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Persistence of Rarefactions under Dafermos Regularization: Blow-Up and an Exchange Lemma for Gain-of-Stability Turning Points*

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Abstract. We construct self-similar solutions of the Dafermos regularization of a system of conservation laws near structurally stable Riemann solutions composed of Lax shocks and rarefactions, with all waves possibly large. The construction requires blowing up a manifold of gain-of-stability turning points in a geometric singular perturbation problem as well as a new exchange lemma to deal with the remaining hyperbolic directions.

Key words. conservation laws, Riemann problem, geometric singular perturbation theory, loss of normal hyperbolicity, blow-up

AMS subject classifications. 35L65, 34E15

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1. Introduction.

This paper is the last in a series of three; the others are [22] and [23]. An introduction to the series is in [22]. We construct self-similar solutions of the Dafermos regularization of a system of conservation laws near structurally stable Riemann solutions composed of Lax shocks and rarefactions, with all waves possibly large. The construction requires blowing up a manifold of gain-of-stability turning points in a geometric singular perturbation problem. In addition, it requires a new exchange lemma to deal with the remaining hyperbolic directions. The latter is a consequence of the general exchange lemma from [23].

In this introduction, we briefly describe the conservation law background, and we describe some solutions near gain-of-stability turning points in order to help the reader's intuition.

A *system of conservation laws* in one space dimension is a partial differential equation of the form

$$u_T + f(u)_X = 0, \tag{1.1}$$

with $X \in \mathbb{R}$, $u \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ a smooth function. For background on this class of equations, see, for example, [26]. An important initial value problem is the *Riemann problem*, which has piecewise constant initial conditions:

$$u(X,0) = \begin{cases} u_L & \text{for } X < 0, \\ u_R & \text{for } X > 0. \end{cases} \tag{1.2}$$

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One looks for a solution of the Riemann problem in the self-similar form $u(x)$, $x = \frac{X}{T}$. Substitution into (1.1) yields the ordinary differential equation (ODE)

$$(A(u) - xI)u_x = 0, \tag{1.3}$$

with $A(u) = Df(u)$, an $n \times n$ matrix. Boundary conditions are $u(-\infty) = u_L$, $u(\infty) = u_R$. Solutions are allowed to have constant parts, continuously changing parts (rarefaction waves), and certain jump discontinuities (shock waves).

The *Dafermos regularization* of (1.1) is

$$u_T + f(u)_X = \epsilon T u_{XX}. \tag{1.4}$$

Solutions that have the self-similar form $u(x)$, $x = \frac{X}{T}$, satisfy the ODE

$$(A(u) - xI)u_x = \epsilon u_{xx}, \tag{1.5}$$

a "viscous perturbation" of (1.3). Solutions of (1.5) that approach constants at $x = \pm\infty$ and have $u'(\pm\infty) = 0$ are called *Riemann–Dafermos solutions*.

The Dafermos regularization was introduced with the expectation that Riemann–Dafermos solutions, for small $\epsilon > 0$, would turn out to be smoothed versions of the Riemann solution with the same boundary values. It is now known under a variety of assumptions that this is true [2, 29, 19, 16, 25]. The conclusion holds, for example, whenever u_L is close to u_R [29]. In addition, it holds for arbitrary u_L and u_R if (1) the Riemann solution consists entirely of shock waves, (2) each shock wave satisfies the viscous profile criterion (see section 2) for the viscosity u_{xx}, and (3) the Riemann solution is structurally stable; see [19]. We shall show that the same conclusion holds for arbitrary u_L and u_R provided (1′) the Riemann solution consists entirely of Lax shock waves and rarefaction waves and (2) and (3) hold. If rarefaction waves are present, this case is not covered by the above results.

Dafermos regularization gives a "holistic" approach to Riemann solutions: rather than piece together shock waves and rarefaction waves to obtain the Riemann solution, as is usually done [26], one constructs (a smoothed version of) the Riemann solution by solving the boundary value problem (1.5), $u(-\infty) = u_L$, $u(\infty) = u_R$, $u'(\pm\infty) = 0$, for a small $\epsilon > 0$. This approach to solving Riemann problems was implemented numerically in [17], but it is not fully justified without a better collection of results relating Riemann and Riemann–Dafermos solutions.

The Dafermos regularization arises naturally in the study of the long-time behavior of viscous conservation laws. To see this, consider the viscous regularization of (1.1)

$$u_T + f(u)_X = u_{XX}. \tag{1.6}$$

The change of variables

$$x = \frac{X}{T}, \quad t = \ln T \tag{1.7}$$

converts (1.6) into the nonautonomous system

$$u_t + (A(u) - xI)u_x = e^{-t}u_{xx}. \tag{1.8}$$

To study solutions of (1.8) for large t, it is natural to begin by studying the autonomous system

$$u_t + (A(u) - xI)u_x = \epsilon u_{xx} \tag{1.9}$$

with $\epsilon > 0$ small. Equation (1.9) is just (1.4) written in the variables (1.7). Riemann–Dafermos solutions are just stationary solutions of (1.9). This point of view on the Dafermos regularization is developed in [14]. Again, to pursue this approach to the long-time behavior of viscous conservation laws, a better collection of results relating Riemann and Riemann–Dafermos solutions is needed.

We remark that if the term u_{XX} in (1.6) is replaced by the more general viscous term $(B(u)u_X)_X$, then one should replace u_{xx} by $(B(u)u_x)_x$ in (1.8). Hence, to obtain the relevant Dafermos regularization, one should replace u_{xx} by $(B(u)u_x)_x$ in (1.9) and (1.5). One should therefore require shock waves in Riemann solutions to satisfy the viscous profile criterion for the new viscosity. We do not pursue this generalization in the present paper.

The ODE (1.5) can be written as the nonautonomous system

$$\begin{aligned} \epsilon u_x &= v, \\ \epsilon v_x &= (A(u) - xI)v. \end{aligned}$$

Setting $x = x_0 + \epsilon t$, and using a dot to denote the derivative with respect to t, we obtain the autonomous system

$$\dot{u} = v, \tag{1.10}$$

$$\dot{v} = (A(u) - xI)v, \tag{1.11}$$

$$\dot{x} = \epsilon, \tag{1.12}$$

with $(u, v, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. The boundary conditions become

$$(u, v, x)(-\infty) = (u_L, 0, -\infty), \quad (u, v, x)(\infty) = (u_R, 0, \infty). \tag{1.13}$$

It turns out that a solution of the Riemann problem (1.1)–(1.2) can be regarded as a singular solution ($\epsilon = 0$) of the boundary value problem (1.10)–(1.13). Riemann–Dafermos solutions, on the other hand, correspond to true solutions of (1.10)–(1.13) with $\epsilon > 0$. Therefore, to show the existence of Riemann–Dafermos solutions near a given Riemann solution, one can try to construct true solutions of (1.10)–(1.13), with $\epsilon > 0$ small, near certain singular solutions.

Note that for every ϵ, ux-space is invariant under (1.10)–(1.12). On ux-space, the system reduces to $\dot{u} = 0$, $\dot{x} = \epsilon$, so for $\epsilon = 0$, ux-space consists of equilibria. The linearization of (1.10)–(1.12) at one of these equilibria has the matrix

$$\begin{pmatrix} 0 & I & 0 \\ 0 & A(u) - xI & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.14}$$

This matrix has an eigenvalue 0 with multiplicity $n+1$ (the eigenspace is ux-space), plus the eigenvalues of $A(u) - xI$.

A common assumption in the study of conservation laws is *strict hyperbolicity*: for all u in a region of interest, $A(u)$ has n distinct real eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$. Under this assumption, the eigenvalues of $A(u) - xI$ are $\lambda_i(u) - x$, $i = 1, \ldots, n$. Therefore, for $\epsilon = 0$, ux-space loses normal hyperbolicity (see section 2) along the codimension-one surfaces $x = \lambda_i(u)$, $i = 1, \ldots, n$. As one crosses one of these surfaces along a line with u constant and x increasing, the eigenvalue $\lambda_i(u) - x$ changes from positive to negative (gain of stability).

For a small $\delta > 0$, let us consider $I_{u_L} = \{(u, v, x) : u = u_L,\ v = 0,\ x < \lambda_1(u) - \delta\}$. See Figure 1. For each ϵ, it is invariant and lies in the normally repelling invariant manifold $\{(u, v, x) : \|u\| < \frac{1}{\delta},\ v = 0,\ x < \lambda_1(u) - \delta\}$. (This manifold extends to $x = -\infty$; however, a compactification argument shows that it can still be regarded as a normally hyperbolic invariant manifold. See [21, Appendix A].) Hence it has an unstable manifold $W^u_\epsilon(I_{u_L})$ of dimension $n+1$ (see section 2). Similarly, $I_{u_R} = \{(u, v, x) : u = u_R,\ v = 0,\ \lambda_n(u) + \delta < x\}$ has a stable manifold $W^s_\epsilon(I_{u_R})$ of dimension $n+1$. For $\epsilon > 0$, solutions of (1.10)–(1.13) lie in $W^u_\epsilon(I_{u_L}) \cap W^s_\epsilon(I_{u_R})$. Notice that two manifolds of dimension $n+1$ in $\mathbb{R}^{2n+1}$, if they intersect, will typically intersect in curves. To find solutions of (1.10)–(1.13), one should follow $W^u_\epsilon(I_{u_L})$ forward by the flow for $\epsilon > 0$ until it meets $W^s_\epsilon(I_{u_R})$ (if it does).

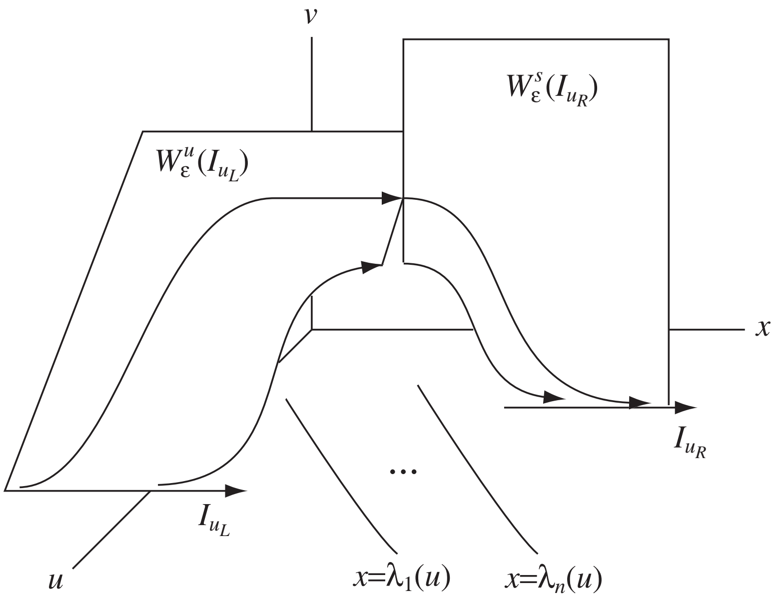

Figure 1. *For $\epsilon > 0$, an intersection of $W^u_\epsilon(I_{u_L})$ and $W^s_\epsilon(I_{u_R})$ gives a solution of the boundary value problem. The figure does not show the complications that typically occur in tracing $W^u_\epsilon(I_{u_L})$ forward.*

If the solution of the Riemann problem (1.1)–(1.2) consists only of shock waves, then for small $\epsilon > 0$, the relevant portion of $W^u_\epsilon(I_{u_L})$ does not pass near any of the surfaces $v = 0$, $x = \lambda_i(u)$, where normal hyperbolicity is lost, so it can be tracked when it passes near $v = 0$ using the usual exchange lemma [19]. If, however, the Riemann solution includes a rarefaction wave of the ith family (see section 2), then the relevant portion of $W^u_\epsilon(I_{u_L})$ passes near the surface $v = 0$, $x = \lambda_i(u)$ [25]. Thus we have the problem of tracking a manifold of solutions as it passes near a surface of gain-of-stability turning points. In the present paper we show how to do this, and we apply the result to finding solutions of the boundary value problem (1.10)–(1.13).

For $\epsilon = 0$, at a point (u, v, x) with $v = 0$ and $x = \lambda_i(u)$, the matrix (1.14) has the eigenvalue 0 with multiplicity $n+2$, and $n-1$ real nonzero eigenvalues. If $n \geq 2$, the analysis of the flow near such a point has two parts: the first part is the analysis of the flow on a collection of normally hyperbolic invariant manifolds K_ϵ of dimension $n+2$, each of which properly contains an open subset of ux-space; the second part is the application of the general exchange lemma from [23] to deal with the hyperbolic directions. For $n = 1$, the second step is not necessary; this was the situation in [25].

To help the reader's intuition, Figure 2 indicates the type of solution in which we are interested in the case $n = 1$, in which case $\lambda_1(u) = f'(u)$. In the figure, $u_L < u_R$, and $\lambda_1'(u) = f''(u) > 0$ for $u_L \leq u \leq u_R$. The figure shows a singular solution, which consists of the lines $u = u_L$, $v = 0$, $x < \lambda_1(u_L)$ and $u = u_R$, $v = 0$, $\lambda_1(u_R) < x$, together with the curve $u_L \leq u \leq u_R$, $v = 0$, $x = \lambda_1(u)$. For small $\epsilon > 0$ there is an actual solution $(u_\epsilon(t), v_\epsilon(t), \epsilon t)$ just above this one that approaches $(u_L, 0, -\infty)$ as $t \to -\infty$ and approaches $(u_R, 0, \infty)$ as $t \to \infty$. Such a solution lies in $W^u_\epsilon(I_{u_L}) \cap W^s_\epsilon(I_{u_R})$. Other solutions in $W^u_\epsilon(I_{u_L})$ with $v > 0$ follow along the curve $x = \lambda_1(u)$ for different lengths before leaving and hence approach different right states. Such solutions can be proved to exist using the blow-up construction discussed below. Intuitively, for small $\epsilon > 0$, if a solution is close to the curve $u_L \leq u \leq u_R$, $v = 0$, $x = \lambda_1(u)$, but slightly above it, x increases slowly (because $\dot{x} = \epsilon$) and u increases slowly (because $\dot{u} = v$), so the solution moves along the curve.

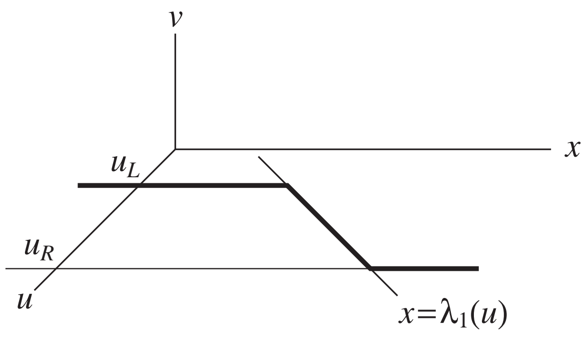

Figure 2. *A singular solution with $n = 1$.*

We begin the paper by constructing self-similar solutions of the Dafermos regularization in section 2. The construction uses the exchange lemma we shall prove. In section 3 we state the exchange lemma to be proved and outline the proof. In section 4 we derive the differential equations on a normally hyperbolic invariant manifold. In section 5 we analyze the reduced flow via the blow-up construction, and in section 6 we use the blow-up construction to track solutions in the normally hyperbolic invariant manifold as they pass the manifold of turning points. In section 7 we use our analysis of the flow on the normally hyperbolic invariant manifold to prove an exchange lemma for dealing with the remaining hyperbolic directions.

2. Construction of Riemann–Dafermos solutions.

2.1. Conservation laws. Consider the system of conservation laws (1.1) and its viscous regularization (1.6). Let $A(u) = Df(u)$. We assume strict hyperbolicity on $\mathbb{R}^n$. We denote the eigenvalues of $A(u)$ by $\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$, and we denote corresponding eigenvectors by $\tilde{r}_i(u)$, $i = 1, \dots, n$.

For notational convenience we let $\lambda_0(u) = -\infty$ and $\lambda_{n+1}(u) = \infty$.

We assume that (1.1) is *genuinely nonlinear*, i.e., $D\lambda_i(u)\tilde{r}_i(u) \neq 0$ for all $i = 1, \dots, n$ and for all $u \in \mathbb{R}^n$. Then we can choose $r_i(u)$ so that

$$D\lambda_i(u)r_i(u) = 1.$$

2.2. Rarefactions. A *rarefaction wave* is a solution of (1.1) of the form $u(x)$, $x = \frac{X}{T} \in [a, b]$, with $a < b$ and $u'(x) \neq 0$ for all $x \in [a, b]$. Then $u(x)$ is a solution of the ODE

$$(A(u) - xI)u_x = 0$$

with $u_x \neq 0$. Notice that each x must be an eigenvalue of $A(u(x))$. In particular, a rarefaction of the *ith family* has $x = \lambda_i(u(x))$. Given u_-, denote the solution of the initial value problem

$$u_x = r_i(u), \quad u(\lambda_i(u_-)) = u_-,$$

by $\psi_i(u_-, x)$. Then a rarefaction of the ith family with left state u_- is just $\psi_i(u_-, x)$, $\lambda_i(u_-) \leq x \leq b$, with $\lambda_i(u_-) < b$.

2.3. Traveling waves. A *traveling wave* with speed s is a solution of (1.6) of the form $u(t)$, $t = X - sT$, $-\infty < t < \infty$. Hence $u(t)$ is a solution of the ODE

$$(A(u) - sI)u_t = u_{tt}. \tag{2.1}$$

We shall always require constant boundary conditions:

$$u(-\infty) = u_-, \quad u(\infty) = u_+, \quad u'(\pm\infty) = 0.$$

Integrating (2.1) from $-\infty$ to t and using the boundary conditions at $-\infty$, we obtain

$$u_t = f(u) - f(u_-) - s(u - u_-). \tag{2.2}$$

The system (2.2) has an equilibrium at u_-, and it has an equilibrium at u_+ provided the *Rankine–Hougoniot condition* is satisfied:

$$f(u_+) - f(u_-) - s(u_+ - u_-) = 0. \tag{2.3}$$

Thus there is a traveling wave solution of (1.6) with left state u_-, speed s, and right state u_+ if and only if (2.3) is satisfied and (2.2) has a heteroclinic solution $u(t)$ from u_- to u_+.

2.4. Shock waves. Let $x = \frac{X}{T}$, let $s \in \mathbb{R}$, and consider the function

$$u(x) = \begin{cases} u_- & \text{if } x < s, \\ u_+ & \text{if } x > s. \end{cases} \tag{2.4}$$

We shall call (2.4) a *shock wave* with speed s, and admit it as a solution of (1.1), if the viscous system (1.6) has a traveling wave solution $u(t)$ with the same left state, speed, and right state. The traveling wave $u(t)$ is a *viscous profile* for the shock wave (2.4), for the viscosity u_{xx}. We associate with each shock wave a fixed viscous profile.

For each $i = 1, \dots, n$, the shock wave (2.4) is a *Lax i-shock* if $\lambda_{i-1}(u_-) < s < \lambda_i(u_-)$ and $\lambda_i(u_+) < s < \lambda_{i+1}(u_+)$. It is *regular* if, for the system (2.2), $W^u(u_-)$ meets $W^s(u_+)$ transversally along the viscous profile $u(t)$. Notice that u_- and u_+ are hyperbolic equilibria of (2.2), $W^u(u_-)$ has dimension $n - i + 1$, and $W^s(u_+)$ has dimension i. Hence a transversal intersection has dimension one.

2.5. Classical Riemann solutions. An *n-wave classical Riemann solution* of (1.1) is a function $u^*(x)$, $x = \frac{X}{T}$, with the following property. Let $s_0^* = -\infty$ and $a_{n+1}^* = \infty$. Then there is a sequence of numbers $a_1^* \le s_1^* < a_2^* \le s_2^* < \cdots < a_n^* \le s_n^*$ and a sequence of points $u_0^*, u_1^*, \dots, u_n^*$ such that the following hold:

(1) For $i = 0, \dots, n$, if $s_i^* < x < a_{i+1}^*$, then $u(x) = u_i^*$.
(2) If $a_i^* < s_i^*$, then $u^*|[a_i^*, s_i^*]$ is a rarefaction of the ith family. Moreover, $u^*(a_i^*) = u_{i-1}^*$ and $u(s_i^*) = u_i^*$.
(3) If $a_i^* = s_i^*$, the triple $(u_{i-1}^*, s_i^*, u_i^*)$ is a Lax i-shock.

Thus $u^*(x)$ has a jump discontinuity whenever $a_i^* = s_i^*$. We will take $u^*(x)$ to be undefined at such points. If $a_i^* = s_i^*$, we denote the corresponding viscous profile by $q_i(t)$. If $u_0^* = u_L$ and $u_n^* = u_R$, then $u^*(x)$ is a solution of the Riemann problem (1.1)–(1.2).

2.6. Structural stability. Given an n-wave classical Riemann solution $u^*(x)$, define functions $G_i : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \dots, n$, as follows:

(1) If $a_i^* < s_i^*$, $G_i(u_-, s, u_+) = u_+ - \psi_i(u_-, s)$.
(2) If $a_i^* = s_i^*$, $G_i(u_-, s, u_+) = f(u_+) - f(u_-) - s(u_+ - u_-)$.

Define $G : \mathbb{R}^{n^2+2n} \to \mathbb{R}^{n^2}$ by

$$G(u_0, s_1, u_1, s_2, u_2, \dots, u_{n-1}, s_n, u_n) = (G_1(u_0, s_1, u_1), G_2(u_1, s_2, u_2), \dots, G_n(u_{n-1}, s_n, u_n)).$$

Let $u^* = (u_0^*, s_1^*, u_1^*, s_2^*, u_2^*, \dots, u_{n-1}^*, s_n^*, u_n^*)$. (We hope this reuse of the symbol u^* will not be confusing.) Then $G(u^*) = 0$. If all shock waves are regular, then nearby solutions of $G = 0$ also define n-wave classical Riemann solutions with the same sequence of rarefaction and shock waves. The Riemann solution $u^*(x)$ is said to be *structurally stable* if all shock waves are regular and the restriction of $DG(u^*)$ to the n^2-dimensional space of vectors with $\bar{u}_0 = \bar{u}_n = 0$ is invertible. In this case, for each (u_0, u_n) near (u_0^*, u_n^*), there is an n-wave classical Riemann solution with left state u_0, right state u_n, and the same sequence of rarefaction and shock waves.

For $i = 0, \dots, n$, let O_i be a small neighborhood of u_i^* in $\mathbb{R}^n$, and for $i = 1, \dots, n$, let I_i be a small neighborhood of s_i^* in $\mathbb{R}$.

For $i = 1, \ldots, n$, define $W_i : O_{i-1} \times I_i \to \mathbb{R}^n$ as follows: $W_i(u_{i-1}, x_i)$ is the solution u_i near u_i^* of the equation $G_i(u_{i-1}, x_i, u_i) = 0$. There is a unique such solution by the implicit function theorem.

For $i = 0, \ldots, n$, we inductively define subsets R_i of O_i as follows:

(1) $R_0 = \{u_0^*\}$.

(2) For $i = 1, \ldots, n$, $u_i \in O_i$ is in R_i provided there exist $u_{i-1} \in R_{i-1}$ and $x_i \in I_i$ such that $W_i(u_{i-1}, x_i) = u_i$.

Proposition 2.1. *Let $u^*(x)$ be an n-wave classical Riemann solution that is structurally stable. Then the following hold:*

(1) *For $i = 0, \ldots, n$, R_i is a manifold of dimension i, and $u_i^* \in R_i$.*

(2) *For $i = 1, \ldots, n$, W_i maps an open subset of $R_{i-1} \times I_i$ diffeomorphically onto R_i.*

Proposition 2.1 is an easy consequence of our assumption on $DG(u^*)$.

Suppose the ith wave of the structurally stable Riemann solution $u^*(x)$ is a shock wave. Then for $(u_{i-1}, x_i) \in R_{i-1} \times I_i$, the traveling wave equation

$$\dot{u} = f(u) - f(u_{i-1}) - x_i(u - u_{i-1})$$

has a connecting orbit $u(t)$ from u_{i-1} to $u_i = W_i(u_{i-1}, x_i)$ near $q_i(t)$; moreover, the $(n-i+1)$-dimensional unstable manifold of u_{i-1} and the i-dimensional stable manifold of u_i meet transversally along this orbit.

2.7. Dafermos regularization. We consider the Dafermos regularization of (1.1) with viscosity u_{XX}, namely, (1.4). We recall that a Riemann–Dafermos solution is a solution of (1.4) of the form $u(x)$, $x = \frac{X}{T}$, with $u(\pm\infty)$ constant and $u'(\pm\infty) = 0$. As shown in the introduction, Riemann–Dafermos solutions correspond to solutions of the autonomous system (1.10)–(1.12) that satisfy analogous boundary conditions.

2.8. Dafermos ODE with $\epsilon = 0$. We consider (1.10)–(1.12) with $\epsilon = 0$:

$$\dot{u} = v, \tag{2.5}$$

$$\dot{v} = (A(u) - xI)v, \tag{2.6}$$

$$\dot{x} = 0. \tag{2.7}$$

We note that the $(n+1)$-dimensional space $v = 0$ consists of equilibria, and the functions x and $f(u) - xu - v$ are first integrals. They have the following significance. Fix a number s. If we restrict (2.5)–(2.6) to the $2n$-dimensional invariant set $x = s$, we obtain the second-order traveling wave equation (2.1), converted to a first-order system by setting $v = u_t$. Now choose u_- and let $w = f(u_-) - su_-$. Then $\{(u, v, x) : x = s \text{ and } w = f(u) - su - v\}$ is invariant and has dimension n. Parameterizing it by u, the system (2.5)–(2.7) reduces to the integrated traveling wave equation (2.2).

In particular, (2.2) has a heteroclinic solution $u(t)$ from u_- to u_+ if and only if the system (2.5)–(2.7) has a heteroclinic solution $(u(t), \dot{u}(t), s)$ from $(u_-, 0, s)$ to $(u_+, 0, s)$.

At an equilibrium $(u, 0, x)$ of (2.5)–(2.7), the matrix (1.14) of the linearization has the eigenvalues $\lambda_i(u) - x$, $i = 1, \ldots, n$, and 0 repeated $n+1$ times. Then ux-space, the set of equilibria for (2.5)–(2.7), decomposes as follows.

- For $i = 0, \ldots, n$, let

$$E_i = \{(u, v, x) : v = 0 \text{ and } \lambda_i(u) < x < \lambda_{i+1}(u)\}.$$

Each E_i is an $(n+1)$-dimensional manifold of equilibria of (2.5)–(2.7). At $(u, 0, x)$ in E_i, the linearization of (2.5)–(2.7) has i negative eigenvalues $\lambda_k(u) - x$, $k = 1, \ldots, i$; $n - i$ positive eigenvalues $\lambda_k(u) - x$, $k = i+1, \ldots, n$; and the semisimple eigenvalue 0 with multiplicity $n + 1$.

- For $i = 1, \ldots, n$, let

$$F_i = \{(u, v, x) : v = 0 \text{ and } x = \lambda_i(u)\}.$$

Each F_i is an n-dimensional manifold of equilibria of (2.5)–(2.7). At $(u, 0, x)$ in E_i, the linearization of (2.5)–(2.7) has $i - 1$ negative eigenvalues, $n - i$ positive eigenvalues, and the semisimple eigenvalue 0 with multiplicity $n + 2$.

2.9. Singular solution. Suppose the Riemann problem (1.1)–(1.2) has the structurally stable n-wave classical Riemann solution $u^*(x)$, with $u_0^* = u_L$ and $u_n^* = u_R$. We define the following curves in uvx-space:

- For $i = 0, \ldots, n$, let

$$J_i = \{(u, v, x) : u = u_i^*, v = 0, s_i^* < x < a_{i+1}^*\}.$$

- For $i = 1, \ldots, n$,
  - if $a_i^* < s_i^*$, let $\Gamma_i = \{(u, v, x) : u = u^*(x),\ v = 0,\ a_i^* \le x \le s_i^*\}$, and
  - if $a_i^* = s_i^*$, let $\Gamma_i = \{(u, v, x) : u = q_i(t), v = \dot{q}_i(t), x = s_i^*\} \cup \{(u_{i-1}^*, 0, s_i^*), (u_i^*, 0, s_i^*)\}$.

Note that for each i, $J_i \subset E_i$, and for each i for which $a_i^* < s_i^*$, $\Gamma_i \subset F_i$.

The singular solution of the boundary value problem (1.10)–(1.13) is then $J_0 \cup \Gamma_1 \cup J_1 \cup \cdots \cup J_{n-1} \cup \Gamma_n \cup J_n$. It corresponds to the Riemann solution, together with the viscous profiles of the shock waves.

2.10. Normally hyperbolic invariant manifolds. Let $\dot{\alpha} = g(\alpha, \gamma)$ be a smooth differential equation with $\alpha \in \mathbb{R}^{k+l+m}$ and γ a vector of parameters. Suppose $\dot{\alpha} = g(\alpha, 0)$ has an m-dimensional manifold of equilibria $\Sigma \subset \mathbb{R}^{k+l+m}$, and at each point of Σ the linearization has k eigenvalues with negative real part and l eigenvalues with positive real part. Then near any open subset of $\Sigma \times \{0\}$ whose closure is a compact subset of $\Sigma \times \{0\}$, there is a smooth change of coordinates $(\xi, \zeta, \theta, \gamma) \to \alpha$, $(\xi, \zeta, \theta) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$, that converts the system into

$$\dot{\xi} = h_1(\xi, \zeta, \theta, \gamma), \quad \dot{\zeta} = h_2(\xi, \zeta, \theta, \gamma), \quad \dot{\theta} = h_3(\xi, \zeta, \theta, \gamma), \tag{2.8}$$

with

$$h_1(0, \zeta, \theta, \gamma) = 0, \quad h_2(\xi, 0, \theta, \gamma) = 0, \quad h_3(0, \zeta, \theta, \gamma) = h_3(\xi, 0, \theta, \gamma) = \hat{h}(\theta, \gamma), \quad \hat{h}(\theta, 0) = 0;$$

moreover, the real parts of eigenvalues of $D_\xi h_1(0, 0, \theta, \gamma)$ are bounded above by a negative number, and the real parts of eigenvalues of $D_\zeta h_2(0, 0, \theta, \gamma)$ are bounded below by a positive number. In the new (Fenichel) coordinates, θ-space is locally invariant for each γ and consists of equilibria for $\gamma = 0$; $\xi\theta$-space and $\zeta\theta$-space are locally invariant for each γ; and, for each γ, the sets $\zeta = 0$, $\theta = \theta_0$ are mapped to one another by the flow on $\xi\theta$-space, as are the sets $\xi = 0$, $\theta = \theta_0$ by the flow on $\zeta\theta$-space. See Figure 3. For each γ, θ-space is called a *normally hyperbolic invariant manifold* (although it is only locally invariant); $\xi\theta$-space is its stable manifold; $\zeta\theta$-space is its unstable manifold; the set $\zeta = 0$, $\theta = \theta_0$ is the *stable fiber* of the point $(0, 0, \theta_0)$; and the set $\xi = 0$, $\theta = \theta_0$ is the *unstable fiber* of the point $(0, 0, \theta_0)$. For $\gamma = 0$ the stable and unstable fibers of points are simply the stable and unstable manifolds of the individual equilibria. The same terms are used for the corresponding sets in α-space.

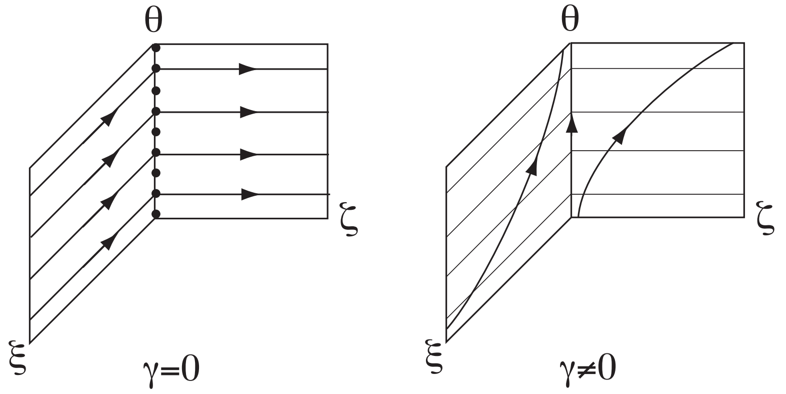

Figure 3. *Fenichel coordinates for a normally hyperbolic invariant manifold.*

The stable manifold of a normally hyperbolic invariant manifold *projects along stable fibers* to the normally hyperbolic invariant manifold itself; in $\xi\zeta\theta$-coordinates, this is just the mapping $(\xi, 0, \theta) \to (0, 0, \theta)$. Similarly, the unstable manifold of a normally hyperbolic invariant manifold projects along unstable fibers to the normally hyperbolic invariant manifold itself.

If g is C^s, $s \geq 1$, there is a C^s change of coordinates $(\xi, \zeta, \theta, \gamma) \to \alpha$ that accomplishes $h_1(0, \zeta, \theta, \gamma) = 0$ and $h_2(\xi, 0, \theta, \gamma) = 0$. If $s \geq 2$, there is a C^{s-1} change of coordinates that also accomplishes the required conditions on h_3 [6]. After this coordinate change, (h_1, h_2, h_3) in (2.8) is C^{s-2}.

Note that for any γ, any invariant subset of θ-space has its own stable and unstable manifolds: the union of the stable and unstable fibers, respectively, of its points. This fact was used in the introduction to define $W^s_\epsilon(I_{u_R})$ and $W^u_\epsilon(I_{u_L})$.

2.11. Riemann–Dafermos solution. Let $\delta > 0$ be small. The following are normally hyperbolic invariant manifolds of equilibria for (1.10)–(1.12) with $\epsilon = 0$: $E_0^\delta = \{(u, v, x) : \|u\| < \frac{1}{\delta},\ v = 0,\ -\infty < x < \lambda_1(u) - \delta\}$; for $i = 1, \dots, n-1$, $E_i^\delta = \{(u, v, x) : \|u\| < \frac{1}{\delta},\ v = 0,\ \lambda_i(u) + \delta < x < \lambda_{i+1}(u) - \delta\}$; and $E_n^\delta = \{(u, v, x) : \|u\| < \frac{1}{\delta},\ v = 0,\ \lambda_n(u) < x < \infty\}$. E_0^δ and E_n^δ extend to $x = -\infty$ and $x = \infty$, respectively, but it is shown in [21, Appendix A] that they can still be regarded as normally hyperbolic invariant manifolds. The sets $E_0^\delta, \dots, E_n^\delta$ remain normally hyperbolic invariant manifolds of (1.10)–(1.12) for $\epsilon \neq 0$. Abusing notation a little, we denote the stable and unstable manifolds of E_i^δ by $W^s_\epsilon(E_i)$ and $W^u_\epsilon(E_i)$.

We continue to consider the Riemann solution $u^*(x)$ of the previous subsection. Let

$N_0 = \{(u,v,x) : u \in R_0,\ v = 0,\ -\infty < x < \lambda_1(u) - \delta\}$. For $i = 1,\dots,n-1$, let $N_i = \{(u,v,x) : u \in R_i,\ v = 0,\ s_i^* + 2\delta < x < \lambda_{i+1}(u) - \delta\}$. Let $N_n = \{(u,v,x) : u \in R_n,\ v = 0,\ s_n^* + 2\delta < x < \infty\}$. Each N_i is contained in E_i^δ.

By Proposition 2.1, each N_i is a manifold of dimension $i+1$. Note that each N_i is locally invariant under (1.10)–(1.12) for any ϵ. By the previous subsection, stable and unstable manifolds of each N_i can be defined. $W_\epsilon^u(N_i)$ has dimension $(i+1)+(n-i) = n+1$.

Proposition 2.2. *For $i = 1,\dots,n$:*

(1) *$W_0^u(u_{i-1}^*, 0, s_i^*)$ meets $W_0^s(E_i)$ transversally along the curve $(u,v,x) = (q_i(t), \dot q_i(t), s_i^*)$.*

(2) *$W_0^u(N_{i-1})$ meets $W_0^s(E_i)$ transversally near the curve $(u,v,x) = (q_i(t), \dot q_i(t), s_i^*)$.*

(3) *Near the curve $(u,v,x) = (q_i(t), \dot q_i(t), s_i^*)$, the projection of $W_0^u(N_{i-1}) \cap W_0^s(E_i)$ to E_i, along stable fibers of $W_0^s(E_i)$, is the i-dimensional manifold $\{(u,v,x) : u \in R_i,\ v = 0,\ x = s_i(u)\}$, where $s_i(u)$ is just the value of x for which there exists $u_{i-1} \in R_{i-1}$ with $W_i(u_{i-1}, x) = u$.*

Proof. (1) follows from the fact that the ith shock wave is regular. Note that $W_0^u(u_{i-1}^*, 0, s_i^*)$ has dimension $n-i+1$ and $W_0^s(E_i)$ has dimension $n+1+i$, so the intersection has dimension $(n-i+1)+(n+1+i)-(2n+1) = 1$: it is the given curve. (2) and (3) are consequences of (1); see also the last paragraph of subsection 2.6. See [19] for details. ■

Theorem 2.4, stated below, is the main result of this paper. The following proposition takes us most of the way there. Our work on rarefaction waves, which comprises the remainder of this paper, is used in its proof.

Recall the sets I_{u_L} and I_{u_R} defined in the introduction. They are subsets of J_0 and J_n, respectively.

For each $i = 0,\dots,n$, let Δ_i be a δ-neighborhood of N_i in $W_0^u(N_i)$, which has dimension $n+1$. Near N_i write uvx-space as the product of Δ_i and an n-dimensional complement Λ_i.

Proposition 2.3. *Let f be C^s with s sufficiently large. For $\delta > 0$ sufficiently small, if $\epsilon_0 > 0$ is sufficiently small, then for each $i = 0,\dots,n$, there is a smooth function $\tilde w_i : \Delta_i \times [0,\epsilon_0) \to \Lambda_i$ such that the following hold:*

(1) *$\tilde w_i = 0$ when $\epsilon = 0$.*

(2) *For $0 < \epsilon < \epsilon_0$, the set of (u,v,x) in the graph of $\tilde w_i(\cdot,\epsilon)$ is an open subset of $W_\epsilon^u(I_{u_L})$.*

See Figure 4. Recall from the introduction that $W_\epsilon^u(I_{u_L})$ is an $(n+1)$-dimensional manifold. Thus the proposition says that for $0 < \epsilon < \epsilon_0$ and for each $i = 0,\dots,n$, an open subset of $W_\epsilon^u(I_{u_L})$ is close to another $(n+1)$-dimensional manifold, namely, Δ_i. Note that $W_\epsilon^u(N_n) = N_n$, so for $i = n$ we are simply saying that an open subset of $W_\epsilon^u(I_{u_L})$ is close to N_n.

Proof. The proof is by induction on i. The statement is clearly true for $i = 0$, because for $0 \le \epsilon < \epsilon_0$, $W_\epsilon^u(N_0) \subset W_\epsilon^u(I_{u_L})$, and a δ-neighborhood of N_0 in $W_\epsilon^u(N_0)$ is close to a δ-neighborhood of N_0 in $W_0^u(N_0)$. Since (1.10)–(1.12) is C^{s-1} when f is C^s, from subsection 2.10, the mapping $\tilde w_0$ can be taken to be C^{s-1}.

Suppose the statement is true for $i = k-1$, with $1 \le k \le n$.

If the kth wave in the Riemann solution is a shock wave, then $W_0^u(N_{k-1})$ meets $W_0^s(E_k)$ transversally by Proposition 2.2, and the statement follows from the Jones–Tin exchange lemma (Theorem 2.3 of [23]). In the Jones–Tin exchange lemma, we can take each M_ϵ, $0 \le \epsilon < \epsilon_0$, to be the graph of $\tilde w_{k-1}(\cdot,\epsilon)$ with x fixed. Assumption (JT3) of the Jones–Tin

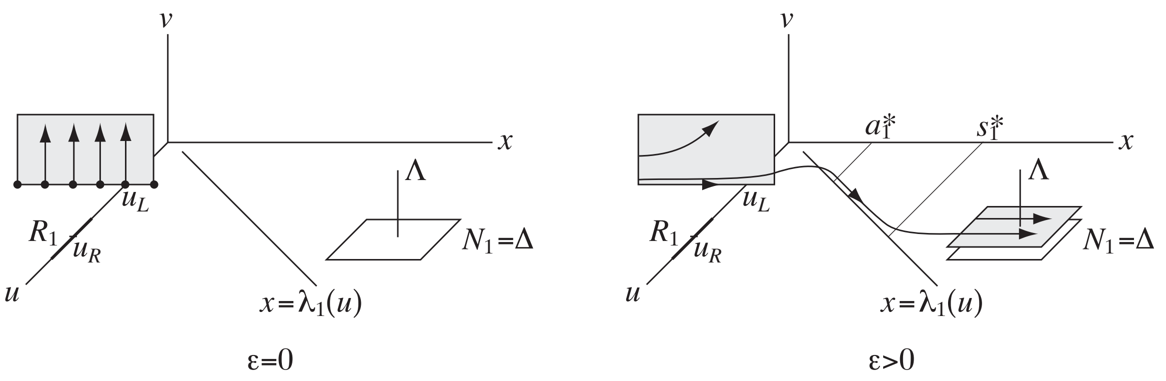

Figure 4. *Graphs of $\tilde{w}_1$ for $\epsilon = 0$ and $\epsilon > 0$, with $n = 1$. Since $n = 1$, $W^u_\epsilon(N_1) = N_1$, which is two-dimensional. A complementary space Λ is one-dimensional. For $\epsilon = 0$, $\tilde{w}_1 = 0$, so the graph of $\tilde{w}_1$ is simply $N_1 = \Delta$ itself. For $\epsilon > 0$, the graph of $\tilde{w}_1$ is an open subset of $W^u_\epsilon(I_{u_L})$, which is grey.*

exchange lemma follows from Proposition 2.2 (1). The Jones–Tin exchange lemma guarantees that $\tilde{w}_k$ is at most three degrees of differentiability weaker than $\tilde{w}_{k-1}$.

If the kth wave in the Riemann solution is a rarefaction wave, the result follows from Theorem 3.1, to be proved in this paper. In that theorem we again take each M_ϵ as in the previous paragraph; U_* is an open subset of E_{k-1}. In assumption (R5) of section 3, M_0 meets the stable fiber of $(u_*, 0, x_*)$ at $(u_*, 0, x_*)$ itself. In fact, since $\tilde{w}_{k-1} = 0$ when $\epsilon = 0$, $M_0 \subset W^u_0(N_{k-1})$. Theorem 3.1 guarantees that $\tilde{w}_k$ is at most 11 degrees of differentiability weaker than $\tilde{w}_{k-1}$.

If the Riemann solution has m shock waves and $n-m$ rarefactions, then all $\tilde{w}_i$ are at least C^1 provided $s \geq 3m + 11(n-m) + 2 = 11n - 8m + 2$. ∎

Theorem 2.4. *Let $u^*(x)$ be a classical Riemann solution of (1.1), with $u(-\infty) = u_L$ and $u(\infty) = u_R$, that has m shock waves and $n - m$ rarefactions and is structurally stable in the sense of subsection 2.6. Assume f is C^s with $s \geq 11n - 8m + 2$. Then for small $\epsilon > 0$, there is, for the same u_L and u_R, a Riemann–Dafermos solution near the singular solution defined in subsection 2.9.*

Proof. By Proposition 2.3 and its proof, for small $\epsilon > 0$, an open subset of $W^u_\epsilon(I_{u_L})$ is C^1-close to N_n, which includes $\{(u, v, x) : u = u^*_n,\ v = 0,\ s^*_n + 2\delta < x < \frac{1}{\delta}\}$. Therefore, $W^u_\epsilon(I_{u_L})$ meets $W^s_\epsilon(I_{u_R})$ transversally. The intersection corresponds to the Riemann–Dafermos solution. ∎

2.12. Extensions. With the aid of [19] one can show that Theorem 2.4 holds, with a different formula for s, for any structurally stable Riemann solution consisting entirely of constant states, classical rarefaction waves, and shock waves (including undercompressive shock waves) with hyperbolic end states.

The theorem also presumably holds for structurally stable Riemann solutions that include composite waves, but we have not gone through this in detail. One scalar case is discussed in [25].

We also have not checked whether the viscosity u_{xx} that is used throughout this paper can be replaced by the more general viscosity $(B(u)u_x)_x$, as is the case for structurally stable

Riemann solutions consisting entirely of constant states and shock waves [19].

3. Exchange lemma. To discuss the passage of a manifold of solutions of (1.10)–(1.12) near a manifold of turning points, we shall slightly generalize the situation previously described and pay closer attention to the degree of differentiability.

We consider the system (1.10)–(1.12) with $(u, v, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $A(u)$ an $n \times n$ matrix that is a C^{r+11} function of u, $r \geq 1$. We do not require that $A(u) = Df(u)$ for some function f.

Let $n = k + l + 1$. Let U be an open subset of u-space with the following properties:

(R1) For all $u \in U$, $A(u)$ has a simple real eigenvalue $\lambda(u)$.

(R2) There are numbers $\tilde{\lambda} < 0 < \tilde{\mu}$ such that for all $u \in U$, $A(u)$ has k eigenvalues with real part less than $\lambda(u) + \tilde{\lambda}$ and l eigenvalues with real part greater than $\lambda(u) + \tilde{\mu}$.

We shall consider (1.10)–(1.12) only on $\{(u, v, x) : u \in U\}$.

Let $E = \{(u, v, x) : u \in U \text{ and } v = 0\}$, which is invariant for each ϵ. For $\epsilon = 0$, E is an $(n + 1)$-dimensional manifold of equilibria. (R1)–(R2) imply that E fails to be normally hyperbolic along the n-dimensional surface $\{(u, v, x) : u \in U,\ v = 0, \text{ and } x = \lambda(u)\}$. More precisely, as one crosses this surface along a line with u constant and x increasing, an eigenvalue $\lambda(u) - x$ changes from positive to negative (gain of stability). On the surface, there are k eigenvalues with real part in $(-\infty, \tilde{\lambda})$ and l eigenvalues with real part in $(\tilde{\mu}, \infty)$.

Let $\tilde{r}(u)$ be an eigenvector of $A(u)$ for the eigenvalue $\lambda(u)$. Assume the following:

(R3) For all $u \in U$, $D\lambda(u)\tilde{r}(u) \neq 0$.

Then for each $u \in U$ we can choose an eigenvector $r(u)$ for the eigenvalue $\lambda(u)$ such that

(R3′) $D\lambda(u)r(u) = 1$.

Let $\phi(t, u)$ be the flow of $\dot{u} = r(u)$. Since $A(u)$ is C^{r+11}, so are $\lambda(u)$, $r(u)$, and $\phi(t, u)$.

Let $u_* \in U$. Choose $t^* > 0$ such that $\phi(t, u_*) \in U$ for $0 \leq t \leq t^*$. Let $u^* = \phi(t^*, u_*)$. By (R3′), $\lambda(u^*) = \lambda(u_*) + t^*$.

Choose a number $\beta_0 > 0$ such that

$$\tilde{\lambda} + \tilde{\mu} + r\beta_0 < 0 < \tilde{\mu} - \max(7, 2r + 2)\beta_0. \tag{3.1}$$

(We may have to first adjust the numbers $\tilde{\lambda}$ and $\tilde{\mu}$ used in (R2) to make this possible.)

Choose numbers x_* and x^* such that $\lambda(u_*) - \beta_0 < x_* < \lambda(u_*)$ and $\lambda(u^*) < x^* < \lambda(u^*) + \beta_0$. See Figure 5.

For a small $\delta > 0$, let

$$U_* = \{(u, v, x) : |u - u_*| < \delta, v = 0, |x - x_*| < \delta\},$$
$$U^* = \{(u, v, x) : |u - u^*| < \delta, v = 0, |x - x^*| < \delta\}.$$

For the system (1.10)–(1.12) with $\epsilon = 0$, U_* and U^* are normally hyperbolic manifolds of equilibria of dimension $n + 1$. For U_*, the stable and unstable manifolds of each point have dimensions k and $l + 1$, respectively; for U^*, the stable and unstable manifolds of each point have dimensions $k + 1$ and l, respectively. In fact, for the system (1.10)–(1.12) with any fixed ϵ, U_* and U^* are normally hyperbolic invariant manifolds. The stable and unstable fibers of points have the dimensions just given.

For each $u_0 \in U_*$ let $I_{u_0} = \{(u, 0, x) \in U_* : u = u_0\}$.

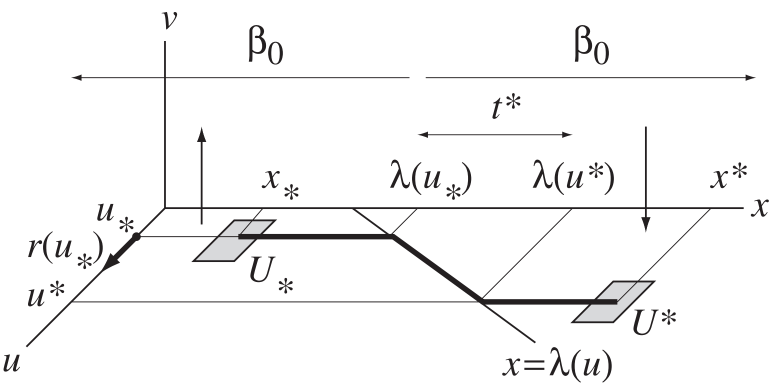

Figure 5. *Notation of this section. For small $\epsilon > 0$, there is a solution near the thick line from $(u, v, x) = (u_*, 0, x_*)$ to $(u, v, x) = (u^*, 0, x^*)$. In the case $p = 1$, Q_0 is the point $(u, v, x) = (u_*, 0, x_*)$; R_0 is the point u_* in u-space; and R_0^* is an interval around u^* in u-space. If in addition $n = 1$, Q_0^* and U^* coincide.*

For each $\epsilon \geq 0$, let M_ϵ be a C^{r+11} submanifold of uvx-space of dimension $l+p$, $1 \leq p \leq n$. Assume the following:

(R4) $M = \{(u, v, x, \epsilon) : (u, v, x) \in M_\epsilon\}$ is itself a C^{r+11} manifold.

(R5) M_0 is transverse to $W_0^s(U_*)$ at a point in the stable fiber of $(u_*, 0, x_*)$.

(R6) The tangent space to M_0 at this point contains no nonzero vectors that are tangent to the stable manifold of I_{u_*}.

Each M_ϵ meets $W_\epsilon^s(U_*)$ transversally in a manifold S_ϵ of dimension $p-1$. S_ϵ projects along the stable fibers of points to a submanifold Q_ϵ of ux-space of dimension $p-1$. The coordinate system in which the projection is done is C^{r+10} (see subsection 2.10), so the family of manifolds Q_ϵ is C^{r+10}. At each point of Q_ϵ, the vector $(\bar{u}, \bar{x}) = (0, 1)$ is not tangent to Q_ϵ. Thus each Q_ϵ projects to a C^{r+10} submanifold R_ϵ of u-space of dimension $p-1$. We assume the following:

(R7) At u_*, $r(u_*)$ is not tangent to R_0.

Under the forward flow of (1.10)–(1.12), each M_ϵ becomes a manifold M_ϵ^* of dimension $l+p+1$.

For a small $\delta > 0$, let

$$R_0^* = \cup_{|t-t^*|<\delta}\phi(t, R_0), \quad Q_0^* = \{(u, v, x) : u \in R_0^*, v = 0, |x - x^*| < \delta\}.$$

R_0^* and Q_0^* have dimensions p and $p+1$, respectively.

Near the point $(u^*, 0, x^*)$ write uvx-space as the product of $W_0^u(Q_0^*)$, which has dimension $l+p+1$, and a complement Λ.

The following is our main result about rarefactions in the Dafermos regularization.

Theorem 3.1. *Assume* (R1)–(R7). *Let Δ be a small open neighborhood of $(u^*, 0, x^*)$ in $W_0^u(Q_0^*)$. Then for $\epsilon_0 > 0$ sufficiently small there is a C^r function $\tilde{w} : \Delta \times [0, \epsilon_0) \to \Lambda$ such that the following hold:*

(1) *$\tilde{w} = 0$ when $\epsilon = 0$.*

(2) *For $0 < \epsilon < \epsilon_0$, the set of (u, v, x) in the graph of $\tilde{w}(\cdot, \epsilon)$ is an open subset of M_ϵ^*.*

Note that Δ and M^*_ϵ are both manifolds of dimension $l+p+1$.

We shall use the general exchange lemma from [23] to prove Theorem 3.1. In outline, the proof goes as follows.

For each ϵ the portion of $(n+1)$-dimensional ux-space with $u \in U$ and x near $\lambda(u)$ lies in a normally hyperbolic invariant manifold K_ϵ of dimension $n+2$. $M_\epsilon \cap W^s(K_\epsilon)$ projects along stable fibers to a p-dimensional submanifold P_ϵ of K_ϵ. We must trace the evolution of the sets P_ϵ, which under the flow of (1.10)–(1.12) become submanifolds P^*_ϵ of K_ϵ of dimension $p+1$. Let $K = \{(u,v,x,\epsilon) : (u,v,x) \in K_\epsilon)$. In order to study the P^*_ϵ, we blow up the surface $v=0$, $x=\lambda(u)$, $\epsilon=0$ within the manifold K. Once we know where the P^*_ϵ lie for (u,v,x) near $(u^*,0,x^*)$, we can verify the hypotheses of the general exchange lemma.

In section 4 we define convenient coordinates for doing the calculations. We do the blow-up in section 5, track the manifolds P^*_ϵ in section 6, and verify the hypotheses of the general exchange lemma in section 7. This requires replacing the manifolds P_ϵ by different cross-sections of P^*_ϵ.

The differentiability loss in Theorem 3.1 is due to several coordinate changes and blow-ups, the use of the Jones–Tin exchange lemma to track the manifolds P^*_ϵ, and the use of the general exchange lemma at the end of the proof.

4. New coordinates.

Let $\chi(w_2,\dots,w_n,\epsilon)$ be a C^{r+10} function that parameterizes an ϵ-dependent cross-section to the flow of $\dot u = r(u)$ near u_*, such that $\chi(0,\dots,0) = u_*$ and $\chi(w_2,\dots,w_p,0,\dots,0,\epsilon)$ is a parameterization of Q_ϵ. Let

$$u(w,\epsilon) = u(w_1,\dots,w_n,\epsilon) = \phi(w_1,\chi(w_2,\dots,w_n,\epsilon)), \quad v = D_w u(w,\epsilon)z, \quad x = \lambda(u(w,\epsilon)) + \sigma.$$

Writing (1.10)–(1.12) in the new variables (w,z,σ), we obtain the system

$$\dot w = z, \tag{4.1}$$

$$\dot z = (B(w,\epsilon) - \sigma I)z + C(w,\epsilon)(z,z), \tag{4.2}$$

$$\dot\sigma = \epsilon - E(w,\epsilon)z, \tag{4.3}$$

with

$$B(w,\epsilon) = (D_w u(w,\epsilon))^{-1}(A(u(w,\epsilon)) - \lambda(u(w,\epsilon))I)D_w u(w,\epsilon),$$
$$C(w,\epsilon) = (D_w u(w,\epsilon))^{-1} D^2_w u(w,\epsilon),$$
$$E(w,\epsilon) = D_w(\lambda \circ u)(w,\epsilon).$$

The functions $B(w,\epsilon)$ and $E(w,\epsilon)$ are C^{r+9}. Since $\dot z = (D_w u(w,\epsilon))^{-1}\dot v$ is also C^{r+9}, the function $C(w,\epsilon)(z,z)$ is the difference of C^{r+9} functions and is therefore C^{r+9} as well. We choose an open set W in w-space such that for $w \in W$ and small ϵ, $u(w,\epsilon) \in U$. Recalling the choice of t^* in section 3, we see that we may assume that W contains $\{w : 0 \le w_1 \le t^*$ and $w_2 = \dots = w_n = 0\}$. We shall consider (4.1)–(4.3) on $\{(w,z,\sigma,\epsilon) : w \in W,\ |\sigma| < \beta_0$, and ϵ small$\}$. Let $\sigma_* = \lambda(u_*) - x_*$ and $\sigma^* = x^* - \lambda(u^*)$. We have

$$-\beta_0 < -\sigma_* < 0 < \sigma^* < \beta_0.$$

Let $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$. Notice that for $w \in W$ and ϵ small, the following hold:

(R1″) $B(w,\epsilon)$ has a simple real eigenvalue 0 with eigenvector e_1.
(R2″) $B(w,\epsilon)$ has k eigenvalues with real part less than $\tilde{\lambda} < 0$ and l eigenvalues with real part greater than $\tilde{\mu} > 0$.
(R3″) $E(w,\epsilon)e_1 = 1$.

For the system (4.1)–(4.3) with $\epsilon = 0$, $w\sigma$-space consists of equilibria. The linearization of (4.1)–(4.3) at one of these equilibria has the matrix

$$\begin{pmatrix} 0 & I & 0 \\ 0 & B(w,0) - \sigma I & 0 \\ 0 & -E(w,0) & 0 \end{pmatrix}. \tag{4.4}$$

For $w \in W$ and $\sigma = 0$, this matrix has the following:

- An eigenvalue 0 with algebraic multiplicity $n+2$. The generalized eigenspace is $wz_1\sigma$-space.
- k eigenvalues with real part less than $\tilde{\lambda} < 0$ and l eigenvalues with real part greater than $\tilde{\mu} > 0$.

For $w \in W$ and $\sigma \neq 0$, one of the zero eigenvalues becomes $-\sigma$. For $w \in W$ and $|\sigma| < \beta_0$, the matrix has the following:

- $n+2$ eigenvalues with real part between $-\beta_0$ and β_0, at least $n+1$ of which are 0, having total algebraic multiplicity $n+2$. The sum of their generalized eigenspaces is $wz_1\sigma$-space.
- k eigenvalues with real part less than $\tilde{\lambda} + \beta_0 < 0$ and l eigenvalues with real part greater than $\tilde{\mu} - \beta_0 > 0$.

The system (4.1)–(4.3) has, for each small ϵ, a normally hyperbolic invariant manifold K_ϵ of dimension $n+2$ that contains the $(n+1)$-dimensional set $\{(w,z,\sigma) : w \in W,\ z = 0, \text{ and } |\sigma| < \beta_0\}$, which is locally invariant for every ϵ. Let $K = \{(w,z,\sigma,\epsilon) : (w,z,\sigma) \in K_\epsilon\}$.

Lemma 4.1. *K is a C^{r+10} normally hyperbolic submanifold of $wz\sigma\epsilon$-space. It has stable fibers of dimension k and unstable fibers of dimension l. Both are C^{r+10} and vary in a C^{r+10} fashion with the base point.*

Proof. K is also a normally hyperbolic invariant manifold for the C^{r+11} system (1.10)–(1.12). By [6] it is C^{r+11} in the $uvx\epsilon$-variables and has stable and unstable fibers that are C^{r+11} and vary in a C^{r+10} fashion with the base point. Applying the C^{r+10} coordinate change to the $wz\sigma\epsilon$-variables, we get the result. ∎

Let $\tilde{z} = (z_2, \dots, z_n)$. K_ϵ has the form $\tilde{z} = g(w, z_1, \sigma, \epsilon)$, with g C^{r+10} by Lemma 4.1. We must have $g(w, 0, \sigma, \epsilon) = 0$, so

$$\tilde{z} = z_1 h(w, z_1, \sigma, \epsilon) \tag{4.5}$$

with h C^{r+9}. K_0 must be tangent at each point of $w\sigma$-space to $wz_1\sigma$-space. Therefore, $h(w, 0, \sigma, 0) = 0$, so

$$h(w, z_1, \sigma, \epsilon) = z_1 h_1(w, z_1, \sigma, \epsilon) + \epsilon h_2(w, z_1, \sigma, \epsilon) \tag{4.6}$$

with h_1 and h_2 C^{r+8}.

On K the system (4.1)–(4.3) reduces to the C^{r+9} system:

$$\dot{w} = z_1(1,h), \tag{4.7}$$

$$\dot{z}_1 = B_1(w,\epsilon)z_1(0,h) - \sigma z_1 + C_1(w,\epsilon)z_1^2((1,h),(1,h)), \tag{4.8}$$

$$\dot{\sigma} = \epsilon - z_1(1 + E(w,\epsilon)(0,h)). \tag{4.9}$$

We append the equation

$$\dot{\epsilon} = 0. \tag{4.10}$$

In (4.8) and (4.9) we have used

$$B_1(w,\epsilon)(1,h) = B_1(w,\epsilon)(1,0) + B_1(w,\epsilon)(0,h) = 0 + B_1(w,\epsilon)(0,h) = B_1(w,\epsilon)(0,h), \tag{4.11}$$

$$E(w,\epsilon)(1,h) = E(w,\epsilon)(1,0) + E(w,\epsilon)(0,h) = 1 + E(w,\epsilon)(0,h). \tag{4.12}$$

5. Blow-up.

As in [25], in $wz_1\sigma\epsilon$-space we shall blow up w-space, which consists of equilibria that are not normally hyperbolic within $wz_1\sigma$-space for (4.7)–(4.9) with $\epsilon = 0$, to the product of w-space with a 2-sphere. The 2-sphere is a blow-up of the origin in $z_1\sigma\epsilon$-space.

The blow-up transformation is a map from $\mathbb{R}^n \times S^2 \times [0,\infty)$ to $wz_1\sigma\epsilon$-space defined as follows. Let $(w,(\bar{z_1},\bar{\sigma},\bar{\epsilon}),\bar{r})$ be a point of $\mathbb{R}^n \times S^2 \times [0,\infty)$; we have $\bar{z_1}^2 + \bar{\sigma}^2 + \bar{\epsilon}^2 = 1$. Then the blow-up transformation is

$$w = w, \tag{5.1}$$

$$z_1 = \bar{r}^2\bar{z_1}, \tag{5.2}$$

$$\sigma = \bar{r}\bar{\sigma}, \tag{5.3}$$

$$\epsilon = \bar{r}^2\bar{\epsilon}. \tag{5.4}$$

We refer to $\mathbb{R}^n \times S^2 \times [0,\infty)$ as *blow-up space*, and we call $\mathbb{R}^n \times S^2 \times \{0\}$ the *blow-up cylinder*. Under the transformation (5.1)–(5.4), the system (4.7)–(4.10) becomes one for which the blow-up cylinder $\bar{r} = 0$ consists entirely of equilibria. The system we shall study is this one divided by $\bar{r}$. Division by $\bar{r}$ desingularizes the system on the blow-up cylinder but leaves it invariant.

Note that from (4.6),

$$h(w,z_1,\sigma,\epsilon) = \bar{r}^2\tilde{h}(w,\bar{z}_1,\bar{\sigma},\bar{\epsilon},\bar{r}), \tag{5.5}$$

with $\tilde{h}$ C^{r+8}.

We shall need three charts.

5.1. Chart for $\bar{\sigma} < 0$.

This chart uses the coordinates w, $z_a = \frac{\bar{z}_1}{\bar{\sigma}^2}$, $r_a = -\bar{r}\bar{\sigma}$, and $\epsilon_a = \frac{\bar{\epsilon}}{\bar{\sigma}^2}$ on the set of points in $\mathbb{R}^n \times S^2 \times [0,\infty)$ with $\bar{\sigma} < 0$. Thus we have

$$w = w, \tag{5.6}$$

$$z_1 = r_a^2 z_a, \tag{5.7}$$

$$\sigma = -r_a, \tag{5.8}$$

$$\epsilon = r_a^2\epsilon_a, \tag{5.9}$$

with $r_a > 0$. After division by r_a (equivalent to division by $\bar{r}$ up to multiplication by a positive function), the system (4.7)–(4.10) becomes the C^{r+8} system

$$\dot{w} = r_a z_a (1, r_a^2 \tilde{h}), \tag{5.10}$$

$$\dot{z_a} = z_a(1 + r_a B_1(w, r_a^2\epsilon_a)(0,\tilde{h}) + r_a z_a C_1(w, r_a^2\epsilon_a)(1, r_a^2\tilde{h})(1, r_a^2\tilde{h}) + 2(\epsilon_a - z_a - r_a^2 z_a E(w, r_a^2\epsilon_a)(0,\tilde{h}))), \tag{5.11}$$

$$\dot{r_a} = r_a(z_a - \epsilon_a + r_a^2 z_a E(w, r_a^2\epsilon_a)(0,\tilde{h})), \tag{5.12}$$

$$\dot{\epsilon_a} = 2\epsilon_a(\epsilon_a - z_a - r_a^2 z_a E(w, r_a^2\epsilon_a)(0,\tilde{h})). \tag{5.13}$$

We consider the system (5.10)–(5.13) with $r_a \geq 0$. We have the following structures:

(1) Codimension-one invariant sets: (1) $z_a = 0$, (2) $r_a = 0$, (3) $\epsilon_a = 0$, (4) $r_a^2\epsilon_a = k$.

(2) Invariant foliations:
  (a) Of $z_a = 0$, each plane $w = w^0$ is invariant.
  (b) Of $r_a = 0$, each plane $w = w^0$ is invariant.

(3) Equilibria: (1) $z_a = \epsilon_a = 0$; (2) $z_a = \frac{1}{2}$, $r_a = \epsilon_a = 0$.

The flow in one of the invariant planes $r_a = 0$, $w = w^0$ is pictured in Figure 6. In this figure, the lines $z_a = 0$ and $\epsilon_a = 0$ are invariant. There are a hyperbolic attractor at $(z_a, \epsilon) = (\frac{1}{2}, 0)$ and a nonhyperbolic equilibrium at the origin. The latter's unstable manifold is the line $\epsilon_a = 0$, and one center manifold is the line $z_a = 0$. The origin is quadratically repelling on the portion of this line with $\epsilon_a > 0$.

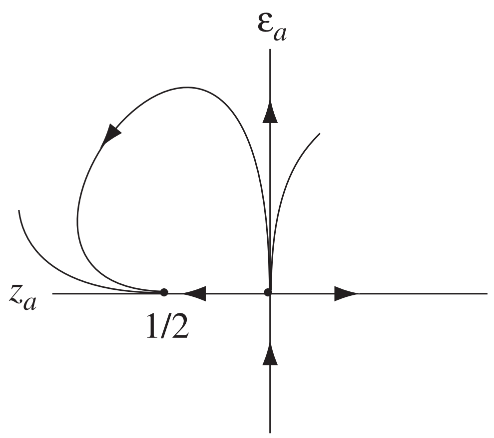

Figure 6. *Flow of (5.10)–(5.13) in the invariant plane $r_a = 0$, $w = w^0$.*

5.2. Chart for $\bar{\epsilon} > 0$.

This chart uses the coordinates w, $z_b = \frac{\bar{z}_1}{\bar{\epsilon}}$, $\sigma_b = \frac{\bar{\sigma}}{\sqrt{\bar{\epsilon}}}$, and $r_b = \bar{r}\sqrt{\bar{\epsilon}}$ on the set of points in $\mathbb{R}^n \times S^2 \times [0,\infty)$ with $\bar{\epsilon} > 0$. Thus we have

$$w = w, \tag{5.14}$$

$$z_1 = r_b^2 z_b, \tag{5.15}$$

$$\sigma = r_b \sigma_b, \tag{5.16}$$

$$\epsilon = r_b^2, \tag{5.17}$$

with $r_b > 0$. After division by r_b (equivalent to division by $\bar{r}$ up to multiplication by a positive function), the system (4.7)–(4.10) becomes the C^{r+8} system

$$\dot{w} = r_b z_b (1, r_b^2 \tilde{h}), \tag{5.18}$$

$$\dot{z}_b = r_b z_b B_1(w, r_b^2)(0, \tilde{h}) - \sigma_b z_b + r_b z_b^2 C_1(w, r_b^2)(1, r_b^2 \tilde{h})(1, r_b^2 \tilde{h}), \tag{5.19}$$

$$\dot{\sigma}_b = 1 - z_b - r_b^2 z_b E(w, r_b^2)(0, \tilde{h}), \tag{5.20}$$

$$\dot{r}_b = 0. \tag{5.21}$$

We consider the system (5.18)–(5.21) with $r_b \geq 0$. We have the following structures:

(1) Codimension-one invariant sets: (1) $z_b = 0$, (2) $r_b = r_b^0$.
(2) Invariant foliations:
  (a) Of $z_b = 0$, each plane $w = w^0$ is invariant.
  (b) Of $r_b = 0$, each plane $w = w^0$ is invariant.
(3) Equilibria: $z_b = 1$, $\sigma_b = r_b = 0$.

The flow in one of the invariant planes $r_b = 0$, $w = w^0$ is pictured in Figure 7. In this figure, the line $z_b = 0$ is invariant, and there is a hyperbolic saddle at $(z_b, \sigma_b) = (1, 0)$.

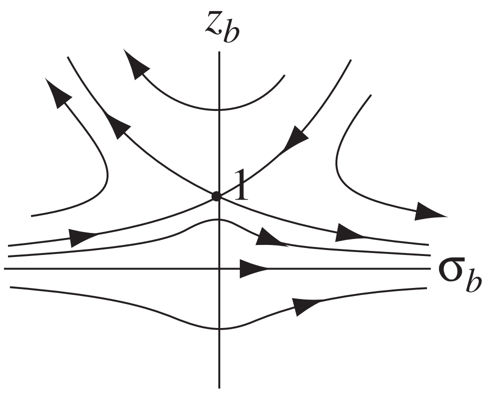

Figure 7. *Chart for $\bar{\epsilon} > 0$. Flow of (5.18)–(5.21) in the invariant plane $r_b = 0$, $w = w^0$.*

5.3. Chart for $\bar{\sigma} > 0$.

This chart uses the coordinates w, $z_c = \frac{\bar{z}_1}{\bar{\sigma}^2}$, $r_c = \bar{r}\bar{\sigma}$, and $\epsilon_c = \frac{\bar{\epsilon}}{\bar{\sigma}^2}$ on the set of points in $\mathbb{R}^n \times S^2 \times [0, \infty)$ with $\bar{\sigma} > 0$. Thus we have

$$w = w, \tag{5.22}$$

$$z_1 = r_c^2 z_c, \tag{5.23}$$

$$\sigma = r_c, \tag{5.24}$$

$$\epsilon = r_c^2 \epsilon_c, \tag{5.25}$$

with $r_c > 0$. After division by r_c (equivalent to division by $\bar{r}$ up to multiplication by a positive function), the system (4.7)–(4.10) becomes the C^{r+8} system

$$\dot{w} = r_c z_c(1, r_c^2 \tilde{h}), \tag{5.26}$$

$$\begin{aligned} \dot{z_c} = z_c(-1 + r_c B_1(w, r_c^2\epsilon_c)(0, \tilde{h}) + r_c z_c C_1(w, r_c^2\epsilon_c)(1, r_c^2\tilde{h})(1, r_c^2\tilde{h}) \\ - 2(\epsilon_c - z_c - r_c^2 z_c E(w, r_c^2\epsilon_c)(0, \tilde{h}))), \end{aligned} \tag{5.27}$$

$$\dot{r_c} = r_c(\epsilon_c - z_c - r_c^2 z_c E(w, r_c^2\epsilon_c)(0, \tilde{h})), \tag{5.28}$$

$$\dot{\epsilon_c} = 2\epsilon_c(-\epsilon_c + z_c + r_c^2 z_c E(w, r_c^2\epsilon_c)(0, \tilde{h})). \tag{5.29}$$

We consider the system (5.26)–(5.29) with $r_c \geq 0$. The description of the flow is similar to that for the chart for $\bar{\sigma} < 0$. Again, within the space $r_c = 0$, each plane $w = w^0$ is invariant. For a fixed w^0, the flow in this plane is pictured in Figure 8. This time there are a hyperbolic repeller at $(z_c, \epsilon) = (\frac{1}{2}, 0)$ and a nonhyperbolic equilibrium at the origin. The latter's stable manifold is the line $\epsilon_c = 0$, and one center manifold is the line $z_c = 0$. The origin is quadratically attracting on the portion of this line with $\epsilon_c > 0$.

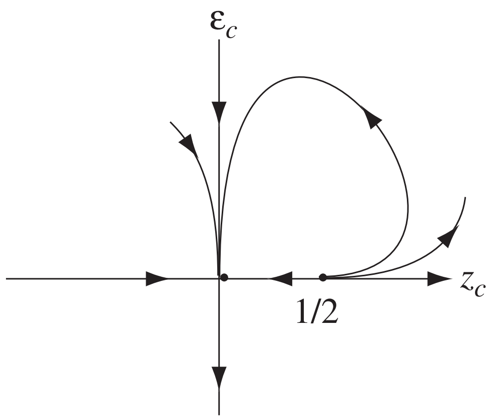

Figure 8. *Flow of (5.26)–(5.29) in the invariant plane $r_c = 0$, $w = w^0$.*

5.4. Summary.

Figure 9 shows the flow in the portion of blow-up space with $\bar{\epsilon} \geq 0$, as reconstructed from these coordinate charts and the corresponding ones for $\bar{z}_1 < 0$ and $\bar{z}_1 > 0$. (The circled numbers in the figure will be discussed in the next section.) A value $w = w^0$ is fixed; in the figure we look straight down the ϵ-axis. We see the top of the sphere $w = w^0$, $\bar{r} = 0$, and, outside it, the plane $w = w^0$, $\epsilon = 0$, in which the origin has been blown up to a circle. In this plane there are two lines of equilibria along the σ-axis and two equilibria elsewhere on the circle. The figure shows as dashed curves stable and unstable manifolds of these equilibria that do not actually lie in $w = w^0$. We also see one other equilibrium on the sphere $w = w^0$, $\bar{r} = 0$; it was identified in subsection 5.2.

To see the flow in all of the blow-up space with $\bar{\epsilon} \geq 0$, one must cross Figure 9 with w-space. Thus the lines of equilibria become $(n+1)$-dimensional planes of equilibria, and the equilibrium identified in subsection 5.2 becomes an n-dimensional plane of equilibria in the blow-up cylinder. This plane is denoted L_0 in the following section.

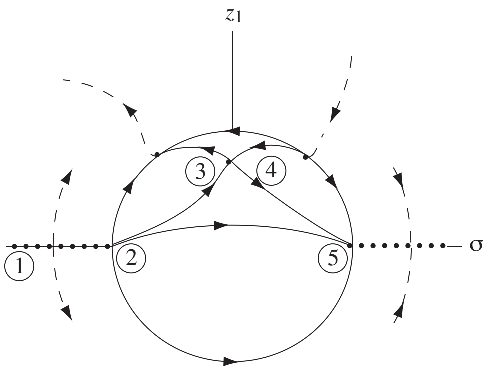

Figure 9. *The blown-up flow. Numbers correspond to subsections of section 6.*

6. Tracking.

We consider the p-dimensional submanifolds P_ϵ of K_ϵ defined at the end of section 3. In $wz_1\sigma$-coordinates on K_ϵ, P_ϵ is given by equations of the form

$$w_i = \hat{w}_i(w_2,\dots,w_p,z_1,\epsilon), \quad i = 1, p+1, \dots, n,$$
$$\sigma = -\hat{\sigma}(w_2,\dots,w_p,z_1,\epsilon),$$

with $\hat{w}^i$ and $\hat{\sigma}$ C^{r+10} by Lemma 4.1; $\hat{w}^i = 0$ if $z_1 = 0$, and $\hat{\sigma}(0,0,0) = \sigma_* > 0$. The sets Q_ϵ defined in section 3 are given by the same equations with $z_1 = 0$.

6.1. P_ϵ^* approaches the unstable manifold of $w_2 \dots w_p$-space.

Let $\delta > 0$ be small. Within $wz_1\sigma$-space, $\{(w,z_1,\sigma) : w \in W,\ z_1 = 0,\ -\beta_0 < \sigma < -\frac{1}{2}\delta\}$ is, for each ϵ, a normally hyperbolic (repelling) invariant manifold. Therefore, as long as $\sigma < -\frac{1}{2}\delta$, we can follow the evolution of the P_ϵ using the usual exchange lemma. After a C^{r+8} coordinate change, the C^{r+9} system (4.7)–(4.9) becomes a C^{r+7} system in which stable fibers are lines. We obtain the following result.

Proposition 6.1. *Let*

$$A = \left\{(w_2,\dots,w_p,z_1,\sigma) : \max(|w_2|,\dots,|w_p|,|z_1|) < \delta \text{ and } -3\delta < \sigma < -\frac{1}{2}\delta\right\}.$$

For $\epsilon_0 > 0$ sufficiently small, there is a C^{r+6} function $(\tilde{w}_1, \tilde{w}_{p+1},\dots,\tilde{w}_n) : A\times[0,\epsilon_0) \to \mathbb{R}^{n-p+1}$ such that the following hold:

(1) *If $z_1 = 0$, then $(\tilde{w}_1,\tilde{w}_{p+1},\dots,\tilde{w}_n) = 0$.*
(2) *For $\epsilon = 0$, the unstable manifold of the subset of $w\sigma$-space given by* $\max(|w_2|,\dots,|w_p|) < \delta$, *$w_1 = w_{p+1} = \cdots = w_n = 0$, and $-3\delta < \sigma < -\frac{1}{2}\delta$ has the equations $(w_1, w_{p+1},\dots, w_n) = (\tilde{w}_1,\tilde{w}_{p+1},\dots,\tilde{w}_n)(w_2,\dots,w_p,z_1,\sigma,0)$.*
(3) *For $0 < \epsilon < \epsilon_0$, $\{(w,z_1,\sigma) : (w_2,\dots,w_p,z_1,\sigma) \in A$ and $(w_1,w_{p+1},\dots,w_n) = (\tilde{w}_1,\tilde{w}_{p+1},\dots,\tilde{w}_n)(w_2,\dots,w_p,z_1,\sigma,\epsilon)\}$ is an open subset of P_ϵ^*.*

Let P^* denote $\{(w,z^1,\sigma,\epsilon) : \epsilon > 0$ and $(w,z^1,\sigma) \in P_\epsilon^*\}$, together with the limit points of this set that have $\epsilon = 0$. Proposition 6.1 describes P^* for $-3\delta < \sigma < -\frac{1}{2}\delta$. We shall use our blow-up to track P^* as σ increases further; we shall denote the preimage of P^* under the blow-up transformation, as well as the corresponding set in a local coordinate system, by P^* as well.

Figure 9 gives an outline of this section. The numbers in the figure correspond to subsections of this section. In subsection 6.1, the present one, we have followed P^* along the plane of equilibria in $\sigma < 0$. In subsection 6.2 P^* "turns the corner" and passes along the blow-up cylinder. In subsection 6.3 P^* approaches the manifold of equilibria L_0 along its stable manifold and then moves along it in the w_1 direction. In subsection 6.4 P^* leaves the manifold of equilibria L_0 at $w_1 = t^*$ along its unstable manifold. In subsection 6.5 P^* again "turns the corner" and passes along the plane of equilibria in $\sigma > 0$.

6.2. P_ϵ^* arrives at the blow-up cylinder. In $wz_ar_a\epsilon_a$-coordinates, the equations for P^* become

$$w_i = \breve{w}_i(w_2, \ldots, w_p, r_a^2 z_a, -r_a, r_a^2\epsilon_a), \quad i = 1, p+1, \ldots, n,$$
$$\max(|w_2|, \ldots, |w_p|, |r_a^2 z_a|) < \delta,\ \frac{1}{2}\delta < r_a < 3\delta,\ 0 \le r_a^2\epsilon_a < \epsilon_0.$$

Equations for P_ϵ^* are obtained by setting $r_a^2\epsilon_a = \epsilon$.

The following proposition describes P^* as it arrives at $r_a = 0$. See Figure 10.

Proposition 6.2. *Let*

$$B = \{(w_2, \ldots, w_p, z_a, r_a, \epsilon_a) : \max(|w_2|, \ldots, |w_p|, |z_a|) < \delta, 0 \le r_a < \delta,\ \textit{and}\ 0 < \epsilon_a < \delta\}.$$

For $\delta > 0$ sufficiently small, there is a C^{r+5} function $(\breve{w}_1, \breve{w}_{p+1}, \ldots, \breve{w}_n) : B \to \mathbb{R}^{n-p+1}$ such that the following hold:

(1) *If $z_a = 0$ or $r_a = 0$, then $(\breve{w}_1, \breve{w}_{p+1}, \ldots, \breve{w}_n) = 0$.*

(2) *$\{(w, z_a, r_a, \epsilon_a) : (w_2, \ldots, w_p, z_a, r_a, \epsilon_a) \in B$ and $(w_1, w_{p+1}, \ldots, w_n) = (\breve{w}_1, \breve{w}_{p+1}, \ldots, \breve{w}_n)$ $(w_2, \ldots, w_p, z_a, r_a, \epsilon_a)\}$ is an open subset of P^*.*

Proof. In $wz_ar_a\epsilon_a$-space, we consider the C^{r+8} system (5.10)–(5.13). For $\delta > 0$ small, the codimension-one set

$$\{(w, z_a, r_a, \epsilon_a) : \max|w_i| < \delta, z_a = 0, 0 \le r_a < 3\delta,\ \text{and}\ 0 \le \epsilon_a < \delta\}$$

is normally hyperbolic (repelling).

The unstable fibers of points in $z_a = 0$ are curves. After a coordinate change of class C^{r+7}, we obtain new coordinates $(\breve{w}, z_a, \breve{r}_a, \breve{\epsilon}_a)$, with

$$\breve{w} = w + r_a z_a \breve{W}, \breve{r}_a = r_a(1 + z_a\breve{R}), \breve{\epsilon}_a = \epsilon_a(1 + z_a\breve{E}), \tag{6.1}$$

in which unstable fibers are lines $(\breve{w}, \breve{r}_a, \breve{\epsilon}_a) =$ constant.

To simplify the notation, we drop the cups in the new coordinates. In the new coordinates, the system becomes

$$\dot{w} = 0, \tag{6.2}$$
$$\dot{z_a} = z_a a(w, z_a, r_a, \epsilon_a), \tag{6.3}$$
$$\dot{r}_a = -r_a\epsilon_a, \tag{6.4}$$
$$\dot{\epsilon}_a = 2\epsilon_a^2. \tag{6.5}$$

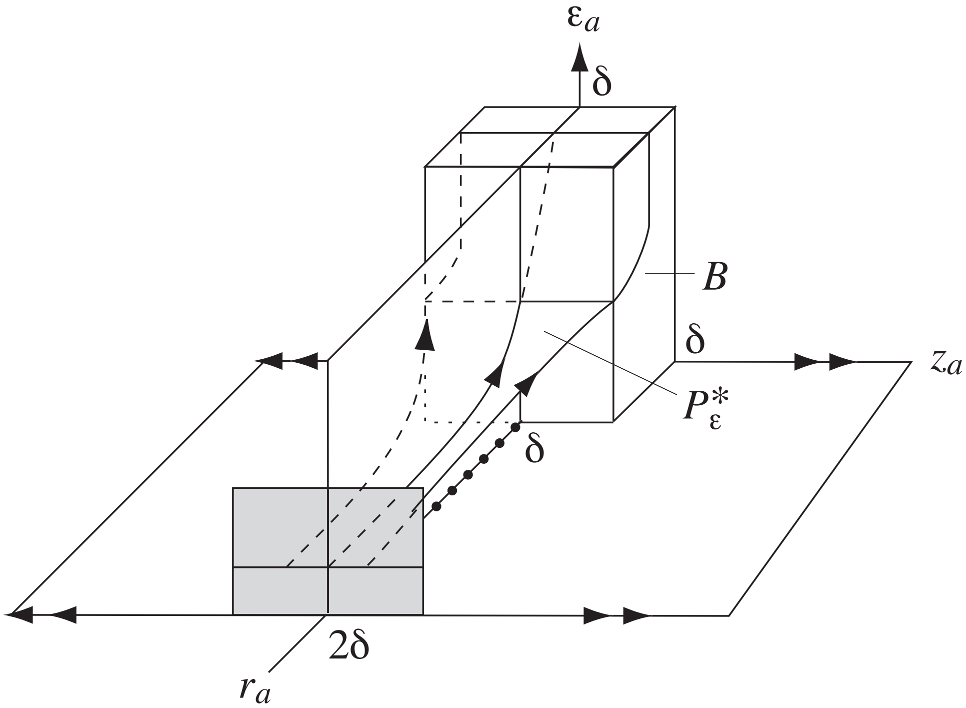

Figure 10. *P_ϵ^* in the coordinate chart for $\bar\sigma < 0$, with w suppressed. The cross-section of P^* defined by (6.6)–(6.7) is shaded.*

It is of class C^{r+6}. Moreover, there is a number $\nu_0 > 0$ such that $a(w, z_a, r_a, \epsilon_a) > \nu_0$.

We shall follow the evolution of a cross-section of P^* parameterized by $(w_2, \dots, w_p, z_a, \epsilon_a)$; the equations of the cross-section have the form

$$w_i = \hat w_i(w_2, \dots, w_p, z_a, \epsilon_a),$$

$$\hat w_i(w_2, \dots, w_p, 0, \epsilon_a) = \hat w_i(w_2, \dots, w_p, z_a, 0) = 0, \quad i = 1, p+1, \dots, n; \tag{6.6}$$

$$r_a = 2\delta; \tag{6.7}$$

from Proposition 6.1, the functions $\hat w_i$ are C^{r+6}.

We denote the solution of (6.2)–(6.5) whose value at $t = \tau$ is $(w^1, z_a^1, r_a^1, \epsilon_a^1)$ by

$$(w, z_a, r_a, \epsilon_a)(t, \tau, w^1, z_a^1, r_a^1, \epsilon_a^1),$$

a C^{r+6} function. This is the solution of a Silnikov problem of the second type, so Deng's lemma (Theorem 2.2 of [22]) applies.

One easily calculates that for $\tau = \frac{1}{2\epsilon_a^1}(4(\frac{\delta}{r_a^1})^2 - 1)$, $r_a(0, \tau, w^1, z_a^1, r_a^1, \epsilon_a^1) = \delta$.

Given $(w_2^1, \dots, w_p^1, z_a^1, r_a^1, \epsilon_a^1)$, we wish to find $(w_1^1, w_{p+1}^1, \dots, w_n^1)$ such that for $i = 1, p+1, \dots, n$,

$$w_i^1 - \hat w_i\left(w_2^1, \dots, w_p^1, z_a\left(0, \frac{1}{2\epsilon_a^1}\left(4\left(\frac{\delta}{r_a^1}\right)^2 - 1\right), w^1, z_a^1, \epsilon_a^1\right), \epsilon_a^1\right) = 0. \tag{6.8}$$

The desired function is then $(\tilde{w}_1, \tilde{w}_{p+1}, \dots, \tilde{w}_n) = (w_1^1, w_{p+1}^1, \dots, w_n^1)$.

For $i = 1, p+1, \dots, n$, we define

$$G_i((w_1^1, w_{p+1}^1, \dots, w_n^1), (w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), z_a^1)$$

to be the left-hand side of (6.8). G_i is a component of a C^{r+6} map G into $\mathbb{R}^{n-p+1}$. The domain of G is $X \times Y \times Z$,

$$X = \{(w_1^1, w_{p+1}^1, \dots, w_n^1) : \max |w_i^1| < \delta\},$$
$$Y = \{(w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1) : \max |w_i^1| < \delta, 0 < r_a^1 < \delta, 0 < \epsilon_a^1 < \delta\},$$
$$Z = \{z_a^1 : |z_a^1| < \delta\}.$$

The proof then proceeds in the following steps. We omit the details; for a similar, but harder, argument, see the proof of the general exchange lemma in [23]. Let $0 < (r+5)\gamma < \nu_0$.

(1) $G(0, (w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), 0) = 0$, and $G(0, (w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), z_a^1)$ is of order $e^{-\nu_0 \tau}$, $\tau = \frac{1}{2\epsilon_a^1}(4(\frac{\delta}{r_a^1})^2 - 1)$.

(2) $D_{(w_1^1, w_{p+1}^1, \dots, w_n^1)} G(0, (w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), 0) = I$.

(3) $D_{(w_1^1, w_{p+1}^1, \dots, w_n^1)} G((w_1^1, w_{p+1}^1, \dots, w_n^1), (w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), z_a^1) - D_{(w_1^1, w_{p+1}^1, \dots, w_n^1)} G(0, (w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), 0)$ is of order $e^{-(\nu_0 - \gamma)\tau}$.

(4) By the implicit function theorem (Theorem 5.1 of [23]), for each $((w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), z_a^1) \in Y \times Z$, there is a unique $(w_1^1, w_{p+1}^1, \dots, w_n^1)$, of order $e^{-\nu_0 \tau}$, such that $G((w_1^1, w_{p+1}^1, \dots, w_n^1), (w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), z_a^1) = 0$. Moreover, $(w_1^1, w_{p+1}^1, \dots, w_n^1)$ is a C^{r+6} function of $((w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), z_a^1)$.

(5) Any partial derivative of order i of G, $1 \le i \le r+5$, with respect to $((w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), z_a^1)$ is of order $e^{-(\nu_0 - i\gamma)\tau}$.

(6) Any partial derivative of order i of $(w_1^1, w_{p+1}^1, \dots, w_n^1)$, $1 \le i \le r+5$, with respect to $((w_2^1, \dots, w_p^1, r_a^1, \epsilon_a^1), z_a^1)$ is of order $e^{-(\nu_0 - i\gamma)\tau}$.

By (1), $(\tilde{w}_1, \tilde{w}_{p+1}, \dots, \tilde{w}_n) = 0$ when $z_a^1 = 0$. By (6), $(\tilde{w}_1, \tilde{w}_{p+1}, \dots, \tilde{w}_n)$ extends to equal 0 for $r_a^1 = 0$, and the extended function has all partial derivatives through order $r+5$ equal to 0 for $r_a^1 = 0$. Returning to the original $wz_a r_a \epsilon_a$-coordinates, the equations for P^* have the properties given in the proposition. ■

6.3. P_ϵ^* arrives at the plane of equilibria L_0.

The transformation from $wz_b\sigma_b r_b$-coordinates to $wz_a r_a \epsilon_a$-coordinates is given by

$$w = w, \quad z_a = \frac{z_b}{\sigma_b^2}, \quad r_a = -\sigma_b r_b, \quad \epsilon_a = \frac{1}{\sigma_b^2}.$$

Using this change of coordinates, Proposition 6.2 yields the following proposition.

Proposition 6.3. *Let*

$$\check{B} = \Big\{ (w_2, \dots, w_p, z_b, \sigma_b, r_b) : |w_i| < \delta \textit{ for } i = 2, \dots, p,$$
$$-\infty < \sigma_b < -\delta^{-\frac{1}{2}}, |z_b| < \delta\sigma_b^2, \textit{ and } 0 \le r_b < -\frac{\delta}{\sigma_b} \Big\}.$$

For $\delta > 0$ sufficiently small, there is a C^{r+5} function

$$(\check{w}_1, \check{w}_{p+1}, \dots, \check{w}_n) : \check{B} \to \mathbb{R}^{n-p+1}$$

for which the following hold:

(1) *If $z_b = 0$ or $r_b = 0$, then $(\check{w}_1, \check{w}_{p+1}, \dots, \check{w}_n) = 0$.*

(2) *$\{(w, z_b, \sigma_b, r_b) : (w_2, \dots, w_p, z_b, \sigma_b, r_b) \in \check{B}$ and $(w_1, w_{p+1}, \dots, w_n) = (\check{w}_1, \check{w}_{p+1}, \dots, \check{w}_n)$ $(w_2, \dots, w_p, z_b, \sigma_b, r_b)\}$ is an open subset of P^*.*

We choose a cross-section C of P^* with $\sigma_b = \sigma_b^* << 0$. Let $C_{r_b} = \{(w, z_b, \sigma_b) : (w, z_b, \sigma_b, r_b) \in C\}$.

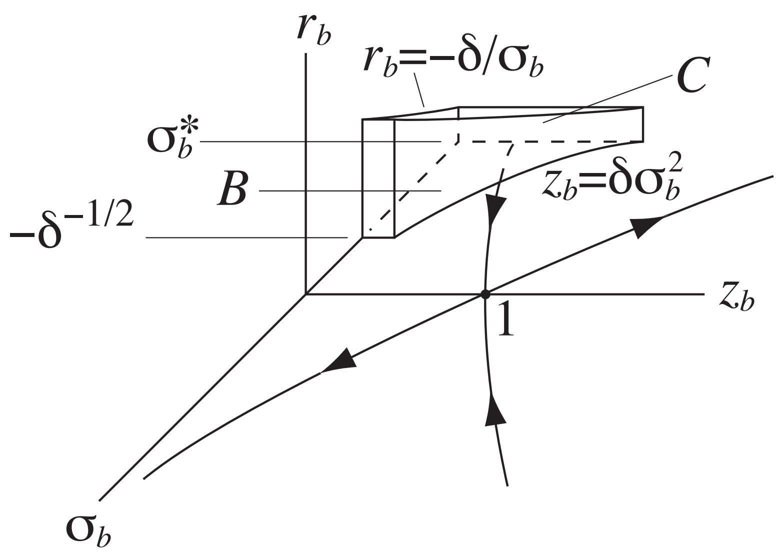

Figure 11. *The portions of B and C with $z_b \geq 0$; w is suppressed. They meet the stable manifold of the equilibrium shown, which lies in $r_b = 0$.*

Note that from (5.21), $\dot{r}_b = 0$, so r_b can be regarded as a parameter in the C^{r+8} system (5.18)–(5.20). (From (5.17), $\epsilon = r_b^2$.) For $r_b = 0$, the system (5.18)–(5.20) has the normally hyperbolic manifold of equilibria $L_0 = \{(w, z_b, \sigma_b) : (z_b, \sigma_b) = (1, 0)\}$. For small $r_b > 0$, L_0 perturbs to a normally hyperbolic invariant manifold L_{r_b}. The stable and unstable manifolds of L_{r_b} are given by

$$W^s(L_{r_b}) = \{(w, z_b, \sigma_b) : z_b = z_b^s(w, \sigma_b, r_b)\},$$
$$W^u(L_{r_b}) = \{(w, z_b, \sigma_b) : z_b = z_b^u(w, \sigma_b, r_b)\};$$

the functions z_b^s and z_b^u are C^{r+8}. For $r_b = 0$, each point $(w^0, 1, 0)$ of L_0 is an equilibrium; its stable fiber is simply its one-dimensional stable manifold, which has the equation $(w, z_b) = (w^0, z_b^s(w^0, \sigma_b, 0))$. The portion of $W^s(L_0)$ in $\sigma_b < 0$ has $0 < z_b < 1$. Therefore, if we choose σ_b^* sufficiently negative in defining C, the surfaces C_0 and $W^s(L_0)$ meet transversally. See Figure 11. The intersection projects along the foliation of $W^s(L_0)$ by stable manifolds of points to the submanifold of L_0 given by $\{(w, z_b, \sigma_b) : w_1 = w_{p+1} = \cdots = w_n = 0, (z_b, \sigma_b) = (1, 0)\}$.

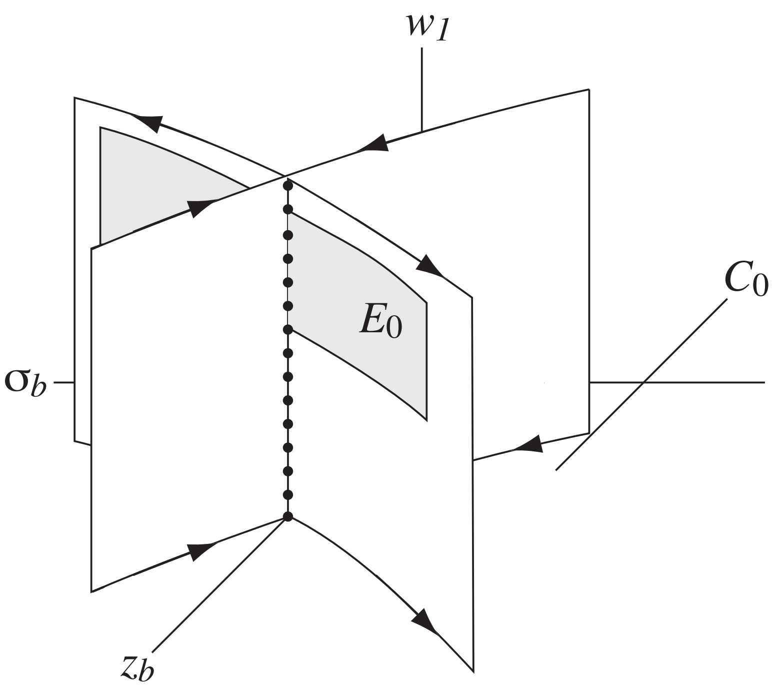

Figure 12. *The invariant space $(w_2,\dots,w_n)$ fixed, $r_b = 0$ for (5.18)–(5.21).*

The flow of (5.18)–(5.20) on L_{r_b} is $\dot{w} = r_b(1,0) + O(r_b^2)$. For small $r_b > 0$ we follow the solution from C_{r_b} until w_1 is close to t^*. From the exchange lemma for normally hyperbolic manifolds of equilibria (Theorem 2.3 of [23]), we have the following result.

Proposition 6.4. *Let $\eta > 0$ be small. Let*

$$D = \{(w_1, w_2, \dots, w_p, \sigma_b) : \max(|w_1 - t^*|, |w_2|, \dots, |w_p|, |\sigma_b|) < \eta\}.$$

For $r_b^* > 0$ sufficiently small, there is a C^{r+2} function $(\breve{w}_{p+1},\dots,\breve{w}_n,\breve{z}_b) : D\times[0,r_b^*) \to \mathbb{R}^{n-p+1}$ such that:

1. *As $r_b \to 0$, $(\breve{w}_{p+1},\dots,\breve{w}_n)(w_1,w_2,\dots,w_p,\sigma_b,r_b)$, and $\breve{z}_b(w_1,w_2,\dots,w_p,\sigma_b,r_b) - z_b^u(w_1,w_2,\dots,w_p,0,\dots,0,\sigma_b,r_b)$ approach 0 exponentially, along with all their partial derivatives of order at most r.*
2. *Let $E_{r_b} = \{(w,z_b,\sigma_b) : (w_1,w_2,\dots,w_p,\sigma_b) \in D$ and $(w_{p+1},\dots,w_n,z_b) = (\breve{w}_{p+1},\dots,\breve{w}_n,\breve{z}_b)(w_1,w_2,\dots,w_p,\sigma_b,r_b)\}$. For $0 < r_b < r_b^*$, E_{r_b} is an open subset of P_ϵ^*, $\epsilon = r_b^2$.*

See Figure 12. Let $E = \{(w,z_b,\sigma_b,r_b) : 0 \le r_b < r_b^*$ and $(w,z_b,\sigma_b) \in E_{r_b}\}$.

6.4. P_ϵ^* leaves the plane of equilibria L_0. The transformation from $wz_b\sigma_b r_b$-coordinates to $wz_c r_c\epsilon_c$-coordinates is given by

$$w = w, \quad z_c = \frac{z_b}{\sigma_b^2}, \quad r_c = \sigma_b r_b, \quad \epsilon_c = \frac{1}{\sigma_b^2}.$$

In $wz_cr_c\epsilon_c$-coordinates, the two-dimensional face of E with $\sigma_b = \eta$ corresponds to

$$F_{\frac{1}{\eta^2}} = \Bigg\{(w, z_c, r_c, \epsilon_c) : \max(|w_1 - t^*|, |w_2|, \dots, |w_p|) < \eta,\ 0 \le r_c < \eta r_b^*,\ \epsilon_c = \frac{1}{\eta^2},$$
$$(w_{p+1}, \dots, w_n) = (\hat{w}_{p+1}, \dots, \hat{w}_n)\left(w_1, w_2, \dots, w_p, \eta, \frac{r_c}{\eta}\right),\ z_c = \frac{1}{\eta^2}\hat{z}_b\left(w_1, w_2, \dots, w_p, \eta, \frac{r_c}{\eta}\right)\Bigg\}.$$

See Figure 13.

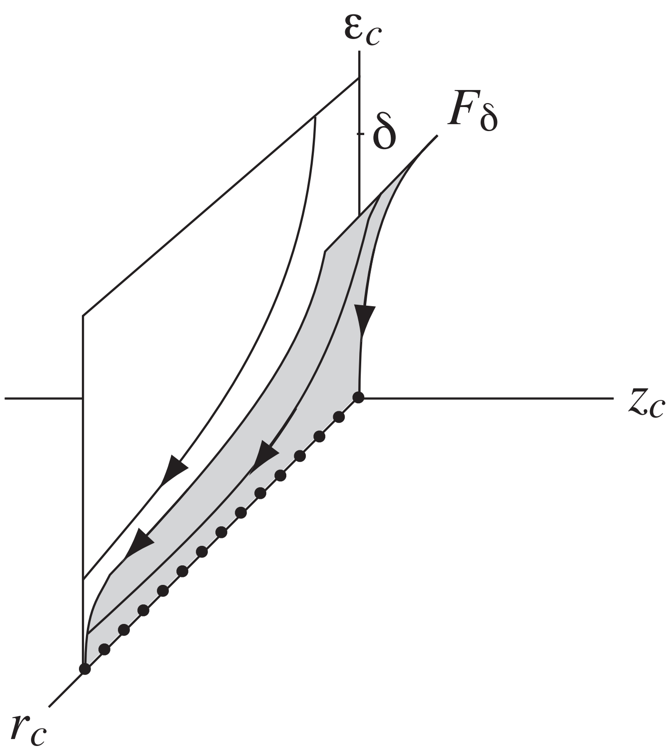

Figure 13. *Flow of (5.26)–(5.29), with w suppressed. The set of points in F_δ with a fixed value of $(w_1, w_2, \dots, w_p)$ is a curve. Solutions through points in this curve are shown.*

We follow the flow of (5.26)–(5.29) until $\epsilon_c = \delta > 0$, arriving at a set F_δ of the form

$$F_\delta = \{w, z_c, r_c, \epsilon_c) : \max(|w_1 - t^*|, |w_2|, \dots, |w_p|) < \delta,\ 0 \le r_c < \delta,\ \epsilon_c = \delta,$$
$$(w_{p+1}, \dots, w_n, z_c) = (w^\sharp_{p+1}, \dots, w^\sharp_n, z^\sharp_c)(w_1, w_2, \dots, w_p, r_c)\}, \tag{6.9}$$

where $(w^\sharp_{p+1}, \dots, w^\sharp_n, z^\sharp_c)$ is C^{r+2}.

6.5. P^*_ϵ leaves the blow-up cylinder. Finally, we follow solutions from F_δ until r_c is close to σ^*.

Proposition 6.5. *Let $\delta > 0$ be small. Let*

$$G = \{(w_1, w_2, \dots, w_p, r_c) : \max(|w_1 - t^*|, |w_2|, \dots, |w_p|, |r_c - \sigma^*|) < \delta\}.$$

For $\epsilon_c^* > 0$ sufficiently small, there is a C^{r+2} function

$$(\tilde{w}_{p+1}, \dots, \tilde{w}_n, \tilde{z}_c) : G \times [0, \epsilon_c^*) \to \mathbb{R}^{n-p+1}$$

for which

(1) $(\tilde{w}_{p+1}, \dots, \tilde{w}_n, \tilde{z}_c) = 0$ *when* $\epsilon_c = 0$, *and*

(2) $\{(w, z_c, r_c, \epsilon_c) : (w_1, w_2, \dots, w_p, r_c, \epsilon_c) \in G \times [0, \epsilon_c^*)$ *and* $(w_{p+1}, \dots, w_n, z_c) = (\tilde{w}_{p+1}, \dots, \tilde{w}_n, \tilde{z}_c)(w_1, w_2, \dots, w_p, r_c, \epsilon_c)\}$ *is an open subset of* P^*.

Proof. In $wz_cr_c\epsilon_c$-space, for $\delta > 0$ small, the codimension-one set

$$\{(w, z_c, r_c, \epsilon_c) : |w| < \delta, z_c = 0, 0 \le r_c < \sigma^* + \delta, \text{ and } 0 \le \epsilon_c < 2\delta\}$$

is normally hyperbolic (attracting) for the C^{r+8} system (5.26)–(5.29).

The stable fibers of points in $z_c = 0$ are curves. In new C^{r+7} coordinates $(\check{w}, z_c, \check{r}_c, \check{\epsilon}_c)$, with

$$\check{w} = w + r_c z_c \check{W}, \quad \check{r}_c = r_c(1 + z_c\check{R}), \quad \check{\epsilon}_c = \epsilon_c(1 + z_c\check{E}), \tag{6.10}$$

they are lines $(\check{w}, \check{r}_c, \check{\epsilon}_c) =$ constant.

To simplify the notation, we drop the checks in the new coordinates. In the new coordinates, the system becomes

$$\dot{w} = 0, \tag{6.11}$$

$$\dot{z}_c = z_c b(w, z_c, r_c, \epsilon_c), \tag{6.12}$$

$$\dot{r}_c = r_c\epsilon_c, \tag{6.13}$$

$$\dot{\epsilon}_c = -2\epsilon_c^2, \tag{6.14}$$

a C^{r+6} system. Moreover, there is a number $\omega_0 < 0$ such that $b(w, z_c, r_c, \epsilon_c) < \omega_0$.

We denote the solution of (6.11)–(6.14) whose value at $t = 0$ is $(w^0, z_c^0, r_c^0, \epsilon_c^0)$ by

$$(w, z_c, r_c, \epsilon_c)(t, 0, w^0, z_c^0, r_c^0, \epsilon_c^0),$$

a C^{r+6} function. This is the solution of an initial value problem; in the terminology of [22], it is also a solution of Silnikov's first boundary value problem, so Deng's lemma (Theorem 2.2 of [22]) applies.

We easily calculate that if $\epsilon_c^0 = \delta$ and $(r_c, \epsilon_c) = (r_c^1, \epsilon_c^1)$ at time t, then $r_c^0 = r_c^1\sqrt{\frac{\epsilon_c^1}{\delta}}$ and $t = \frac{\delta - \epsilon_c^1}{2\delta\epsilon_c^1}$.

To avoid proliferation of notation, we shall use the description (6.9) of F_δ in the new coordinates.

The desired function $(\tilde{w}_{p+1}, \dots, \tilde{w}_n, \tilde{z}_c)(w_1^1, w_2^1, \dots, w_p^1, r_c^1, \epsilon_c^1)$ is as follows. Given $(w_1^1, w_2^1, \dots, w_p^1, r_c^1, \epsilon_c^1)$, with $\max(|w_1^1 - t^*|, |w_2^1|, \dots, |w_p^1|, |r_c^1 - \sigma^*|) < \delta$ and $0 < \epsilon_c^1 < \epsilon_c^*$, where ϵ_c^* is small, let $\epsilon_c^0 = \delta$, $r_c^0 = r_c^1\sqrt{\frac{\epsilon_c^1}{\delta}}$, and $t = \frac{\delta - \epsilon_c^1}{2\delta\epsilon_c^1}$. Then

$$\tilde{w}_i^1(w_1^1, w_2^1, \dots, w_p^1, r_c^1, \epsilon_c^1) = w_i^\sharp(w_1^1, \dots, w_p^1, r_c^0), \quad i = p+1, \dots, n,$$
$$z_c^1(w_1^1, w_2^1, \dots, w_p^1, r_c^1, \epsilon_c^1) = z_c(t, 0, w_1^1, \dots, w_p^1, (w_{p+1}^\sharp, \dots, w_n^\sharp, z_c^\sharp)(w_1^1, \dots, w_p^1, r_c^0), r_c^0, \epsilon_c^0).$$

The functions $w_i^\sharp$ and $z_c^\sharp$ are of class C^{r+2}.

From Proposition 6.4 it follows that in the coordinates we are using, all partial derivatives of $(w_{p+1}^\sharp, \dots, w_n^\sharp, z_c^\sharp)$ of order $i \le r+2$ go to 0 exponentially as $r_c^0 \to 0$. Choose $\gamma > 0$ such that $\omega_0 + (r+5)\gamma < 0$. By Deng's lemma, all partial derivatives of $z_c(t, 0, w^0, z_c^0, r_c^0, \epsilon_c^0)$ of order $i \le r+5$ are of order $e^{(\omega_0 + i\gamma)t}$. It follows that as $\epsilon_c \to 0$, $(\tilde{w}_{p+1}, \dots, \tilde{w}_n, \tilde{z}_c) \to 0$ exponentially, along with its derivatives through order $r+2$ with respect to all variables. Returning to the original $wz_cr_c\epsilon_c$-coordinates, the equations for P^* have the properties given in the proposition. ■

7. Completion of the proof. We are now ready to prove Theorem 3.1 by verifying the hypotheses of the general exchange lemma from [23].

We have seen that (4.1)–(4.3) has, for each small ϵ, a normally hyperbolic invariant manifold K_ϵ of dimension $n+2$ that contains $\{(w, z, \sigma) : w \in W,\ z = 0, \text{ and } |\sigma| < \beta_0\}$. Let $\lambda_0 = \tilde{\lambda} + \beta_0 < 0$ and $\tilde{\mu}_0 = \tilde{\mu} - \beta_0 > 0$. For $w \in W$ and $|\sigma| < \beta_0$, the matrix (4.4) has k eigenvalues with real part less than λ_0, l eigenvalues with real part greater than μ_0, and $n+2$ real eigenvalues between $-\beta_0$ and β_0. From (3.1),

$$\lambda_0 + \mu_0 + r\beta_0 = \tilde{\lambda} + \tilde{\mu} + r\beta_0 < 0 < \tilde{\mu} - \max(7, 2r+2)\beta_0 = \mu_0 - \max(6, 2r+1)\beta_0.$$

It follows easily that hypotheses (E1) and (E2) of the general exchange lemma are satisfied on a neighborhood of K in $wz\sigma\epsilon$-space.

Let Σ be a codimension-one submanifold of $wz\sigma\epsilon$-space defined by an equation of the form $\sigma = \sigma(w, z, \epsilon)$, with $\sigma(w, 0, 0) = -\delta$. From (R4)–(R6), for $\epsilon > 0$ we can use the usual exchange lemma to follow M until it meets Σ. Let $\tilde{M} = M^* \cap \Sigma$, $\tilde{M}_\epsilon = \{(w, z, \sigma) : (w, z, \sigma, \epsilon) \in \tilde{M}\}$. Instead of the manifolds M_ϵ described by (R4)–(R6), in verifying hypotheses (E3)–(E5) of the general exchange lemma, we will use the manifolds $\tilde{M}_\epsilon$. Since M is C^{r+11}, the usual exchange lemma implies that $\tilde{M}$ is C^{r+8}. Each $\tilde{M}_\epsilon$ has dimension $l+p$, and hypotheses (E3)–(E5) of the general exchange lemma are satisfied. Since $\tilde{M}_0$ is contained in $W_0^u(\{(w, z, \sigma) : w^1 = w^{p+1} = \cdots = w^n = 0,\ z = 0,\ \sigma \text{ near } -\delta\})$, in hypothesis (E4) we have $x_* = 0$.

The choice of Σ determines the sets P_ϵ; the choice of P_ϵ determines whether a coordinate system on K in which (E6)–(E8) hold can be found. We shall first describe convenient coordinates on K in which Σ can be defined. We shall then define coordinates on K in which (E6)–(E8) hold.

On $\{(w, z_1, \sigma, \epsilon) : \max(|w_i|, |z_1|) < \delta,\ -3\delta < \sigma < -\frac{1}{2}\delta,\ 0 \le \epsilon < \delta\}$, we can make a C^{r+8} change of coordinates such that the C^{r+9} system (4.7)–(4.10) becomes

$$\dot{w} = 0, \tag{7.1}$$
$$\dot{z}_1 = z_1 a(w, z_1, \sigma, \epsilon), \tag{7.2}$$
$$\dot{\sigma} = \epsilon, \tag{7.3}$$

a C^{r+7} system with $a(w, z_1, \sigma, \epsilon) > \mu_0 > 0$. In these coordinates, we let Σ be defined by $\sigma = -2\delta$. Then a cross-section of P^* is given by

$$w_i = \hat{w}_i(w_2, \dots, w_p, z_1, \epsilon), \quad \hat{w}_i(w_2, \dots, w_p, 0, \epsilon) = 0, \quad i = 1, p+1, \dots, n,$$
$$\sigma = -2\delta,$$

with $\hat{w}_i$ C^{r+8}. Let the solution of (7.1)–(7.3) with $(w, z_1, \sigma)(\tau) = (w^1, z_1^1, \sigma^1)$ be $(w, z_1, \sigma)(t, \tau, w^1, z_1^1, \sigma^1, \epsilon)$; the mapping is C^{r+7}. Note that $\sigma(0, \frac{\sigma^1+2\delta}{\epsilon}, w^1, z_1^1, \sigma^1, \epsilon) = -2\delta$.

For $-2\delta < \sigma^1 < -\frac{1}{2}\delta$, we define new C^{r+7} coordinates $y_i(w^1, z_1^1, \sigma^1, \epsilon)$, $i = 1, p+1, \ldots, n$, by

(7.4)
$$y_i(w^1, z_1^1, \sigma^1, \epsilon) = w_i^1 - \hat{w}_i\left(w_2^1, \ldots, w_p^1, z_1\left(0, \frac{\sigma^1 + 2\delta}{\epsilon}, w^1, z_1^1, \sigma^1, \epsilon\right), \epsilon\right), \quad i = 1, p+1, \ldots, n.$$

Proposition 7.1. *$y_i - (w_i^1 - \hat{w}_i(w_2^1, \ldots, w_p^1, 0, \epsilon))$ and its derivatives through order $r + 6$ go to 0 exponentially as $\epsilon \to 0$. If we use $(y_1, w_2, \ldots, w_p, y_{p+1}, \ldots, y_n, z_1, \sigma, \epsilon)$ as coordinates on $\{(w, z_1, \sigma, \epsilon) : \max(|w_i|, |z_1|) < \delta,\ -\frac{3}{2}\delta < \sigma < -\frac{1}{2}\delta,\ 0 \le \epsilon < \delta\}$, the system takes the form*

$$\dot{w}_i = 0, \quad i = 2, \ldots, p, \tag{7.5}$$
$$\dot{y}_i = 0, \quad i = 1, p+1, \ldots, n, \tag{7.6}$$
$$\dot{z}_1 = z_1 a(w, y, z_1, \sigma, \epsilon), \tag{7.7}$$
$$\dot{\sigma} = \epsilon, \tag{7.8}$$

a C^{r+6} system with $a(w, y, z_1, \sigma, \epsilon) > \mu_0 > 0$ and P^* given by $y = 0$.

Proof. From its definition, y_i is constant on orbits and equals 0 on P^*. Since y_i is constant on orbits, the new system has the form (7.5)–(7.8). By Deng's lemma (Theorem 2.2 of [22]), for $-\frac{3}{2}\delta < \sigma < -\frac{1}{2}\delta$, $z_1(0, \frac{\sigma^1+2\delta}{\epsilon}, w^1, z_1^1, \sigma^1, \epsilon)$ and its derivatives through order $r + 6$ go to 0 exponentially as $\epsilon \to 0$. Therefore,

$$y_i - (w_i^1 - \hat{w}_i(w_2^1, \ldots, w_p^1, 0, \epsilon)) = \hat{w}_i(w_2^1, \ldots, w_p^1, 0, \epsilon) - \hat{w}_i\left(w_2^1, \ldots, w_p^1, z_1\left(0, \frac{\sigma^1 + 2\delta}{2}, w^1, z_1^1, \sigma^1, \epsilon\right), \epsilon\right)$$

and its derivatives through order $r + 6$ go to 0 exponentially as $\epsilon \to 0$. ■

In the coordinates $(y_1, w_2, \ldots, w_p, y_{p+1}, \ldots, y_n, z_1, \sigma, \epsilon)$, Σ is just the set $\sigma = \delta$, and each P_ϵ, in the coordinates $(y_1, w_2, \ldots, w_p, y_{p+1}, \ldots, y_n, z_1, \sigma)$, is given by $(y_1 = y_{p+1} = \cdots = y_n = 0, \sigma = \delta)$. The coordinates (u^0, v^0, w^0) in which hypotheses (E6)–(E8) of the general exchange lemma hold are given by

$$u^0 = \sigma + \delta, \quad v^0 = (w_2, \ldots, w_p, z_1), \quad w^0 = (y_1, y_{p+1}, \ldots, y_n).$$

In (E7) we use $a = 1$.

Let $V^* = \{(w, z_1, \sigma) : \max(|w_1 - t^*|, |w_2|, \ldots, |w_n|, |z_1|, |\sigma - \sigma^*|) < \delta\}$. The coordinate system in which hypothesis (E9) holds is essentially given by Proposition 6.5; w^1 is the C^{r+2} function $(\tilde{w}_{p+1}, \ldots, \tilde{w}_n, \tilde{z}_c)$. In these C^{r+2} coordinates, the system is C^{r+1}, so (E10) is satisfied. Since, for the original differential equation, $\dot{x} = \epsilon$, (E11) is satisfied with $a = 1$ for δ sufficiently small.

Since all hypotheses of the general exchange lemma are satisfied, Theorem 3.1 follows.

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