardner, R., and Kapitula, T. (1993). Stability of travelling waves for nonconvex scalar viscous conservation laws. *Comm. Pure Appl. Math.* **46**, 505–526.
19. Jones, C. K. R. T., and Kopell, N. (1994). Tracking invariant manifolds with differential forms in singularly perturbed systems. *J. Diff. Eq.* **108**, 64–89.
20. Kreiss, G., and Kreiss, H.-O. (1998). Stability of systems of viscous conservation laws. *Comm. Pure Appl. Math.* **51**, 1397–1424.
21. M.Lewicka, in preparation.
22. Lin, X.-B. (2006). Slow eigenvalues of self-similar solutions of the Dafermos regularization of a system of conservation laws: an analytic approach. *J. Dyn. Diff. Eq.* DOI: 10.1007/s10884-005-9001-2.
23. Lin, X.-B., and Schecter, S. (2004). Stability of self-similar solutions of the Dafermos regularization of a system of conservation laws. *SIAM J. Math. Anal.* **35**, 884–921.
24. Liu, T.-P. (1985). Nonlinear stability of shock waves for viscous conservation laws. *Mem. Amer. Math. Soc.* **56**(328), 1–108.
25. Liu, T.-P. (1997). Pointwise convergence to shock waves for viscous conservation laws. *Comm. Pure Appl. Math.* **50**, 1113–1182.
26. Liu, T.-P., and Zumbrun, K. (1995). Nonlinear stability of an undercompressive shock for complex Burgers equation. *Comm. Math. Phys.* **168**, 163–186.
27. Liu, T.-P., and Zumbrun, K. (1995). On nonlinear stability of general undercompressive viscous shock waves. *Comm. Math. Phys.* **174**, 319–345.
28. Liu, W. (2004). Multiple viscous wave fan profiles for Riemann solutions of hyperbolic systems of conservation laws. *Discrete Contin. Dyn. Syst.* **10**, 871–884.
29. Matsumura, A., and Nishihara, K. (1985). On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas *Jpn. J. Appl. Math.* **2**, 17–25.
30. Nishihara, K., and Zhao, H. (2002). Convergence rates to viscous shock profile for general scalar viscous conservation laws with large initial disturbance. *J. Math. Soc. Jpn* **54**, 447–466.

31. Plaza, R., and Zumbrun, K. (2004). An Evans function approach to spectral stability of small-amplitude shock profiles. *Discrete Contin. Dyn. Syst.* **10**, 885–924.
32. Sattinger, D. H. (1976). On the stability of waves of nonlinear parabolic systems. *Adv. Math.* **22**, 312–355.
33. Schecter, S. (2002). Undercompressive shock waves and the Dafermos regularization. *Nonlinearity* **15**, 1361–1377.
34. Schecter, S., and Szmolyan, P. (2004). Composite waves in the Dafermos regularization. *J. Dyn. Diff. Eq.* **16**, 847–867.
35. Szepessy, A., and Xin, Z. P. (1993). Nonlinear stability of viscous shock waves. *Arch. Ration. Mech. Anal.* **122**, 53–103.
36. Szepessy, A., and Zumbrun, K., (1996). Stability of rarefaction waves in viscous media. *Arch. Ration. Mech. Anal.* **133**, 249–298.
37. P. Szmolyan, in preparation.
38. Tupčiev, V. A. (1964). On the splitting of an arbitrary discontinuity for a system of two first-order quasilinear equations. *Ž. VyČisl. Mat. i Mat. Fiz.* **4**, 817–825. English translation: *USSR Comput. Math. Math. Phys.* **4**, 36–48.
39. Tupčiev, V. A. (1973). The method of introducing a viscosity in the study of a problem of decay of a discontinuity. *Dokl. Akad. Nauk SSSR* **211**, 55–58. English translation: *Soviet Math. Dokl.* **14**, 978–982.
40. Tzavaras, A. E. (1996). Wave interactions and variation estimates for self-similar zero-viscosity limits in systems of conservation laws. *Arch. Ration. Mech. Anal.* **135**, 1–60.
41. Zumbrun, K., and Howard, P. (1998). Pointwise semigroup methods and stability of viscous shock waves. *Indiana Univ. Math. J.* **47**, 741–871.