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J. Differential Equations 245 (2008) 392–410

Journal of Differential Equations

www.elsevier.com/locate/jde

Exchange lemmas 1: Deng's lemma ☆

Stephen Schecter

Mathematics Department, North Carolina State University, Box 8205, Raleigh, NC 27695, USA

Received 9 July 2007; revised 27 August 2007

Available online 29 September 2007

Abstract

Deng's lemma gives estimates on the behavior of solutions of ordinary differential equations in the neighborhood of a partially hyperbolic equilibrium. We prove a generalization in which "partially hyperbolic equilibrium" is replaced by "normally hyperbolic invariant manifold."

MSC: 34E15; 34B15; 34C30

Keywords: Geometric singular perturbation theory; Normally hyperbolic invariant manifold

1. Introduction

Boundary value problems for ordinary differential equations are ubiquitous in applied mathematics. Consider one of the form

$$\dot{\xi} = F(\xi, \epsilon), \quad \xi(t_-) \in A_-(\epsilon),\ \xi(t_+) \in A_+(\epsilon), \tag{1.1}$$

in which $\xi \in \mathbb{R}^n$; $\epsilon \geqslant 0$ is a small parameter; $A_-(\epsilon)$ and $A_+(\epsilon)$ are manifolds; t_- and t_+ may be specified functions of ϵ or may be left unspecified, in which case we simply want a solution that goes from $A_-(\epsilon)$ to $A_+(\epsilon)$. See Fig. 1. For example, if $A_-(\epsilon)$ is part of the unstable manifold of an equilibrium $\xi_-(\epsilon)$, and $A_+(\epsilon)$ is part of the stable manifold of an equilibrium $\xi_+(\epsilon)$, then a solution of (1.1), when extended to the time interval $-\infty < t < \infty$, is a heteroclinic solution from $\xi_-(\epsilon)$ to $\xi_+(\epsilon)$. Such a solution may be of interest because it represents a traveling wave of a related partial differential equation.

☆ This work was supported in part by the National Science Foundation under grant DMS-0406016.
E-mail address: schecter@math.ncsu.edu.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.08.011

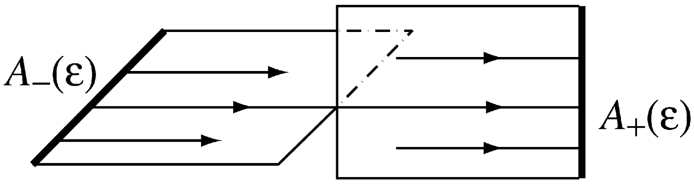

Fig. 1. A boundary value problem and its solution.

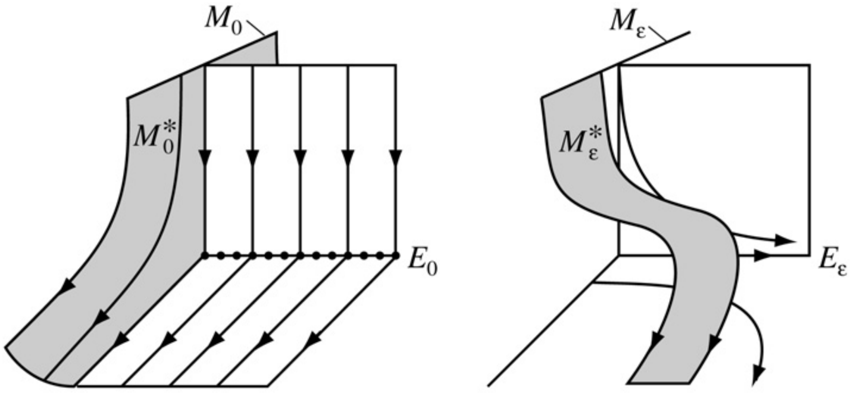

Fig. 2. Unperturbed and perturbed flows.

To show the existence of a solution of (1.1) with $\epsilon > 0$, one often uses a perturbation argument from $\epsilon = 0$ to show that the manifold of solutions that start on $A_-(\epsilon)$ and the manifold of solutions that end on $A_+(\epsilon)$ meet transversally. See Fig. 1.

Frequently, the problem (1.1) with $\epsilon = 0$ is degenerate in some way, and is only of interest insofar as it helps to solve (1.1) with $\epsilon > 0$. Such problems are typically referred to a *singularly perturbed*. The geometric approach to such problems, which focuses on tracking manifolds of potential solutions rather than on asymptotic expansions of solutions, is called *geometric singular perturbation theory* [7,8].

Suppose, for example, that (1.1) with $\epsilon = 0$ has an m-dimensional manifold of normally hyperbolic equilibria E_0, and that, after following $A_-(0)$ forward, we have a manifold M_0 that is transverse to the stable manifold of E_0. If we follow M_0 forward it becomes a manifold M_0^* as pictured in Fig. 2. For small $\epsilon > 0$, following $A_-(\epsilon)$ forward leads to a manifold M_ϵ near M_0 that is transverse to the stable manifold of E_ϵ, the perturbed normally hyperbolic invariant manifold near E_0. Since E_ϵ typically does not consist of equilibria, in forward time M_ϵ becomes a manifold M_ϵ^* as pictured in Fig. 2. M_ϵ^* is far from M_0^*.

The differential equation on the normally hyperbolic invariant manifold E_ϵ locally reduces to $\dot{c} = \epsilon G(c, \epsilon)$, $c \in \mathbb{R}^n$. The flow of $c' = G(c, 0)$, the limiting rescaled differential equation, is called the *slow flow*. The most common situation is *rectifiable slow flow*: on the region of interest, $c' = G(c, 0)$ can be put in the form $c_1' = 1$, $c_2' = \cdots = c_m' = 0$. In this case, the *Exchange Lemma* [9–11,24] asserts that M_ϵ^* is close to part of the unstable manifold of E_ϵ, which is in turn close to part of the unstable manifold of E_0. Thus transversality to the stable manifold of E_0 has been "exchanged" for closeness to part of the unstable manifold of E_0. This information can then be used to follow $A_-(\epsilon)$ forward farther and thus to solve the boundary value problem.

At present, much work in geometric singular perturbation theory deals with manifolds of equilibria E_0 that fail to be normally hyperbolic at some points. If there are no normally hyperbolic directions at such points, the flow near E_0 for small ϵ can often be understood using the "blowing up" construction [4,5,13,17,21,23].

If there are normally hyperbolic directions, a recipe for analyzing the flow near E_0 for small ϵ is as follows. One imbeds E_0 in a larger manifold K_0 that contains the directions along which normal hyperbolicity is lost. K_0 is itself normally hyperbolic, and hence perturbs to nearby normally hyperbolic manifolds K_ϵ. The flow on K_ϵ can by analyzed by blowing up. One then needs a generalization of the Exchange Lemma to relate this flow to the flow on a neighborhood of K_ϵ. Since K_0 is not a manifold of equilibria, the Exchange Lemma just described does not apply.

One type of loss of normal hyperbolicity is the *turning point*: a manifold of equilibria E_0 is known to perturb to a family of invariant manifolds E_ϵ, but normal hyperbolicity is lost along a codimension-one submanifold of E_0. At a *loss-of-stability turning point*, a real eigenvalue changes from negative to positive as one crosses the codimension-one submanifold in the direction of the slow flow. Exchange lemmas for loss-of-stability turning points have been proved by Weishi Liu [16].

My motivation to work in this area comes from *gain-of-stability turning points*: a real eigenvalue changes from positive to negative as one crosses the codimension-one submanifold in the direction of the slow flow. Gain-of-stability turning points occur when one looks for a self-similar solution of the Dafermos regularization of a system of conservation laws near a Riemann solution of the underlying system of conservation laws that includes a rarefaction wave [21]. For information about the Dafermos regularization, its possible relevance to the long-time behavior of solutions of viscous conservation laws, its self-similar solutions, and their stability, see [2,25] and [15].

It turned out that instead of proving an exchange lemma for gain-of-stability turning points, one can state and prove a General Exchange Lemma that encompasses all these situations (normally hyperbolic invariant manifold with rectifiable slow flow, loss-of-stability turning points, gain-of-stability turning point) and perhaps others. This General Exchange Lemma and its application to self-similar solutions of the Dafermos regularization are the subject of the present series of papers.

In the literature, there are three ways to prove exchange lemmas: (1) Jones and Kopell's approach [10,11,16], which is to follow the tangent space to M_ϵ forward using the extension of the linearized differential equation to differential forms; (2) Brunovský's approach [1,18,19], which is to locate M_ϵ^* by solving a boundary value problem in Silnikov variables; and (3) Krupa, Sandstede, and Szmolyan's approach [12], which uses Lin's method [14].

We follow Brunovský's approach, which is in turn based on work of Deng [3]. Brunovský generalized a lemma of Deng that gives estimates on solutions of boundary value problems in Silnikov variables.

In Deng's work, the boundary data lie near an equilibrium that may be nonhyperbolic. In Brunovský's work, the boundary data lie near a solution of a rectifiable slow flow on a normally hyperbolic invariant manifold. Our work requires us to consider more general flows on normally hyperbolic invariant manifolds.

The present paper is devoted to the required generalization of Deng's lemma, which we state in Section 2 and prove in Section 3.

In the second paper in this series [20], we state and prove the General Exchange Lemma, and explain how it easily implies versions of existing exchange lemmas for rectifiable slow flows and loss-of-stability turning points. In the third paper [22], which is joint work with Peter Szmolyan, we use the General Exchange Lemma to prove an exchange lemma for gain-of-stability turning points and to study self-similar solutions of the Dafermos regularization.

2. Generalized Deng's lemma

On $\mathbb{R}^n$ we use coordinates $\xi = (x, y, c)$, with $x \in \mathbb{R}^k$, $y \in \mathbb{R}^l$, $c \in \mathbb{R}^m$, $k + l + m = n$. Let V be an open subset of $\mathbb{R}^m$. We consider a C^{r+1}, $r \geqslant 1$, differential equation $\dot{\xi} = F(\xi)$ on a neighborhood of $\{0\} \times \{0\} \times V$ in $\mathbb{R}^n$ of the following form:

$$\dot{x} = \tilde{A}(x, y, c)x, \tag{2.1}$$

$$\dot{y} = \tilde{B}(x, y, c)y, \tag{2.2}$$

$$\dot{c} = \tilde{C}(c) + \tilde{E}(x, y, c)xy. \tag{2.3}$$

Thus we assume $\tilde{A}x$, $\tilde{B}y$, $\tilde{C}$, and $\tilde{E}xy$ are C^{r+1}. Let $\phi(t, c)$ be the flow of $\dot{c} = \tilde{C}(c)$. For each $c \in V$ there is a maximal interval I_c containing 0 such that $\phi(t, c) \in V$ for all $t \in I_c$. Let the linearized solution operator of (2.1)–(2.3) along the solution $(0, 0, \phi(t, c^0))$ be

$$\begin{pmatrix} \bar{x}(t) \\ \bar{y}(t) \\ \bar{c}(t) \end{pmatrix} = \begin{pmatrix} \Phi^s(t, s, c^0) & 0 & 0 \\ 0 & \Phi^u(t, s, c^0) & 0 \\ 0 & 0 & \Phi^c(t, s, c^0) \end{pmatrix} \begin{pmatrix} \bar{x}(s) \\ \bar{y}(s) \\ \bar{c}(s) \end{pmatrix}. \tag{2.4}$$

We assume:

(E1) There are numbers $\lambda_0 < 0 < \mu_0$, $\beta > 0$, and $M > 0$ such that for all $c^0 \in V$ and $s, t \in I_{c^0}$,

$$\left\| \Phi^s\left(t, s, c^0\right) \right\| \leqslant M e^{\lambda_0 (t-s)} \quad \text{if } t \geqslant s, \tag{2.5}$$

$$\left\| \Phi^u\left(t, s, c^0\right) \right\| \leqslant M e^{\mu_0 (t-s)} \quad \text{if } t \leqslant s, \tag{2.6}$$

$$\left\| \Phi^c\left(t, s, c^0\right) \right\| \leqslant M e^{\beta |t-s|} \quad \text{for all } t, s. \tag{2.7}$$

In addition, we assume one of the following:

(D1) $\lambda_0 + r\beta < 0 < \lambda_0 + \mu_0 - r\beta$.
(D2) $\lambda_0 + \mu_0 + r\beta < 0 < \mu_0 - r\beta$.

We wish to study solutions of Silnikov's boundary value problem, which is (2.1)–(2.3) on an interval $0 \leqslant t \leqslant \tau$, together with one of the following sets of boundary conditions:

$$x(0) = x^0, \quad y(\tau) = y^1, \quad c(0) = c^0 \tag{2.8}$$

or

$$x(0) = x^0, \quad y(\tau) = y^1, \quad c(\tau) = c^1. \tag{2.9}$$

We denote the solution of (2.1)–(2.3) with boundary conditions (2.8) by $(x, y, c)(t, \tau, x^0, y^1, c^0)$, and the solution of (2.1)–(2.3) with boundary conditions (2.9) by $(x, y, c)(t, \tau, x^0, y^1, c^1)$.

We shall use the following notation. Let $f : \mathbb{R}^p \to \mathbb{R}^q$ be a function, and let $\mathbf{i} = i_1, \dots, i_{|\mathbf{i}|}$ be a sequence of $|\mathbf{i}|$ integers between 1 and p. Then

$$D_{\mathbf{i}} f = \frac{\partial^{|\mathbf{i}|} f}{\partial u_{i_1} \cdots \partial u_{i_{|\mathbf{i}|}}}.$$

We shall allow $|\mathbf{i}| = 0$; in this case $\mathbf{i}$ is the empty sequence, and $D_{\mathbf{i}} f = f$. Since the ordering of the sequence is irrelevant when $D_{\mathbf{i}} f$ is continuous, which will always be the case, we will reorder $\mathbf{i}$ whenever it is convenient.

Theorem 2.1 *(Deng's lemma for Silnikov's first boundary value problem). Let V_0 and V_1 be compact subsets of V such that $V_0 \subset \mathrm{Int}(V_1)$. For each $c^0 \in V_0$ let J_{c^0} be the maximal interval such that $\phi(t, c^0) \in \mathrm{Int}(V_1)$ for all $t \in J_{c^0}$. Choose numbers λ and μ such that $\lambda_0 < \lambda < 0 < \mu < \mu_0$, and (E1) and (D1) hold with (λ, μ) replacing (λ_0, μ_0). Then there is a number $\delta_0 > 0$ such that if $\|x^0\| \leqslant \delta_0$, $\|y^1\| \leqslant \delta_0$, $c^0 \in V_0$, and $\tau > 0$ is in J_{c^0}, then Silnikov's first boundary value problem (2.8) has a solution $(x, y, c)(t, \tau, x^0, y^1, c^0)$ on the interval $0 \leqslant t \leqslant \tau$. Moreover, there is a number $K > 0$ such that for all (t, τ, x^0, y^1, c^0) as above,*

$$\big\|x\big(t, \tau, x^0, y^1, c^0\big)\big\| \leqslant K e^{\lambda t}, \tag{2.10}$$

$$\big\|y\big(t, \tau, x^0, y^1, c^0\big)\big\| \leqslant K e^{\mu(t-\tau)}, \tag{2.11}$$

$$\big\|c\big(t, \tau, x^0, y^1, c^0\big) - \phi\big(t, c^0\big)\big\| \leqslant K e^{\lambda t + \mu(t-\tau)}. \tag{2.12}$$

In addition, if $\mathbf{i}$ is any $|\mathbf{i}|$-tuple of integers between 1 and $2 + n$, with $1 \leqslant |\mathbf{i}| \leqslant r$, then

$$\big\|D_{\mathbf{i}} x\big(t, \tau, x^0, y^1, c^0\big)\big\| \leqslant K e^{(\lambda + |\mathbf{i}|\beta)t}, \tag{2.13}$$

$$\big\|D_{\mathbf{i}} y\big(t, \tau, x^0, y^1, c^0\big)\big\| \leqslant K e^{(\mu - |\mathbf{i}|\beta)(t-\tau)}, \tag{2.14}$$

$$\big\|D_{\mathbf{i}} c\big(t, \tau, x^0, y^1, c^0\big) - D_{\mathbf{i}} \phi\big(t, c^0\big)\big\| \leqslant K e^{(\lambda + |\mathbf{i}|\beta)t + (\mu - |\mathbf{i}|\beta)(t-\tau)}. \tag{2.15}$$

In (2.12) and (2.15), note that

$$\phi\big(t, c^0\big) = c\big(t, \tau, 0, 0, c^0\big) = c\big(t, \tau, x^0, 0, c^0\big) = c\big(t, \tau, 0, y^1, c^0\big).$$

Cases of this result were proved by Deng [3] and Brunovský [1].

Theorem 2.2 *(Deng's lemma for Silnikov's second boundary value problem). Let V_0 and V_1 be compact subsets of V such that $V_0 \subset \mathrm{Int}(V_1)$. For each $c^1 \in V_0$ let J_{c^1} be the maximal interval such that $\phi(t, c^1) \in \mathrm{Int}(V_1)$ for all $t \in J_{c^1}$. Choose numbers λ and μ such that $\lambda_0 < \lambda < 0 < \mu < \mu_0$, and (E1) and (D2) hold with (λ, μ) replacing (λ_0, μ_0). Then there is a number $\delta_0 > 0$ such that if $\|x^0\| \leqslant \delta_0$, $\|y^1\| \leqslant \delta_0$, $c^1 \in V_0$, and $-\tau < 0$ is in J_{c^1}, then Silnikov's second boundary value problem (2.9) has a solution $(x, y, c)(t, \tau, x^0, y^1, c^1)$ on the interval $0 \leqslant t \leqslant \tau$. Moreover, there is a number $K > 0$ such that for all (t, τ, x^0, y^1, c^1) as above,*

$$\big\|x\big(t, \tau, x^0, y^1, c^1\big)\big\| \leqslant K e^{\lambda t}, \tag{2.16}$$

$$\big\|y\big(t, \tau, x^0, y^1, c^1\big)\big\| \leqslant K e^{\mu(t-\tau)}, \tag{2.17}$$

$$\big\|c\big(t, \tau, x^0, y^1, c^1\big) - \phi\big(t - \tau, c^1\big)\big\| \leqslant K e^{\lambda t + \mu(t-\tau)}. \tag{2.18}$$

In addition, if $\mathbf{i}$ *is any* $|\mathbf{i}|$*-tuple of integers between* 1 *and* $2+n$*, with* $1 \leqslant |\mathbf{i}| \leqslant r$*, then*

$$\left\| D_{\mathbf{i}} x\left(t, \tau, x^0, y^1, c^1\right)\right\| \leqslant K e^{(\lambda+|\mathbf{i}|\beta)t}, \tag{2.19}$$

$$\left\| D_{\mathbf{i}} y\left(t, \tau, x^0, y^1, c^1\right)\right\| \leqslant K e^{(\mu-|\mathbf{i}|\beta)(t-\tau)}, \tag{2.20}$$

$$\left\| D_{\mathbf{i}} c\left(t, \tau, x^0, y^1, c^1\right) - D_{\mathbf{i}}\phi\left(t-\tau, c^1\right)\right\| \leqslant K e^{(\lambda+|\mathbf{i}|\beta)t+(\mu-|\mathbf{i}|\beta)(t-\tau)}. \tag{2.21}$$

In (2.18) and (2.21), note that

$$\phi\left(t-\tau, c^1\right) = c\left(t, \tau, 0, 0, c^1\right) = c\left(t, \tau, x^0, 0, c^1\right) = c\left(t, \tau, 0, y^1, c^1\right).$$

Remark 2.3 *(Normally hyperbolic invariant manifolds).* Suppose M is a C^s normally hyperbolic compact invariant manifold of dimension m for the C^s differential equation $\dot{\zeta} = G(\zeta)$ on $\mathbb{R}^n$. This means:

(N1) There is a splitting of the tangent bundle to R^n along M into subbundles of dimension k, l, and m, $k+l+m=n$, with the last being the tangent bundle of M: $T_M\mathbb{R}^n = S + U + TM$.
(N2) This splitting is invariant under the linearized solution operator along M.
(N3) Let $\psi(t,\zeta)$ be the flow of $\dot{\zeta} = G(\zeta)$, and let $\Psi(t,s,\zeta)$ be the linearized solution operator along $\psi(t,\zeta)$: $\Psi(t,s,\zeta) = D\psi(t,\zeta) \circ D\psi(-s, \psi(s,\zeta))$. Then for each $\zeta^0 \in M$, there are numbers $\lambda_0 < 0 < \mu_0$, $0 < \beta < \min(|\lambda_0|, \mu_0)$, and $M > 0$, *all depending on* ζ^0, such that

$$\left\|\Psi\left(t, s, \zeta^0\right)\bar{v}(s)\right\| \leqslant M e^{\lambda_0(t-s)}\left\|\bar{v}(s)\right\| \quad \text{if } \bar{v}(s) \in S_{\psi(s,\zeta^0)} \text{ and } t \geqslant s, \tag{2.22}$$

$$\left\|\Psi\left(t, s, \zeta^0\right)\bar{v}(s)\right\| \leqslant M e^{\mu_0(t-s)}\left\|\bar{v}(s)\right\| \quad \text{if } \bar{v}(s) \in U_{\psi(s,\zeta^0)} \text{ and } t \leqslant s, \tag{2.23}$$

$$\left\|\Psi\left(t, s, \zeta^0\right)\bar{v}(s)\right\| \leqslant M e^{\beta|t-s|}\left\|\bar{v}(s)\right\| \quad \text{if } \bar{v}(s) \in T_{\psi(s,\zeta^0)}M, \text{ for all } t, s. \tag{2.24}$$

(N4) $\sup_M \lambda_0 < 0 < \inf_M \mu_0$.

Suppose in addition that there is $r' \leqslant s$ such that at *each* point of M,

$$\lambda_0 + r'\beta < 0 < \mu_0 - r'\beta. \tag{2.25}$$

Then M is covered by open sets U in $\mathbb{R}^n$ on each of which there are $C^{r'-1}$ coordinates $\xi = \xi(\zeta)$ in which $\dot{\zeta} = G(\zeta)$ has the form (2.1)–(2.3); $\{0\} \times \{0\} \times V$ corresponds to $U \cap M$ [6]. In the new coordinates, the differential equation is $C^{r'-2}$. However, $(\lambda_0, \mu_0, \beta)$ cannot necessarily be chosen independent of c^0.

Our statement and proof of Theorems 2.1 and 2.2 require uniform, not pointwise, assumptions. In addition, we require (D2) or (D3) rather than an inequality like (2.25). Thus our assumptions are a little stronger than normal hyperbolicity.

Remark 2.4. Notice that all components of c must be given at $t = 0$, or all components of c must be given at $t = \tau$. This is true in Deng's and Brunovský's work as well. Thus the proof of the Corner Lemma in [18] is wrong and must be reworked.

3. Proof of the generalized Deng's lemma

3.1. Introduction

We shall prove Theorem 2.1 only.

Let $c = \phi(t, c^0) + z$. The system (2.1)–(2.3) becomes

$$\dot{x} = A\big(t, c^0\big)x + f\big(t, c^0, x, y, z\big), \tag{3.1}$$

$$\dot{y} = B\big(t, c^0\big)y + g\big(t, c^0, x, y, z\big), \tag{3.2}$$

$$\dot{z} = C\big(t, c^0\big)z + \theta\big(t, c^0, z\big) + h\big(t, c^0, x, y, z\big), \tag{3.3}$$

with

$$\begin{aligned}
A(t, c_0) &= \tilde{A}\big(0, 0, \phi\big(t, c^0\big)\big),\\
f\big(t, c^0, x, y, z\big) &= \big(\tilde{A}\big(x, y, \phi\big(t, c^0\big) + z\big) - \tilde{A}\big(0, 0, \phi\big(t, c^0\big)\big)\big)x,\\
B(t, c_0) &= \tilde{B}\big(0, 0, \phi\big(t, c^0\big)\big),\\
g\big(t, c^0, x, y, z\big) &= \big(\tilde{B}\big(x, y, \phi\big(t, c^0\big) + z\big) - \tilde{B}\big(0, 0, \phi\big(t, c^0\big)\big)\big)y,\\
C(t, c_0) &= D\tilde{C}\big(\phi\big(t, c^0\big)\big),\\
\theta\big(t, c^0, z\big) &= \tilde{C}\big(\phi\big(t, c^0\big) + z\big) - \tilde{C}\big(\phi\big(t, c^0\big)\big) - D\tilde{C}\big(\phi\big(t, c^0\big)\big)z,\\
h\big(t, c^0, x, y, z\big) &= \tilde{E}\big(x, y, \phi\big(t, c^0\big) + z\big)xy.
\end{aligned}$$

The first six of these functions are C^r; the last is C^{r+1}. To see that the last is C^{r+1}, let $E(x, y, z) = \tilde{E}(x, y, z)xy$. Then E is C^{r+1}, and

$$h\big(t, c^0, x, y, z\big) = E\big(x, y, \phi\big(t, c^0\big) + z\big). \tag{3.4}$$

The solution operator of the linear equation

$$(\dot{x}, \dot{y}, \dot{z}) = \operatorname{diag}\big(A(t, c_0), B(t, c_0), C(t, c_0)\big)(x, y, z)$$

is

$$\big(\bar{x}(t), \bar{y}(t), \bar{z}(t)\big) = \operatorname{diag}\big(\Phi^s\big(t, s, c^0\big), \Phi^u\big(t, s, c^0\big), \Phi^c\big(t, s, c^0\big)\big)\big(\bar{x}(s), \bar{y}(s), \bar{z}(s)\big).$$

Then $(x(t), y(t), c(t))$ is a solution of Silnikov's problem (2.1)–(2.3), (2.8), if and only if $c(t) = \phi(t, c^0) + z(t)$ and $\eta(t) = (x(t), y(t), z(t))$ satisfy

$$x(t) = \Phi^s\big(t, 0, c^0\big)x^0 + \int_0^t \Phi^s\big(t, s, c^0\big) f\big(s, c^0, \eta(s)\big)\, ds, \tag{3.5}$$

$$y(t) = \Phi^u\big(t, \tau, c^0\big)y^1 + \int_\tau^t \Phi^u\big(t, s, c^0\big)g\big(s, c^0, \eta(s)\big)\,ds, \tag{3.6}$$

$$z(t) = \int_0^t \Phi^c\big(t, s, c^0\big)\big(\theta\big(s, c^0, z(s)\big) + h\big(s, c^0, \eta(s)\big)\big)\,ds. \tag{3.7}$$

For a fixed $\tau > 0$, let $\mathcal{X}$ be the set of continuous functions $\eta : [0, \tau] \to \mathbb{R}^n$, $\eta(t) = (x(t), y(t), z(t))$. On $\mathcal{X}$ we will use several different norms: for $j = 0, \ldots, r$,

$$\|\eta\|_j = \sup_{0 \leqslant t \leqslant \tau} \big(e^{-(\lambda + j\beta)t}\big\|x(t)\big\|, e^{-(\mu - j\beta)(t-\tau)}\big\|y(t)\big\|, e^{-(\lambda + j\beta)t-(\mu - j\beta)(t-\tau)}\big\|z(t)\big\|\big).$$

Let N_0 and N_1 be positive constants defined below, and let

$$\sigma = \min\big(\beta, \lambda - \lambda_0, \mu_0 - \mu, |\lambda + \beta|, \mu - \beta, \lambda + \mu - \beta\big) > 0,$$
$$\delta_0 = \min\left(1, \frac{\sigma}{4M^2 \max(N_0, 4N_1)}\right) > 0,$$
$$\Sigma = \big\{\eta \in \mathcal{X}\colon\ \|\eta\|_0 \leqslant 2M\delta_0\big\}.$$

Given (τ, x^0, y^1, c^0), define $T : \Sigma \to \mathcal{X}$ by the right-hand side of (3.5)–(3.7).

Proposition 3.1. *If $\|x^0\| \leqslant \delta_0$, $\|y^1\| \leqslant \delta_0$, $c^0 \in V_0$, and $\tau > 0$ is in J_{c^0}, then T is a contraction of Σ in the norm $\|\ \|_0$ with contraction constant at most $\frac{1}{2}$.*

To prove Theorem 2.1, we shall first derive, in Section 3.2, some useful estimates. Then, in Section 3.3, we shall prove Proposition 3.1. We shall also show that for $\eta \in \Sigma$, $DT(\eta)$ has norm at most $\frac{1}{2}$ in each norm $\|\cdot\|_j$, $j = 0, \ldots, r$. Finally, in Section 3.4, we study partial derivatives of the fixed point $\eta(t)$ of T with respect to t and the parameters (τ, x^0, y^1, c^0). Each is a fixed point of a nonhomogeneous linear equation. The solution can be estimated using the results of Section 3.2 and the estimate of the norm of $DT(\eta)$.

Actually, the framework we have presented does not allow study of partial derivatives with respect to τ, since τ is used in the definition of the space $\mathcal{X}$ and therefore cannot be treated as a parameter. To get around this difficulty, one can, for example, use a larger τ' in the definition of $\mathcal{X}$, and treat the value τ at which boundary conditions are posed as a parameter; the solution is then defined on $0 \leqslant t \leqslant \tau'$. As is common in studies of this sort, we shall ignore this technicality in the rest of the paper.

3.2. Estimates

Proposition 3.2. *There are constants K_j, $j = 1, \ldots, r + 1$, such that if $\mathbf{i}$ is a j-tuple of integers between 1 and $1 + m$, $c^0 \in V_0$, and $t \in J_{c^0}$, then*

$$\big\|D_{\mathbf{i}}\phi\big(t, c^0\big)\big\| \leqslant K_j e^{j\beta|t|}.$$

Proof. We shall give the proof for $t \geqslant 0$. We have $D_t\phi(t,c^0) = \tilde{C}(\phi(t,c^0))$. Therefore, if i is an integer between 1 and $1+m$,

$$D_t D_i \phi(t,c^0) = D\tilde{C}(\phi(t,c^0)) D_i\phi(t,c^0).$$

The solution of this differential equation is

$$D_i\phi(t,c^0) = \Phi^c(t,0,c^0) D_i\phi(0,c^0),$$

where $D_i\phi(0,c^0)$ is the ith column of the $m \times (1+m)$ matrix

$$\begin{pmatrix} \tilde{C}(c^0) & I \end{pmatrix}.$$

Therefore,

$$\|D_i\phi(t,c^0)\| \leqslant Me^{\beta t} \max(\|\tilde{C}\|_0, 1).$$

Thus the proposition is true for $j=1$.

Assume $2 \leqslant p \leqslant r+1$ and the proposition is true for $j = 1, \ldots, p-1$. Let $\mathbf{i}$ be a p-tuple of integers between 1 and $1+m$. We have

$$D_t D_{\mathbf{i}}\phi(t,c^0) = D\tilde{C}(\phi(t,c^0)) D_{\mathbf{i}}\phi(t,c^0) + \Gamma_{\mathbf{i}}(t,c^0), \tag{3.8}$$

$$\Gamma_{\mathbf{i}}(t,c^0) = \sum a_{j\mathbf{i}^1\ldots\mathbf{i}^j} D^j\tilde{C}(\phi(t,c^0)) D_{\mathbf{i}^1}\phi(t,c^0)\cdots D_{\mathbf{i}^j}\phi(t,c^0) \tag{3.9}$$

for certain constants $a_{j\mathbf{i}^1\ldots\mathbf{i}^j}$; $j = 2,\ldots,p$; $|\mathbf{i}^1|,\ldots,|\mathbf{i}^j|$ are each between 1 and $p-1$; and $\mathbf{i}^1\ldots\mathbf{i}^j$ is a permutation of $\mathbf{i}$, so $|\mathbf{i}^1| + \cdots + |\mathbf{i}^j| = p$. The solution of the differential equation (3.8) is

$$D_{\mathbf{i}}\phi(t,c^0) = \Phi^c(t,0,c^0) D_{\mathbf{i}}\phi(0,c^0) + \int_0^t \Phi(t,s,c^0)\Gamma_{\mathbf{i}}(s,c^0)\,ds.$$

Therefore

$$\|D_{\mathbf{i}}\phi(t,c^0)\| \leqslant Me^{\beta t}\|D_{\mathbf{i}}\phi(0,c^0)\| + \int_0^t Me^{\beta(t-s)}\|\Gamma_{\mathbf{i}}(s,c^0)\|\,ds. \tag{3.10}$$

By the inductive hypothesis,

$$\|D_{\mathbf{i}^1}\phi(t,c^0)\|\cdots\|D_{\mathbf{i}^j}\phi(t,c^0)\| \leqslant K_{|\mathbf{i}^1|}e^{|\mathbf{i}^1|\beta t}\cdots K_{|\mathbf{i}^j|}e^{|\mathbf{i}^j|\beta t} = K_{|\mathbf{i}^1|}\cdots K_{|\mathbf{i}^j|}e^{p\beta t}. \tag{3.11}$$

From (3.11) and (3.9), we see that $\|\Gamma_{\mathbf{i}}(s,c^0)\|$ in (3.10) is bounded by a constant times $e^{p\beta s}$. Therefore the integral in (3.10) is bounded by a constant times $e^{p\beta t}$. If the sequence $\mathbf{i}$ contains no 1's, then $D_{\mathbf{i}}\phi(0,c^0) = 0$. Otherwise $D_{\mathbf{i}}\phi(0,c^0)$ can be calculated from an equation like (3.8) and is bounded by a constant times $e^{(p-1)\beta t}$. The result follows. $\square$

Proposition 3.3. *There are constants M_j, $j = 1, \dots, r$, such that if* $\mathbf{i}$ *is a j-tuple of integers between* 1 *and* $2+m$, $c^0 \in V_0$, *and* $t, s \in J_{c^0}$,

$$\left\| D_{\mathbf{i}} \Phi^s\left(t, s, c^0\right)\right\| \leqslant M_j e^{\lambda_0(t-s)+j\beta t} \quad \text{for } t \geqslant s, \tag{3.12}$$

$$\left\| D_{\mathbf{i}} \Phi^u\left(t, s, c^0\right)\right\| \leqslant M_j e^{\mu_0(t-s)+j\beta t} \quad \text{for } t \leqslant s, \tag{3.13}$$

$$\left\| D_{\mathbf{i}} \Phi^c\left(t, s, c^0\right)\right\| \leqslant M_j e^{\beta(t-s)+j\beta t} \quad \text{for } t \geqslant s. \tag{3.14}$$

Proof. We will prove only (3.12). Let $\mathbf{k}$ be a k-tuple of integers between 1 and $1+m$, with $1 \leqslant k \leqslant r$. We have

$$\begin{aligned} D_{\mathbf{k}} A\left(t, c^0\right) &= D_{\mathbf{k}} \tilde{A}\left(0, 0, \phi\left(t, c^0\right)\right) \\ &= \sum a_{j\mathbf{k}^1\dots\mathbf{k}^j} D_c^j \tilde{A}\left(0, 0, \phi\left(t, c^0\right)\right) D_{\mathbf{k}^1}\phi\left(t, c^0\right) \cdots D_{\mathbf{k}^j}\phi\left(t, c^0\right) \end{aligned}$$

for certain constants $a_{j\mathbf{k}^1\dots\mathbf{k}^j}$; $j = 1, \dots, k$; $|\mathbf{k}^1|, \dots, |\mathbf{k}^j|$ are each between 1 and k; and $\mathbf{k}^1 \dots \mathbf{k}^j$ is a permutation of $\mathbf{k}$, so $|\mathbf{k}^1| + \dots + |\mathbf{k}^j| = k$. Then Proposition 3.2 implies that there are constants $L_1, \dots, L_r$ such that for $k = 1, \dots, r$,

$$\left\| D_{\mathbf{k}} A\left(t, c^0\right)\right\| \leqslant L_k e^{k\beta t}. \tag{3.15}$$

Let i be an integer between 1 and $2+m$. We have $D_t \Phi^s(t, s, c^0) = A(t, c^0)\Phi^s(t, s, c^0)$. Therefore

$$D_t D_i \Phi^s\left(t, s, c^0\right) = A\left(t, c^0\right) D_i \Phi^s\left(t, s, c^0\right) + D_i A\left(t, c^0\right)\Phi^s\left(t, s, c^0\right).$$

The solution is

$$D_i \Phi^s\left(t, s, c^0\right) = \Phi^s\left(t, s, c^0\right) D_i \Phi^s\left(s, s, c^0\right) + \int_s^t \Phi^s\left(t, r, c^0\right) D_i A\left(r, c^0\right)\Phi^s\left(r, s, c^0\right) dr.$$

Therefore

$$\left\| D_i \Phi^s\left(t, s, c^0\right)\right\| \leqslant M e^{\lambda_0(t-s)} \left\| D_i \Phi^s\left(s, s, c^0\right)\right\| + \int_s^t M e^{\lambda_0(t-r)} L_1 e^{\beta r} M e^{\lambda_0(r-s)} dr,$$

where $D_i \Phi^s(s, s, c^0)$ is the ith column of the $m \times (2+m)$ matrix

$$\begin{pmatrix} \tilde{C}\left(c^0\right) & -\tilde{C}\left(c^0\right) & I \end{pmatrix}.$$

Thus (3.12) is true for $j = 1$.

Assume $2 \leqslant p \leqslant r$ and the proposition is true for $j = 1, \dots, p-1$. Let $\mathbf{i}$ be a p-tuple of integers between 1 and $2+m$. We have

$$D_t D_{\mathbf{i}} \Phi^s\big(t,s,c^0\big) = A\big(t,c^0\big) D_{\mathbf{i}} \Phi^s\big(t,s,c^0\big) + \Gamma_{\mathbf{i}}\big(t,s,c^0\big), \tag{3.16}$$

$$\Gamma_{\mathbf{i}}\big(t,s,c^0\big) = \sum a_{\mathbf{kl}} D_{\mathbf{k}} A\big(t,c^0\big) D_{\mathbf{l}} \Phi^s\big(t,s,c^0\big) \tag{3.17}$$

for certain constants $a_{\mathbf{kl}}$; $|\mathbf{k}| \geqslant 0$, $|\mathbf{l}| \geqslant 1$, $\mathbf{kl}$ is a permutation of $\mathbf{i}$. The solution is

$$D_{\mathbf{i}} \Phi^s\big(t,s,c^0\big) = \Phi^s\big(t,s,c^0\big) D_{\mathbf{i}} \Phi^s\big(s,s,c^0\big) + \int_s^t \Phi^s\big(t,r,c^0\big) \Gamma_{\mathbf{i}}\big(r,s,c^0\big)\, dr.$$

Therefore

$$\big\| D_{\mathbf{i}} \Phi^s\big(t,s,c^0\big) \big\| \leqslant M e^{\lambda_0(t-s)} \big\| D_{\mathbf{i}} \Phi^s\big(s,s,c^0\big) \big\| + \int_s^t M e^{\lambda_0(t-r)} \big\| \Gamma_{\mathbf{i}}\big(r,s,c^0\big) \big\|\, ds. \tag{3.18}$$

From (3.15) and the inductive hypothesis,

$$\big\| D_{\mathbf{k}} A\big(r,c^0\big) D_{\mathbf{l}} \Phi^s\big(r,s,c^0\big) \big\| \leqslant L_{|\mathbf{k}|} e^{|\mathbf{k}|\beta r} M_{|\mathbf{l}|} e^{\lambda_0(r-s)+|\mathbf{l}|\beta r} = L_{|\mathbf{k}|} M_{|\mathbf{l}|} e^{\lambda_0(r-s)+p\beta r}. \tag{3.19}$$

From (3.19) and (3.17), we see that $\|\Gamma_{\mathbf{i}}(r,s,c^0)\|$ in (3.18) is bounded by a constant times $e^{\lambda_0(r-s)+p\beta r}$. Therefore the integral in (3.18) is bounded by a constant times $e^{\lambda_0(t-s)+p\beta t}$. If the sequence $\mathbf{i}$ contains no 1's or 2's, then $D_{\mathbf{i}}\Phi^s(s,s,c^0) = 0$. Otherwise $D_{\mathbf{i}}\Phi^s(s,s,c^0)$ can be calculated from an equation like (3.16) and is bounded by a constant times $e^{(p-1)\beta t}$. The result follows. □

Proposition 3.4. *There is a constant N_0 such that for all $c^0 \in V_0$, $t \in J_{c^0}$, and η in a bounded set*:

(1) $\|f(t,c^0,\eta)\| \leqslant N_0 \|\eta\| \|x\|$.
(2) $\|g(t,c^0,\eta)\| \leqslant N_0 \|\eta\| \|y\|$.
(3) $\|\theta(t,c^0,z)\| \leqslant N_0 \|z\|^2$.
(4) $\|h(t,c^0,\eta)\| \leqslant N_0 \|x\| \|y\|$.

Proposition 3.5. *There is a constant N_1 such that the following is true. Let i be an integer between 1 and $1+m+n$, let $c^0 \in V_0$, let $t \in J_{c^0}$, and let η belong to a bounded set. Then*:

(1) *If $i \leqslant 1+m$, then $\|D_i f(t,c^0,\eta)\| \leqslant N_1 \|x\| e^{\beta t}$. If $2+m \leqslant i \leqslant 1+m+k$, then $\|D_i f(t,c^0,\eta)\| \leqslant N_1 \|\eta\|$. For other i, $\|D_i f(t,c^0,\eta)\| \leqslant N_1 \|x\|$.*
(2) *If $i \leqslant 1+m$, then $\|D_i g(t,c^0,\eta)\| \leqslant N_1 \|y\| e^{\beta t}$. If $2+m+k \leqslant i \leqslant 1+m+k+l$, then $\|D_i g(t,c^0,\eta)\| \leqslant N_1 \|\eta\|$. For other i, $\|D_i g(t,c^0,\eta)\| \leqslant N_1 \|y\|$.*
(3) *If $i \leqslant 1+m$, then $\|D_i \theta(t,c^0,z)\| \leqslant N_1 \|z\| e^{\beta t}$. If $2+m+k+l \leqslant i \leqslant n$, then $\|D_{\mathbf{i}} \theta(t,c^0,z)\| \leqslant N_1 \|z\|$. For other i, $\|D_{\mathbf{i}} \theta(t,c^0,z)\| = 0$.*
(4) *If $i \leqslant 1+m$, then $\|D_i h(t,c^0,\eta)\| \leqslant N_1 \|x\| \|y\| e^{\beta t}$. If $2+m \leqslant i \leqslant 1+m+k$, then $\|D_i h(t,c^0,\eta)\| \leqslant N_1 \|y\|$. If $2+m+k \leqslant i \leqslant 1+m+k+l$, then $\|D_i h(t,c^0,\eta)\| \leqslant N_1 \|x\|$. Otherwise, $\|D_i h(t,c^0,\eta)\| \leqslant N_1 \|x\| \|y\|$.*

Proof. We shall only discuss parts (1) and (4) of both propositions. In the definition of $f(t, c^0, \eta)$, the expression $\tilde{A}(x, y, \phi(t, c^0) + z) - \tilde{A}(0, 0, \phi(t, c^0))$ is $O(\eta)$ because $\tilde{A}$ is C^1; this justifies (1) in the first proposition. To treat (1) in the second proposition, note that

$$D_i f(t, c^0, \eta) = D_i\bigl(\tilde{A}\bigl(x, y, \phi\bigl(t, c^0\bigr) + z\bigr) - \tilde{A}\bigl(0, 0, \phi\bigl(t, c^0\bigr)\bigr)\bigr)x + \bigl(\tilde{A}\bigl(x, y, \phi\bigl(t, c^0\bigr) + z\bigr) - \tilde{A}\bigl(0, 0, \phi\bigl(t, c^0\bigr)\bigr)\bigr)D_i x.$$

If $i \leqslant 1 + m$, we see from Proposition 3.2 that the first summand is of order $\|x\|e^{\beta t}$. The second summand is 0. If $2 + m \leqslant i \leqslant 1 + m + k$, the first summand is the product of a bounded term and one of order $\|x\|$, and the second is the product of a term of order $\|\eta\|$ and one that is bounded. Otherwise, the first summand is the product of a bounded term and one of order $\|x\|$, and the second is 0.

To treat (4) in the second proposition, one uses (3.4), noting that E and ϕ are at least C^2, and $E(0, y, z) = E(x, 0, z) = 0$. □

For an integer j with $2 \leqslant j \leqslant r$, let $\mathbf{i}$ be a j-tuple of integers between 1 and $1 + m + n$. Write $\mathbf{i} = \mathbf{kn}$, where $\mathbf{k}$ is all terms that are between 1 and $1 + m$, and $\mathbf{n}$ is all terms that are between $2 + m$ and $1 + m + n$. Similar arguments yield:

Proposition 3.6. *There are constants N_j, $j = 2, \dots, r$, such that the following is true. Let $\mathbf{i} = \mathbf{kn}$ be any j-tuple of integers between 1 and $1 + m + n$, decomposed as above, let $c^0 \in V_0$, and let $t \in J_{c^0}$. Then:*

(1) $\|D_{\mathbf{i}} f(t, c^0, \eta)\| \leqslant N_j \|x\|^{\alpha} e^{|\mathbf{k}|\beta t}$, *where $\alpha = 1$ if no i is between $2 + m$ and $1 + m + k$, and $\alpha = 0$ otherwise.*
(2) $\|D_{\mathbf{i}} g(t, c^0, \eta)\| \leqslant N_j \|y\|^{\gamma} e^{|\mathbf{k}|\beta t}$, *where $\gamma = 1$ if no i is between $2+m+k$ and $1+m+k+l$, and $\gamma = 0$ otherwise.*
(3) $\|D_{\mathbf{i}} \theta(t, c^0, z)\| \leqslant N_j \|z\|^{\alpha} e^{|\mathbf{k}|\beta t}$, *where α is 1 if no i is between $2 + m + k + l$ and n, and $\alpha = 0$ otherwise.*
(4) $\|D_{\mathbf{i}} h(t, c^0, \eta)\| \leqslant N_j \|x\|^{\alpha} \|y\|^{\gamma} e^{|\mathbf{k}|\beta t}$, *where $\alpha = 1$ if no i is between $2 + m$ and $1 + m + k$, and $\alpha = 0$ otherwise; $\gamma = 1$ if no i is between $2 + m + k$ and $1 + m + k + l$, and $\gamma = 0$ otherwise.*

3.3. Proof that T is a contraction

Let (τ, x^0, y^1, c^0) be as above, let $(x, y, z) \in \Sigma$, and let $(\hat{x}, \hat{y}, \hat{z}) = T(x, y, z)$. From the definition of T and Proposition 3.4(1), we have, for $0 \leqslant t \leqslant \tau$,

$$\begin{aligned}\|\hat{x}(t)\| &\leqslant Me^{\lambda_0 t}\|x^0\| + \int_0^t Me^{\lambda_0(t-s)} N_0 \|\eta(s)\| \|x(s)\| \, ds \\ &\leqslant Me^{\lambda_0 t}\delta_0 + \int_0^t Me^{\lambda_0(t-s)} N_0 \cdot 2M\delta_0 \cdot 2M\delta_0 e^{\lambda s} \, ds \\ &\leqslant Me^{\lambda t}\delta_0 + 4M^3 N_0 \delta_0^2 e^{\lambda_0 t}(\lambda - \lambda_0)^{-1} e^{(\lambda - \lambda_0)t} \\ &= Me^{\lambda t}\delta_0\bigl(1 + 4M^2 N_0 \delta_0 \sigma^{-1}\bigr) \leqslant 2M\delta_0 e^{\lambda t}.\end{aligned}$$

Therefore

$$e^{-\lambda t}\left\|\hat{x}(t)\right\| \leqslant 2M\delta_0. \tag{3.20}$$

Similarly,

$$e^{-\mu(t-\tau)}\left\|\hat{y}(t)\right\| \leqslant 2M\delta_0. \tag{3.21}$$

Finally, using Proposition 3.5(3) and (4),

$$\begin{aligned}
\left\|\hat{z}(t)\right\| &\leqslant \int_0^t Me^{\beta(t-s)}\left(N_0\left\|z(s)\right\|^2 + N_0\left\|x(s)\right\|\left\|y(s)\right\|\right)ds \\
&\leqslant \int_0^t Me^{\beta(t-s)}N_0(2M\delta_0)^2\left(e^{2\lambda s+2\mu(s-\tau)} + e^{\lambda s+\mu(s-\tau)}\right)ds \\
&\leqslant \int_0^t 8M^3N_0\delta_0^2e^{\beta(t-s)}e^{\lambda s+\mu(s-\tau)}\,ds \\
&\leqslant 8M^3N_0\delta_0^2e^{\beta t-\mu\tau}(\lambda+\mu-\beta)^{-1}e^{(\lambda+\mu-\beta)t} \\
&\leqslant 8M^3N_0\delta_0^2\sigma^{-1}e^{\lambda t+\mu(t-\tau)} \leqslant 2M\delta_0e^{\lambda t+\mu(t-\tau)}.
\end{aligned}$$

Therefore

$$e^{-\lambda t-\mu(t-\tau)}\left\|\hat{z}(t)\right\| \leqslant 2M\delta_0. \tag{3.22}$$

From (3.20)–(3.22) we see that F maps Σ into itself. F is a contraction by the case $j=0$ of Proposition 3.7 below.

The linearization of $T : \Sigma \to \mathcal{X}$ at $\eta = (x, y, z)$, applied to $\bar{\eta} = (\bar{x}, \bar{y}, \bar{z})$, is the map $DT(\eta)\bar{\eta} = \hat{\bar{\eta}}$ given by

$$\hat{\bar{x}}(t) = \int_0^t \Phi^s\left(t, s, c^0\right)D_\eta f\left(s, c^0, \eta(s)\right)\bar{\eta}(s)\,ds, \tag{3.23}$$

$$\hat{\bar{y}}(t) = \int_\tau^t \Phi^u\left(t, s, c^0\right)D_\eta g\left(s, c^0, \eta(s)\right)\bar{\eta}(s)\,ds, \tag{3.24}$$

$$\hat{\bar{z}}(t) = \int_0^t \Phi^c\left(t, s, c^0\right)\left(D_z\theta\left(s, c^0, z(s)\right)\bar{z}(s) + D_\eta h\left(s, c^0, \eta(s)\right)\bar{\eta}(s)\right)ds. \tag{3.25}$$

Proposition 3.7. *Let $\eta \in \Sigma$ and let $\mathcal{X}$ have one of the norms $\| \|_j$, $j = 0, \ldots, r$. Then $\|DT(\eta)\| \leqslant \frac{1}{2}$.*

Proof. From Proposition 3.5(1),

$$\begin{aligned}
\|\hat{\bar{x}}(t)\| &\leqslant \int_0^t Me^{\lambda_0(t-s)}N_1\big(\|\eta(s)\|\|\bar{x}(s)\| + \|x(s)\|\|\bar{y}(s)\| + \|x(s)\|\|\bar{z}(s)\|\big)\,ds \\
&\leqslant \int_0^t Me^{\lambda_0(t-s)}N_1\cdot 2M\delta_0\cdot e^{(\lambda+j\beta)s}\|\bar{\eta}\|_j\,ds \\
&\quad + \int_0^t Me^{\lambda_0(t-s)}N_1\cdot 2M\delta_0\big(e^{\lambda s}+e^{\lambda s}\big)\cdot\|\bar{\eta}\|_j\,ds \\
&\leqslant 2M^2N_1\delta_0 e^{\lambda_0 t}(\lambda-\lambda_0+j\beta)^{-1}e^{(\lambda-\lambda_0+j\beta)t}\|\bar{\eta}\|_j \\
&\quad + 4M^2N_1\delta_0 e^{\lambda_0 t}(\lambda-\lambda_0)^{-1}e^{(\lambda-\lambda_0)t}\|\bar{\eta}\|_j \\
&\leqslant 6M^2N_1\delta_0\sigma^{-1}e^{\lambda t}\|\bar{\eta}\|_j \leqslant \frac{3}{8}e^{\lambda t}\|\bar{\eta}\|_j.
\end{aligned}$$

Therefore

$$e^{-(\lambda+j\beta)t}\|\hat{\bar{x}}(t)\| \leqslant \frac{3}{8}\|\bar{\eta}\|_j. \tag{3.26}$$

Similarly,

$$e^{-(\mu-j\beta)(t-\tau)}\|\hat{\bar{y}}(t)\| \leqslant \frac{3}{8}\|\bar{\eta}\|_j. \tag{3.27}$$

Finally,

$$\begin{aligned}
\|\hat{\bar{z}}(t)\| &\leqslant \int_0^t Me^{\beta(t-s)}\big(N_1\|z(s)\|\|\bar{z}(s)\| \\
&\quad + N_1\big(\|y(s)\|\|\bar{x}(s)\| + \|x(s)\|\|\bar{y}(s)\| + \|x(s)\|\|y(s)\|\|\bar{z}(s)\|\big)\big)\|\bar{\eta}\|_j\,ds \\
&\leqslant \int_0^t Me^{\beta(t-s)}N_1\cdot 2M\delta_0\big(e^{\lambda s+\mu(s-\tau)}e^{(\lambda+j\beta)s+(\mu-j\beta)(s-\tau)} + e^{\mu(s-\tau)}e^{(\lambda+j\beta)s} \\
&\quad + e^{\lambda s}e^{(\mu-j\beta)(s-\tau)} + e^{\lambda s}e^{\mu(s-\tau)}e^{(\lambda+j\beta)s+(\mu-j\beta)(s-\tau)}\big)\|\bar{\eta}\|_j\,ds \\
&\leqslant \int_0^t 8M^2N_1\delta_0 e^{\beta(t-s)}e^{(\lambda+j\beta)s+(\mu-j\beta)(s-\tau)}\|\bar{\eta}\|_j\,ds \\
&\leqslant 8M^2N_1\delta_0 e^{\beta t-(\mu-j\beta)\tau}(\lambda+\mu-\beta)^{-1}e^{(\lambda+\mu-\beta)t}\|\bar{\eta}\|_j \\
&\leqslant 8M^2N_1\delta_0\sigma^{-1}e^{\lambda t+(\mu-j\beta)(t-\tau)}\|\bar{\eta}\|_j \leqslant \frac{1}{2}e^{\lambda t+(\mu-j\beta)(t-\tau)}\|\bar{\eta}\|_j.
\end{aligned}$$

Therefore

$$e^{-(\lambda+j\beta)t-(\mu-j\beta)(t-\tau)}\left\|\hat{\hat{z}}(t)\right\| \leqslant \frac{1}{2}\|\eta\|_j. \tag{3.28}$$

The result follows from (3.26)–(3.28). □

3.4. Differentiability

Let $\mathbf{i}$ be an $|\mathbf{i}|$-tuple of integers between 1 and $2+n$, with $1 \leqslant |\mathbf{i}| \leqslant r$. From (3.5)–(3.7), $D_{\mathbf{i}}\eta(t,\tau,x^0,y^1,c^0)$ satisfies the following system:

$$D_{\mathbf{i}}x\big(t,\tau,x^0,y^1,c^0\big) = \int_0^t \Phi^s\big(t,s,c^0\big)D_\eta f\big(s,c^0,\eta\big(t,\tau,x^0,y^1,c^0\big)\big)D_{\mathbf{i}}\eta\big(t,\tau,x^0,y^1,c^0\big)\,ds + \Gamma_{\mathbf{i}1}\big(t,\tau,x^0,y^1,c^0\big), \tag{3.29}$$

$$D_{\mathbf{i}}y\big(t,\tau,x^0,y^1,c^0\big) = \int_\tau^t \Phi^u\big(t,s,c^0\big)D_\eta g\big(s,c^0,\eta\big(t,\tau,x^0,y^1,c^0\big)\big)D_{\mathbf{i}}\eta\big(t,\tau,x^0,y^1,c^0\big)\,ds + \Gamma_{\mathbf{i}2}\big(t,\tau,x^0,y^1,c^0\big), \tag{3.30}$$

$$\begin{aligned} D_{\mathbf{i}}z\big(t,\tau,x^0,y^1,c^0\big) = \int_0^t \Phi^c\big(t,s,c^0\big)\big(&D_z\theta\big(s,c^0,z\big(t,\tau,x^0,y^1,c^0\big)\big)D_{\mathbf{i}}z\big(t,\tau,x^0,y^1,c^0\big) \\ &+ D_\eta h\big(s,c^0,\eta\big(t,\tau,x^0,y^1,c^0\big)\big)D_{\mathbf{i}}\eta\big(t,\tau,x^0,y^1,c^0\big)\big)\,ds \\ &+ \Gamma_{\mathbf{i}3}\big(t,\tau,x^0,y^1,c^0\big). \end{aligned} \tag{3.31}$$

We have

$$\begin{aligned} &\Gamma_{\mathbf{i}1}\big(t,\tau,x^0,y^1,c^0\big) \\ &\quad = D_{\mathbf{i}}\big(\Phi^s\big(t,0,c^0\big)x^0\big) + \int_0^t \sum a_{\mathbf{jkl}^1\ldots\mathbf{l}^{|\mathbf{n}|}} D_{\mathbf{j}}\Phi^s\big(t,s,c^0\big)D_{\mathbf{k}}f\big(s,c^0,\eta\big(s,\tau,x^0,y^1,c^0\big)\big) \\ &\quad\quad \times D_{\mathbf{l}^1}\eta_{n_1}\big(s,\tau,x^0,y^1,c^0\big)\cdots D_{\mathbf{l}^{|\mathbf{n}|}}\eta_{n_{|\mathbf{n}|}}\big(s,\tau,x^0,y^1,c^0\big)\,ds \end{aligned} \tag{3.32}$$

for certain constants $a_{\mathbf{jkl}^1\ldots\mathbf{l}^{|\mathbf{n}|}}$, where

(C1) $\mathbf{j}$ is a $|\mathbf{j}|$-tuple of integers between 1 and $2+m$, none of which is 2;
(C2) $\mathbf{k}$ is a $|\mathbf{k}|$-tuple of integers between 2 and $1+m+n$;
(C3) $\mathbf{k}=\mathbf{mn}$, where $\mathbf{m}$ is all terms that are between 2 and $1+m$, and $\mathbf{n}$ is all terms that are between $2+m$ and $1+m+n$;
(C4) $\mathbf{n}'=(n_1,\ldots,n_{|\mathbf{n}|})$ is $\mathbf{n}$ with the numbers decreased by $1+m$, so that they are all between 1 and n;
(C5) $\mathbf{l}^1\ldots\mathbf{l}^{|\mathbf{n}|}$ is each a sequence of integers between 2 and $2+n$;

(C6) $|\mathbf{j}|+|\mathbf{m}|+|\mathbf{l}^1|+\cdots+|\mathbf{l}^{|\mathbf{n}|}|=|\mathbf{i}|$;
(C7) $|\mathbf{j}|+|\mathbf{k}|\leqslant|\mathbf{i}|$;
(C8) if $\mathbf{j}=\mathbf{m}=\emptyset$ and $|\mathbf{n}|=1$, in which case we must have $\mathbf{l}^1=\mathbf{i}$, then $a_{\mathbf{jki}}=0$.

Similarly,

$$\begin{aligned}
&\Gamma_{\mathbf{i}2}\big(t,\tau,x^0,y^1,c^0\big)\\
&\quad= D_{\mathbf{i}}\big(\Phi^u\big(t,\tau,c^0\big)y^1\big)+\int_\tau^t\sum a_{\mathbf{jkl}^1\ldots\mathbf{l}^{|\mathbf{n}|}}D_{\mathbf{j}}\Phi^u\big(t,s,c^0\big)D_{\mathbf{k}}g\big(s,c^0,\eta\big(s,\tau,x^0,y^1,c^0\big)\big)\\
&\qquad\times D_{\mathbf{l}^1}\eta_{n_1}\big(s,\tau,x^0,y^1,c^0\big)\cdots D_{\mathbf{l}^{|\mathbf{n}|}}\eta_{n_{|\mathbf{n}|}}\big(s,\tau,x^0,y^1,c^0\big)\,ds
\end{aligned}\tag{3.33}$$

and

$$\begin{aligned}
&\Gamma_{\mathbf{i}3}\big(t,\tau,x^0,y^1,c^0\big)\\
&\quad=\int_0^t\sum a_{\mathbf{jkl}^1\ldots\mathbf{l}^{|\mathbf{n}|}}D_{\mathbf{j}}\Phi^c\big(t,s,c^0\big)D_{\mathbf{k}}\big(\theta\big(s,c^0,z\big(s,\tau,x^0,y^1,c^0\big)\big)+h\big(s,c^0,\eta\big(s,\tau,x^0,y^1,c^0\big)\big)\big)\\
&\qquad\times D_{\mathbf{l}^1}\eta_{n_1}\big(s,\tau,x^0,y^1,c^0\big)\cdots D_{\mathbf{l}^{|\mathbf{n}|}}\eta_{n_{|\mathbf{n}|}}\big(s,\tau,x^0,y^1,c^0\big)\,ds
\end{aligned}\tag{3.34}$$

with similar provisos.

Thus $D_{\mathbf{i}}\eta$ satisfies the linear equation

$$U=AU+\Gamma_{\mathbf{i}}(t),\tag{3.35}$$

with $U=(X,Y,Z)$,

$$\begin{aligned}
AU(t)=\Bigg(&\int_0^t\Phi^s\big(t,s,c^0\big)D_\eta f\big(s,c^0,\eta\big(t,\tau,x^0,y^1,c^0\big)\big)U(t)\,ds,\\
&\int_\tau^t\Phi^u\big(t,s,c^0\big)D_\eta g\big(s,c^0,\eta\big(t,\tau,x^0,y^1,c^0\big)\big)U(t)\,ds,\\
&\int_0^t\Phi^c\big(t,s,c^0\big)\big(D_z\theta\big(s,c^0,z\big(t,\tau,x^0,y^1,c^0\big)\big)Z(t)\\
&\quad+D_\eta h\big(s,c^0,\eta\big(t,\tau,x^0,y^1,c^0\big)\big)U(t)\big)\,ds\Bigg)
\end{aligned}$$

and $\Gamma_{\mathbf{i}}(t)=(\Gamma_{\mathbf{i}1}(t),\Gamma_{\mathbf{i}2}(t),\Gamma_{\mathbf{i}3}(t))$.

To complete the proof of Theorem 2.1, we consider the following statements (A_k), (B_k), $k=1,\ldots,r$:

(A_k) There is a constant P_k such that if $\|x^0\| \leqslant \delta_0$, $\|y^1\| \leqslant \delta_0$, $c^0 \in V_0$, $\tau > 0$ is in J_{c^0}, and $|\mathbf{i}| = k$ then $\|\Gamma_{\mathbf{i}}\|_k \leqslant P_k$.

(B_k) Under the same assumptions, $\|D_{\mathbf{i}}\eta\|_k \leqslant 2P_k$.

We first show (A_1). We will consider only $\Gamma_{\mathbf{i}1}(t)$ given by (3.32), with $|\mathbf{i}| = 1$. From (3.12) it is easy to see that $\|D_{\mathbf{i}}(\Phi^s(t,0,c^0)x^0)\|$ is at most a multiple of $e^{(\lambda_0+\beta)t}$. To estimate the integral, we note that there are two types of summands: (1) $|\mathbf{j}| = 1$, $\mathbf{m} = \emptyset$, and (2) $\mathbf{j} = \emptyset$, $|\mathbf{m}| = 1$. (The case $\mathbf{j} = \mathbf{m} = \emptyset$ is ruled out by (C8).)

For a summand of the first type, $\|D_{\mathbf{j}}\Phi^s(t,s,c^0)\| \leqslant M_1 e^{\lambda_0(t-s)+\beta t}$ by (3.12), and

$$\bigl\| f\bigl(s, c^0, \eta\bigl(s, \tau, x^0, y^1, c^0\bigr)\bigr)\bigr\| \leqslant N_0 \bigl\|\eta(s)\bigr\| \bigl\|x(s)\bigr\|$$

by Proposition 3.4. The other terms are not present. Therefore, since $\delta_0 \leqslant 1$, the integral of one summand is at most

$$\int_0^t M_1 e^{\lambda_0(t-s)+\beta t} N_0 e^{\lambda s}\, ds \leqslant M_1 N_0 e^{(\lambda_0+\beta)t}(\lambda - \lambda_0)^{-1} e^{(\lambda-\lambda_0)t} \leqslant M_1 N_0 \sigma^{-1} e^{(\lambda+\beta)t}.$$

The second case is similar. From these estimates, it follows that $e^{-(\lambda+\beta)t}\|\Gamma_{\mathbf{i}1}(t)\|$ is bounded.

Next we show that (A_k) implies (B_k). Let $\mathcal{X}$ have the norm $\|\ \|_k$, and regard the right-hand side of (3.35) as an affine linear map from $\mathcal{X}$ to itself. By Proposition 3.7, $\|A\| \leqslant \frac{1}{2}$. The result follows.

Finally we prove that for $p = 2, \dots, r$, (B_1), $\dots$, (B_{p-1}) together imply (A_p). Then all (A_k) and (B_k) are true, and Theorem 2.1 is proved.

Assume (B_1), $\dots$, (B_{p-1}) and let $|\mathbf{i}| = p$. We first estimate $\|\Gamma_{\mathbf{i}1}(t)\|$ given by (3.32). From Proposition 3.3 and the assumption that $\|x^0\| \leqslant \delta_0 \leqslant 1$,

$$\bigl\| D_{\mathbf{i}}\bigl(\Phi^s\bigl(t, 0, c^0\bigr)x^0\bigr)\bigr\| \leqslant M_{|\mathbf{i}|} e^{\lambda_0(t-s)+|\mathbf{i}|t}. \tag{3.36}$$

To estimate the integral in (3.32), we must estimate

$$\int_0^t \bigl\| D_{\mathbf{j}}\Phi^s\bigl(t, s, c^0\bigr)\bigr\| \bigl\| D_{\mathbf{k}} f\bigl(s, c^0, \eta\bigl(s, \tau, x^0, y^1, c^0\bigr)\bigr)\bigr\| \\ \times \bigl\| D_{\mathbf{l}^1}\eta_{n_1}\bigl(s, \tau, x^0, y^1, c^0\bigr)\bigr\| \cdots \bigl\| D_{\mathbf{l}^{|\mathbf{n}|}}\eta_{n_{|\mathbf{n}|}}\bigl(s, \tau, x^0, y^1, c^0\bigr)\bigr\|\, ds. \tag{3.37}$$

From Proposition 3.3,

$$\bigl\| D_{\mathbf{j}}\Phi^s\bigl(t, s, c^0\bigr)\bigr\| \leqslant M_{|\mathbf{j}|} e^{\lambda_0(t-s)+|\mathbf{j}|\beta t}. \tag{3.38}$$

By the induction hypothesis,

$$\bigl\| D_{\mathbf{l}^1}\eta_{n_1}\bigl(s, \tau, x^0, y^1, c^0\bigr)\bigr\| \leqslant 2P_{|\mathbf{l}^1|}, \quad \dots, \quad \bigl\| D_{\mathbf{l}^{|\mathbf{n}|}}\eta_{n_{|\mathbf{n}|}}\bigl(s, \tau, x^0, y^1, c^0\bigr)\bigr\| \leqslant 2P_{|\mathbf{l}^{|\mathbf{n}|}|}. \tag{3.39}$$

If no n_i is between 1 and k, then by Proposition 3.6(1),

$$\bigl\| D_{\mathbf{k}} f\bigl(s, c^0, \eta\bigl(s, \tau, x^0, y^1, c^0\bigr)\bigr)\bigr\| \leqslant N_{|\mathbf{k}|} \bigl\|x(s)\bigr\| e^{|\mathbf{k}|\beta s} \leqslant N_{|\mathbf{k}|} e^{(\lambda+|\mathbf{k}|\beta)s}. \tag{3.40}$$

Let $P = 2^{|\mathbf{n}|} M_{|\mathbf{j}|} N_{|\mathbf{k}|} P_{|\mathbf{l}^1|} \cdots P_{|\mathbf{l}^{|\mathbf{n}|}|}$. Then (3.37) is less than or equal to

$$\int_0^t P e^{\lambda_0 (t-s)+|\mathbf{j}|\beta t} e^{(\lambda+|\mathbf{k}|\beta)s}\, ds \leqslant P e^{(\lambda_0+|\mathbf{j}|\beta)t} \big(\lambda + |\mathbf{k}|\beta - \lambda_0\big)^{-1} e^{(\lambda+|\mathbf{k}|\beta-\lambda_0)t}$$

$$\leqslant P\sigma^{-1} e^{(\lambda+(|\mathbf{j}|+|\mathbf{k}|)\beta)t} \leqslant P\sigma^{-1} e^{(\lambda+p\beta)t}.$$

If some n_i is between 1 and k, then by Proposition 3.6(1), (3.40) must be replaced by

$$\big\| D_{\mathbf{k}} f\big(s, c^0, \eta\big(s, \tau, x^0, y^1, c^0\big)\big)\big\| \leqslant N_{|\mathbf{k}|} e^{|\mathbf{k}|\beta s}.$$

Suppose, for example, that $n_1 \leqslant k$. Then, fortunately, the first estimate in (3.39) can be replaced by

$$\big\| D_{\mathbf{l}^1} \eta_{n_1}\big(s, \tau, x^0, y^1, c^0\big)\big\| \leqslant P_{|\mathbf{l}^1|} e^{(\lambda+|\mathbf{l}^1|\beta)s}.$$

We obtain the same result.

Finally we estimate $\|\Gamma_{\mathbf{i}3}(t)\|$ given by (3.34). We must estimate

$$\int_0^t \big\| D_{\mathbf{j}} \Phi^c\big(t, s, c^0\big)\big\| \big(\big\| D_{\mathbf{k}} \theta\big(s, c^0, z\big(s, \tau, x^0, y^1, c^0\big)\big)\big\| + \big\| D_{\mathbf{k}} h\big(s, c^0, \eta\big(s, \tau, x^0, y^1, c^0\big)\big)\big\|\big)$$
$$\times \big\| D_{\mathbf{l}^1} \eta_{n_1}\big(s, \tau, x^0, y^1, c^0\big)\big\| \cdots \big\| D_{\mathbf{l}^{|\mathbf{n}|}} \eta_{n_{|\mathbf{n}|}}\big(s, \tau, x^0, y^1, c^0\big)\big\|\, ds. \tag{3.41}$$

From Proposition 3.3,

$$\big\| D_{\mathbf{j}} \Phi^c\big(t, s, c^0\big)\big\| \leqslant M_{|\mathbf{j}|} e^{\beta(t-s)+|\mathbf{j}|\beta t}. \tag{3.42}$$

By the induction hypothesis, we again have (3.39). If no n_i is greater than $k + l$, then by Proposition 3.6(3),

$$\big\| D_{\mathbf{k}} \theta\big(s, c^0, z\big(s, \tau, x^0, y^1, c^0\big)\big)\big\| \leqslant N_{|\mathbf{k}|} \big\| z(s)\big\| e^{|\mathbf{k}|\beta s} \leqslant N_{|\mathbf{k}|} e^{\lambda s+\mu(s-\tau)+|\mathbf{k}|\beta s}. \tag{3.43}$$

If some n_i is greater than $k + l$, then (3.43) must be replaced by

$$\big\| D_{\mathbf{k}} \theta\big(s, c^0, z\big(s, \tau, x^0, y^1, c^0\big)\big)\big\| \leqslant N_{|\mathbf{k}|} e^{|\mathbf{k}|\beta s}.$$

Suppose, for example, that $n_{|\mathbf{n}|} > k + l$. Then, fortunately, the last estimate in (3.39) can be replaced by

$$\big\| D_{\mathbf{l}^{|\mathbf{n}|}} \eta_{n_{|\mathbf{n}|}}\big(s, \tau, x^0, y^1, c^0\big)\big\| \leqslant P_{|\mathbf{l}^{|\mathbf{n}|}|} e^{\lambda s+\mu(s-\tau)+|\mathbf{l}^1|\beta s}.$$

We obtain the same result.

The term $\|D_{\mathbf{k}}h(s, c^0, \eta(s, \tau, x^0, y^1, c^0))\|$ is dealt similarly, using Proposition 3.6(4) and separately considering the cases (1) no n_i is less than or equal to $k+l$; (2) at least one n_i is less than or equal to k, but none is greater than k and less than or equal to $k+l$; (3) no n_i is less than or equal to k, but at least one is greater than k and less than or equal to $k+l$; (4) at least one n_i is less than or equal to k, and at least one is greater than k and less than or equal to $k+l$.

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