Journal of Dynamics and Differential Equations, Vol. 14, No. 2, April 2002 (© 2002)

Codimension-One Riemann Solutions: Missing Rarefactions Adjacent to Doubly Sonic Transitional Waves

Stephen Schecter[1]

Received January 11, 2000

This paper is the fifth in a series that undertakes a systematic investigation of Riemann solutions of systems of two conservation laws in one spatial dimension. In this paper, three degeneracies that can occur only in Riemann solutions that contain doubly sonic transitional shock waves, together with the degeneracies that pair with them, are studied in detail. Conditions for a codimension-one degeneracy are identified in each case, as are conditions for folding of the Riemann solution surface. Simple examples are given, including a numerically computed Riemann solution that contains a doubly sonic transitional shock wave.

KEY WORDS: Conservation law; Riemann problem; viscous profile.

1. INTRODUCTION

We consider systems of two conservation laws in one space dimension, partial differential equations of the form

$$U_t + F(U)_x = 0 \tag{1.1}$$

with $t > 0$, $x \in \mathbb{R}$, $U(x, t) \in \mathbb{R}^2$, and $F: \mathbb{R}^2 \to \mathbb{R}^2$ a smooth map. The most basic initial-value problem for Eq. (1.1) is the *Riemann problem*, in which the initial data are piecewise constant with a single jump at $x = 0$:

$$U(x, 0) = \begin{cases} U_L & \text{for} \quad x < 0 \\ U_R & \text{for} \quad x > 0 \end{cases} \tag{1.2}$$

[1] Mathematics Department, North Carolina State University, Box 8205 Raleigh, North Carolina 27695 919-515-6533. E-mail: schecter@math.ncsu.edu

1040-7294/02/0400-0295/0

This paper is the fifth in a series of papers on the structure of solutions of Riemann problems; the previous ones are [6–9].

We seek piecewise continuous weak solutions of Riemann problems in the scale-invariant form $U(x, t) = \hat{U}(x/t)$ consisting of a finite number of constant parts, continuously changing parts *(rarefaction waves)*, and jump discontinuities *(shock waves)*. Shock waves occur when

$$\lim_{\xi \to s-} \hat{U}(\xi) = U_- \neq U_+ = \lim_{\xi \to s+} \hat{U}(\xi) \tag{1.3}$$

They are required to satisfy the following *viscous profile admissibility criterion*: a shock wave is admissible provided that the ordinary differential equation

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \tag{1.4}$$

has a heteroclinic solution, or a finite sequence of such solutions, leading from the equilibrium U_- to a second equilibrium U_+. If there is a single heteroclinic solution of (1.4) from U_- to U_+, then the viscous regularization of (1.1),

$$U_t + F(U)_x = \varepsilon U_{xx}$$

has a traveling wave with speed s that connects U_- to U_+.

In [6] it is shown that a structurally stable Riemann solution can include *doubly sonic transitional shock waves*, which are represented by saddle-attractor to repeller-saddle heteroclinic orbits; see Fig. 1. These shock waves play a unique role in structurally stable Riemann solutions: they separate sequences of waves that typically form complete Riemann solutions. As far as I know, doubly sonic transitional shock waves have not yet been observed in Riemann solutions to systems that have arisen in applications. Moreover, they cannot occur when the flux function F is

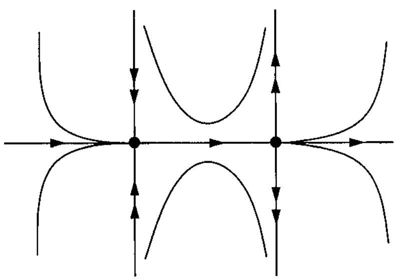


Fig. 1. A saddle-attractor to repeller-saddle heteroclinic orbit.

quadratic, a case that has been intensively studied (see [1, 2], and references cited therein). However, [7] identifies two bifurcations by which a structurally stable Riemann solution containing a transitional wave group can metamorphose, as the left or right state of the Riemann solution varies, into a Riemann solution that contains a doubly sonic transitional shock wave. Transitional wave groups, the composite analogue of saddle-to-saddle "undercompressive" shock waves, arise in mathematical models for three-phase flow in a porous medium [4, 5, and 10].

In this paper we do three things:

- We study in detail the codimension-one Riemann solutions that occur when a rarefaction that precedes or follows a doubly sonic transitional shock wave in a structurally stable Riemann solution shrinks to zero strength. At the same time we treat the degeneracies that pair with these to continue the Riemann solution. In two cases the continuation of the Riemann solution includes a transitional wave group but does not include a doubly sonic transitional shock wave; these are the two bifurcations just mentioned. This study completes the the program begun in [8, 9] to study codimension-one Riemann solutions in which a rarefaction in a structurally stable Riemann solution shrinks to zero strength.
- We give a simple example of a structurally stable Riemann solution that contains a doubly sonic transitional shock wave, and we show that the Riemann solution computed by a standard upwind scheme contains this shock wave. Such an example should be useful since these waves are so unfamiliar.
- We give simple examples of the Riemann solution bifurcations studied in this paper.

In a structurally stable Riemann solution, a doubly sonic transitional shock wave is preceded by a 2-rarefaction and followed by a 1-rarefaction. Either may shrink to zero strength; the two possibilities are dual and need not both be studied. If the 2-rarefaction shrinks to zero strength, there are three possibilities:

- The 2-rarefaction is itself a complete 2-wave group in the Riemann solution.
- The 2-rarefaction is part of a 2-wave group consisting of two waves. Thus it is preceded by a shock wave represented by a saddle to saddle-attractor connection.
- The 2-rarefaction is part of a 2-wave group consisting of more than two waves. It is thus preceded by a shock wave represented by a saddle-attractor to saddle-attractor connection.

The structurally stable Riemann solutions into which these metamorphose when the codimension-one boundary is crossed are as follows [7]. In the first case, the 2-rarefaction and the doubly sonic wave are replaced by a single saddle to repeller-saddle shock wave, which now begins a transitional wave group. In the second case, the saddle to saddle-attractor wave, the 2-rarefaction, and the doubly sonic wave are replaced by a single saddle to repeller-saddle shock wave, which now begins a transitional wave group. In the third case, the saddle-attractor to saddle-attractor shock wave, the 2-rarefaction, and the doubly sonic transitional shock wave are replaced by a single doubly sonic transitional shock wave. The 2-wave group is thus two waves shorter.

The third case is necessarily an *F-boundary*: the degenerate Riemann solution persists under perturbation of U_L and U_R, but exists only along a codimension-one surface in F-space. To my knowledge this type of codimension-one Riemann solution, whose existence was pointed out in [7], has not previously been analyzed. The first and second cases, by contrast, can occur as U_L-boundaries. In the second and third cases, it can happen that both types of Riemann solutions are defined for Riemann data on the same side of the bifurcation surface, so that we do not have local existence and uniqueness of Riemann solutions in (U_L, U_R, F)-space.

The remainder of the paper is organized as follows. In Sections 2 and 3 we review terminology and results about structurally stable Riemann solutions and codimension-one Riemann solutions from [6, 7]. In Sections 4–6 we treat in detail the three missing rarefaction cases. (Section 6 actually deals with the dual of the third case mentioned above.) In Section 7 we give a number of examples.

2. BACKGROUND ON STRUCTURALLY STABLE RIEMANN SOLUTIONS

We consider the system (1.1) with $t \in \mathbb{R}^+$, $x \in \mathbb{R}$, $U(x, t) \in \mathbb{R}^2$, and $F: \mathbb{R}^2 \to \mathbb{R}^2$ a C^2 map. Let

$$\mathcal{U}_F = \{U \in \mathbb{R}^2 : DF(U) \text{ has distinct real eigenvalues}\} \tag{2.1}$$

be the *strictly hyperbolic region* in state-space. We shall call a Riemann solution $\hat{U}$ *strictly hyperbolic* if $\hat{U}(\xi) \in \mathcal{U}_F$ for all $\xi \in \mathbb{R}$. In this paper, all Riemann solutions are assumed to be strictly hyperbolic. For $U \in \mathcal{U}_F$, let $\lambda_1(U) < \lambda_2(U)$ denote the eigenvalues of $DF(U)$, and let $\ell_i(U)$ and $r_i(U)$, $i = 1, 2$, denote corresponding left and right eigenvectors with $\ell_i(U)\, r_j(U) = \delta_{ij}$.

A *rarefaction wave* of *type* R_i is a differentiable map $\hat{U}: [a, b] \to \mathcal{U}_F$, where $a < b$, such that $\hat{U}'(\xi)$ is a multiple of $r_i(\hat{U}(\xi))$ and $\xi = \lambda_i(\hat{U}(\xi))$ for each $\xi \in [a, b]$. The states $U = \hat{U}(\xi)$ with $\xi \in [a, b]$ comprise the *rarefaction curve* $\bar{\Gamma}$. The definition of rarefaction wave implies that if $U \in \bar{\Gamma}$, then

$$D\lambda_i(U)\, r_i(U) = \ell_i(U)\, D^2F(U)(r_i(U), r_i(U)) \neq 0 \tag{2.2}$$

Condition (2.2) is *genuine nonlinearity* of the ith characteristic line field at U. Assuming (2.2), we can choose $r_i(U)$ such that

$$D\lambda_i(U)\, r_i(U) = 1 \tag{2.3}$$

In this paper we shall assume that wherever (2.2) holds, $r_i(U)$ is chosen so that (2.3) holds. The *speed* s of a rarefaction wave of type R_1 is $s = \lambda_1(U_+)$; for a rarefaction wave of type R_2, $s = \lambda_2(U_-)$.

A *shock wave* consists of a *left state* $U_- \in \mathscr{U}_F$, a *right state* $U_+ \in \mathscr{U}_F$, a *speed* s, and a *connecting orbit* Γ, which corresponds to a solution of the ordinary differential equation (1.4) from U_- to U_+. For any equilibrium $U \in \mathscr{U}_F$ of (1.4), the eigenvalues of the linearization at U are $\lambda_i(U) - s$, $i = 1, 2$. We shall use the terminology defined in Table I for such an equilibrium. The *type* of a shock wave is determined by the equilibrium types of its left and right states. (For example, w is of type $R \cdot S$ if its connecting orbit joins a repeller to a saddle.)

An *allowed sequence of elementary waves* or a *Riemann solution* consists of a sequence of waves $(w_1, w_2, \ldots, w_n)$ with increasing wave speeds. We write

$$(w_1, w_2, \ldots, w_n)\colon U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n \tag{2.4}$$

The *type* of $(w_1, w_2, \ldots, w_n)$ is $(T_1, T_2, \ldots, T_n)$ if w_i has type T_i.

Let

$$(w_1^*, w_2^*, \ldots, w_n^*)\colon U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{2.5}$$

Table I. Types of Equilibria

name	symbol	eigenvalues
Repeller	R	$+ \; +$
Repeller-Saddle	RS	$0 \; +$
Saddle	S	$- \; +$
Saddle-Attractor	SA	$- \; 0$
Attractor	A	$- \; -$

be a Riemann solution for $U_t+F^*(U)_x=0$. The Riemann solution (2.5) is *structurally stable* if there are neighborhoods $\mathscr{U}_i$ of U_i^*, $\mathscr{I}_i$ of s_i^*, and $\mathscr{F}$ of F^* in an appropriate Banach space (see [6]), a compact set $\mathscr{K}$ in $\mathbb{R}^2$, and a C^1 map

$$G\colon \mathscr{U}_0 \times \mathscr{I}_1 \times \mathscr{U}_1 \times \mathscr{I}_2 \times \cdots \times \mathscr{I}_n \times \mathscr{U}_n \times \mathscr{F} \to \mathbb{R}^{3n-2}$$

with $G(U_0^*, s_1^*, U_1^*, s_2^*, \dots, s_n^*, U_n^*, F^*)=0$ such that:

(P1) $G(U_0, s_1, U_1, s_2, \dots, s_n, U_n, F)=0$ implies that there exists a Riemann solution

$$(w_1, w_2, \dots, w_n)\colon U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n$$

for $U_t+F(U)_x=0$ with successive waves of the same types as those of the wave sequence (2.5) and with each $\bar{\Gamma}_i$ contained in $\operatorname{Int} \mathscr{K}$;

(P2) $DG(U_0^*, s_1^*, U_1^*, s_2^*, \dots, s_n^*, U_n^*, F^*)$, restricted to the $(3n-2)$-dimensional space of vectors $\{(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dots, \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = 0 = \dot{U}_n, \dot{F}=0\}$, is an isomorphism onto $\mathbb{R}^{3n-2}$.

Condition (P2) implies, by the implicit function theorem, that $G^{-1}(0)$ is a graph over $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$; $(s_1, U_1, \dots, U_{n-1}, s_n)$ is determined by (U_0, U_n, F). We also require that

(P3) $(w_1, w_2, \dots, w_n)$ can be chosen so each $\bar{\Gamma}_i$ depends continuously on (U_0, U_n, F).

Associated with each type of elementary wave is a *local defining map*, which we use to construct maps G that exhibit structural stability. Let $w^*\colon U_-^* \xrightarrow{s^*} U_+^*$ be an elementary wave of type T for $U_t+F^*(U)_x=0$. The local defining map G_T has as its domain a set of the form $\mathscr{U}_- \times \mathscr{I} \times \mathscr{U}_+ \times \mathscr{F}$ (with $\mathscr{U}_\pm$ being neighborhoods of $U_\pm^*$, $\mathscr{I}$ a neighborhood of s^*, and $\mathscr{F}$ a neighborhood of F^*). The range is some $\mathbb{R}^e$; the number e depends only on the wave type T. The local defining map is such that $G_T(U_-^*, s^*, U_+^*, F^*) = 0$. Moreover, if certain *wave nondegeneracy conditions* are satisfied at (U_-^*, s^*, U_+^*, F^*), then there is a neighborhood $\mathscr{N}$ of $\bar{\Gamma}^*$ such that:

(D1) $G_T(U_-, s, U_+, F)=0$ if and only if there exists an elementary wave $w\colon U_- \xrightarrow{s} U_+$ of type T for $U_t+F(U)_x=0$ with $\bar{\Gamma}$ contained in $\mathscr{N}$;

(D2) $DG_T(U_-^*, s^*, U_+^*, F^*)$, restricted to the space $\{(\dot{U}_-, \dot{s}, \dot{U}_+, \dot{F}) : \dot{F} = 0\}$, is surjective.

Condition (D2) implies, by the implicit function theorem, that $G_T^{-1}(0)$ is a manifold of codimension e. In fact,

(D3) w can be chosen so that $\bar{\Gamma}$ varies continuously on $G_T^{-1}(0)$.

We now discuss local defining maps and nondegeneracy conditions for the types of elementary waves that occur in this paper.

First we consider rarefactions. For $i = 1, 2$, let

$$\mathcal{U}_i = \{U \in \mathcal{U} : D\lambda_i(U)\, r_i(U) \neq 0\}$$

In $\mathcal{U}_i$ we assume that Eq. (2.3) holds. For each $U_- \in \mathcal{U}_1$, define ψ_1 to be the solution of

$$\frac{\partial \psi_1}{\partial s}(U_-, s) = r_1(\psi_1(U_-, s))$$

$$\psi_1(U_-, \lambda_1(U_-)) = U_-$$

By (2.3), if $\psi_1(U_-, s) = U$, then $s = \lambda_1(U)$. Thus there is a rarefaction wave of type R_1 for $U_t + F(U)_x = 0$ from U_- to U_+ with speed s if and only if

$$U_+ - \psi_1(U_-, s) = 0 \tag{2.9}$$

$$s = \lambda_1(U_+) > \lambda_1(U_-) \tag{2.10}$$

Similarly, for each $U_+ \in \mathcal{U}_2$, define ψ_2 to be the solution of

$$\frac{\partial \psi_2}{\partial s}(s, U_+) = r_2(\psi_2(s, U_+))$$

$$\psi_2(\lambda_2(U_+), U_+) = U_+$$

By (2.3), if $\psi_2(s, U_+) = U$, then $s = \lambda_2(U)$. Thus there is a rarefaction wave of type R_2 for $U_t + F(U)_x = 0$ from U_- to U_+ with speed s if and only if

$$U_- - \psi_2(s, U_+) = 0 \tag{2.11}$$

$$s = \lambda_2(U_-) < \lambda_2(U_+) \tag{2.12}$$

Equations (2.9) (resp. (2.11)) are defining equations for rarefaction waves of type R_1 (resp. R_2). The nondegeneracy conditions for rarefaction waves of type R_1 (resp. R_2), which are implicit in our definition of rarefaction, are the speed inequality (2.10) (resp. (2.12)), and the genuine nonlinearity condition (2.2).

Next we consider shock waves. If there is to be a shock wave solution of $U_t + F(U)_x = 0$ from U_- to U_+ with speed s, we must have that:

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0 \tag{E0}$$

the differential equation (1.4) has an orbit from U_- to U_+ (C0)

The two-component equation (E0) is a defining equation. In the context of structurally stable Riemann solutions, condition (C0) is an open condition, and therefore is regarded as a nondegeneracy condition, for all but transitional shock waves (those of types $S \cdot S$, $S \cdot RS$, $SA \cdot S$, or $SA \cdot RS$). For these waves, a *separation function* must be defined. Let us consider $S \cdot RS$ waves. Suppose Eq. (1.1) has an $S \cdot RS$ shock wave w^*: $U_-^* \xrightarrow{s^*} U_+^*$. Then for (U_-, s) near (U_-^*, s^*), the differential equation (1.4) has a saddle at U_- with unstable manifold $W_-(U_-, s)$. For $(U_-, s) = (U_-^*, s^*)$, (1.4) has a saddle-node at U_+^*; we denote its center manifold by $W_+(U_-^*, s^*)$. This center manifold perturbs to a family of invariant manifolds $W_+(U_-, s)$.

Let $\tilde{U}(\tau)$ be the connection of (1.4), with $(U_-, s) = (U_-^*, s^*)$, from U_-^* to U_+^*. Let Σ be a line segment through $\tilde{U}(0)$ transverse to $\dot{\tilde{U}}(0)$, in the direction V. Then $W_\pm(U_-, s)$ meet Σ in points $\bar{U}_\pm(U_-, s)$, and we define $S(U_-, s)$ by

$$\bar{U}_-(U_-, s) - \bar{U}_+(U_-, s) = S(U_-, s)\, V$$

See Fig. 2.

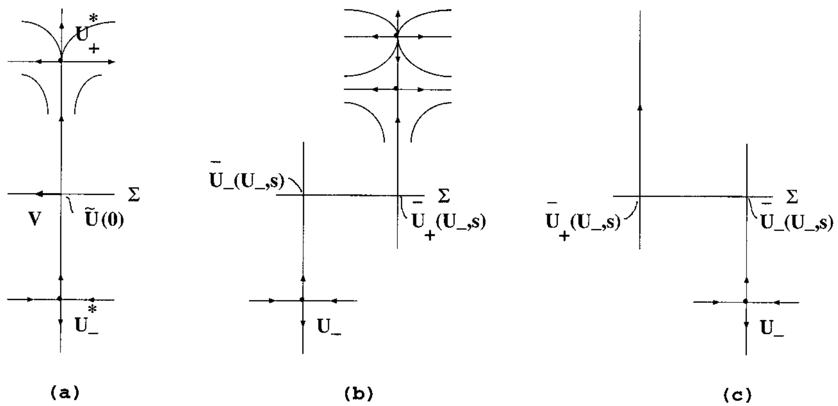


Fig. 2. Geometry of the separation function for $S \cdot RS$ shock waves. (a) Phase portrait of $\dot{U} = F(U) - F(U_-^*) - s^*(U - U_-^*)$. (b) Phase portrait of a nearby vector field $\dot{U} = F(U) - F(U_-) - s(U - U_-)$ for which the equilibrium at U_+^* has split into a saddle and a repeller, and for which $S(U_-, s)$ is positive. (c) Phase portrait of a nearby vector field $\dot{U} = F(U) - F(U_-) - s(U - U_-)$ for which the equilibrium at U_+^* has disappeared, and for which $S(U_-, s)$ is negative.

The family of unstable manifolds $W_-(U_-, s)$ is as smooth as F. The center manifolds $W_+(U_-, s)$ are not uniquely defined. However, if F is C^k, $k < \infty$, then $W_+(U_-, s)$ can be chosen to depend in a C^k manner on (U_-, s) in a neighborhood of (U_-^*, s^*). More precisely, while $\bar{U}_+(U_-, s)$ is uniquely defined for those (U_-, s) for which the differential equation (1.4) has equilibria near U_+^*, $\bar{U}_+(U_-, s)$ is not uniquely defined for other (U_-, s). However, the derivatives of S at (U_-^*, s^*) through order k are independent of the choice.

The partial derivatives of S are given as follows [6]. The linear differential equation

$$\dot{\phi} + \phi[DF(\tilde{U}(\xi)) - s^* I] = 0$$

has, up to constant multiple, a unique bounded solution. For the correct choice of this constant,

$$\frac{\partial S}{\partial s}(U_-^*, s^*) = -\int_{-\infty}^{\infty} \phi(\xi)(\tilde{U}(\xi) - U_-^*)\, d\xi \tag{2.15}$$

$$D_{U_-} S(U_-^*, s^*) = -\left(\int_{-\infty}^{\infty} \phi(\xi)\, d\xi\right)\{DF(U_-^*) - s^* I\} \tag{2.16}$$

The construction of a separation function for $SA \cdot RS$ shock waves is exactly the same, except $W_-(U_-^*, s^*)$ is a center manifold of (1.4) at U_-^*, which perturbs to a family of invariant manifolds $W_-(U_-, s)$ that is not uniquely defined.

In Table II we list additional defining equations and nondegeneracy conditions for the types of shock waves that occur in this paper; the labeling

Table II. Additional Defining Equations and Nondegeneracy Conditions for Various Shock Waves

type of shock wave	additional defining equations		nondegeneracy conditions	
$RS \cdot RS$	$\lambda_1(U_-) - s = 0$	(E3)	$D\lambda_1(U_-)\, r_1(U_-) \neq 0$	(G3)
	$\lambda_1(U_+) - s = 0$	(E4)	$D\lambda_1(U_+)\, r_1(U_+) \neq 0$	(G4)
			$\ell_1(U_+)(U_+ - U_-) \neq 0$	(B2)
			not distinguished connection	(C2)
$S \cdot SA$	$\lambda_2(U_+) - s = 0$	(E6)	$D\lambda_2(U_+)\, r_2(U_+) \neq 0$	(G6)
			not distinguished connection	(C3)
$S \cdot RS$	$\lambda_1(U_+) - s = 0$	(E13)	$D\lambda_1(U_+)\, r_1(U_+) \neq 0$	(G13)
	$S(U_-, s) = 0$	(S2)	transversality	(T2)
$SA \cdot RS$	$\lambda_2(U_-) - s = 0$	(E15)	$D\lambda_2(U_1)\, r_2(U_1) \neq 0$	(G15)
	$\lambda_1(U_+) - s = 0$	(E16)	$D\lambda_1(U_+)\, r_2(U_+) \neq 0$	(G16)
	$S(U_-, s) = 0$	(S4)	transversality	(T4)

of the conditions is from [6]. The wave nondegeneracy conditions are open conditions. Conditions (C2) and (C3) are that the connection Γ is not *distinguished*; for an $RS \cdot RS$ (resp. $S \cdot SA$) shock wave, this means that the connection Γ should not lie in the unstable manifold of U_- (resp. the stable manifold of U_+). The transversality condition (T2) is that there is a vector V in $\mathbb{R}^2$ such that the vectors

$$\begin{pmatrix} \ell_1(U_+) \\ \int_{-\infty}^{\infty} \phi(\xi)\, d\xi \end{pmatrix} (DF(U_-) - sI)\, V \qquad \text{and} \qquad \begin{pmatrix} \ell_1(U_+)(U_+ - U_-) \\ \int_{-\infty}^{\infty} \phi(\xi)(U(\xi) - U_-)\, d\xi \end{pmatrix} \tag{2.17}$$

are linearly independent. The transversality condition (T4) is that

$$\begin{pmatrix} \ell_1(U_+)\, r_1(U_-) & \ell_1(U_+)(U_+ - U_-) \\ \left(\int_{-\infty}^{\infty} \phi(\xi)\, d\xi\right) r_1(U_-) & \int_{-\infty}^{\infty} \phi(\xi)(U(\xi) - U_-)\, d\xi \end{pmatrix} \text{ is invertible} \tag{2.18}$$

The geometric import of (T2) is as follows. The system (1.1) has an $S \cdot RS$ shock wave $U_- \xrightarrow{s} U_+$ near a given $S \cdot RS$ shock wave $U_-^* \xrightarrow{s^*} U_+^*$ provided the following system of local defining equations is satisfied:

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0 \tag{2.19}$$

$$\lambda_1(U_+) - s = 0 \tag{2.20}$$

$$S(U_-, s) = 0 \tag{2.21}$$

The left-hand side of this system is a map from 5-dimensional $U_- s U_+$-space to $\mathbb{R}^4$. If the nondegeneracy conditions (G13) and (T2) hold at (U_-^*, s^*, U_+^*), then this map has surjective derivative there, so the set of solutions is a curve in $U_- s U_+$-space; moreover, this curve projects regularly to curves $\mathscr{E}_1$ in U_--space and $\mathscr{E}_2$ in U_+-space. These are the curves of possible left and right states for $S \cdot RS$ shock waves.

Similarly, the system (1.1) has an $SA \cdot RS$ shock wave $U_- \xrightarrow{s} U_+$ near a given $SA \cdot RS$ shock wave $U_-^* \xrightarrow{s^*} U_+^*$ provided an appropriate system of local defining equations is satisfied. The system is Eqs. (2.19)–(2.21) together with

$$\lambda_2(U_-) - s = 0$$

The left-hand side of this system is, for fixed F, a map from 5-dimensional U_-sU_+-space to $\mathbb{R}^5$. If the nondegeneracy conditions (G15), (G16), and (T4) hold at (U_-^*, s^*, U_+^*), then this map has surjective derivative there. Therefore for each F near F^* there is a unique $SA \cdot RS$ shock wave $U_- \xrightarrow{s} U_+$ near $U_-^* \xrightarrow{s^*} U_+^*$. The triple (U_-, s, U_+) depends smoothly on F.

For the Riemann solution (2.5), let w_i^* have type T_i and local defining map G_{T_i}, with range $\mathbb{R}^{e_i}$. For appropriate neighborhoods $\mathscr{U}_i$ of U_i^*, $\mathscr{I}_i$ of s_i^*, $\mathscr{F}$ of F^*, and $\mathscr{N}_i$ of Γ_i^*, we can define a map G: $\mathscr{U}_0 \times \mathscr{I}_1 \times \cdots \times \mathscr{I}_n \times \mathscr{U}_n \times \mathscr{F} \to \mathbb{R}^{e_1 + \cdots + e_n}$ by $G = (G_1, \ldots, G_n)$, where

$$G_i(U_0, s_1, \ldots, s_n, U_n, F) = G_{T_i}(U_{i-1}, s_i, U_i, F)$$

The map G is called the *local defining map* of the wave sequence (2.5).

We define the *Riemann number* of an elementary wave type T to be

$$\rho(T) = 3 - e(T)$$

where $e(T)$ is the number of defining equations for a wave of type T. For convenience, if w is an elementary wave of type T, we shall write $\rho(w)$ instead of $\rho(T)$.

A *1-wave group* is either a single $R \cdot S$ shock wave or an allowed sequence of elementary waves of the form

$$(R \cdot RS)(R_1\ RS \cdot RS) \cdots (R_1\ RS \cdot RS)\ R_1(RS \cdot S) \tag{2.25}$$

where the terms in parentheses are optional. If any of the terms in parentheses are present, the group is termed *composite*.

A *transitional wave group* is either a single $S \cdot S$ shock wave or an allowed sequence of elementary waves of the form

$$S \cdot RS(R_1\ RS \cdot RS) \cdots (R_1\ RS \cdot RS)\ R_1(RS \cdot S) \tag{2.26}$$

or

$$(S \cdot SA)\ R_2(SA \cdot SA\ R_2) \cdots (SA \cdot SA\ R_2)\ SA \cdot S \tag{2.27}$$

the terms in parentheses being optional. In cases (2.26) and (2.27), the group is termed *composite*.

A *2-wave group* is either a single $S \cdot A$ shock wave or an allowed sequence of elementary waves of the form

$$(S \cdot SA)\ R_2(SA \cdot SA\ R_2) \cdots (SA \cdot SA\ R_2)(SA \cdot A) \tag{2.28}$$

where again the terms in parentheses are optional. If any of the terms in parentheses are present, the group is termed *composite*.

In [6] the following are proved.

Theorem 2.1 (Wave Structure). *Let (2.5) be an allowed sequence of elementary waves. Then*

(1) $\sum_{i=1}^{n} \rho(w_i^*) \leqslant 2$;

(2) $\sum_{i=1}^{n} \rho(w_i^*) = 2$ *if and only if the following conditions are satisfied.*

(1) Suppose that the wave sequence (2.5) includes no $SA \cdot RS$ shock waves. Then it consists of one 1-wave group, followed by an arbitrary number of transitional wave groups (in any order), followed by one 2-wave group.

(2) Suppose that the wave sequence (2.5) includes $m \geqslant 1$ shock waves of type $SA \cdot RS$. Then these waves separate $m+1$ wave sequences $g_0, \ldots, g_m$. Each g_i is exactly as in (1) with the restrictions that:

(a) if $i < m$, the last wave in the group has type R_2;

(b) if $i > 0$, the first wave in the group has type R_1.

Theorem 2.2 (Structural Stability). *Suppose that the allowed sequence of elementary waves (2.5) has $\sum_{i=1}^{n} \rho(w_i^*) = 2$. Assume that:*

(H1) each wave satisfies the appropriate wave nondegeneracy conditions;

(H2) the wave group interaction conditions, as stated precisely in [6], are satisfied;

(H3) if w_i^* is a $* \cdot S$ shock wave and w_{i+1}^* is an $S \cdot *$ shock wave, then $s_i^* < s_{i+1}^*$.

Then the wave sequence (2.5) is structurally stable.

In the remainder of the paper, by a structurally stable Riemann solution we shall mean a sequence of elementary waves that satisfies the hypotheses of Theorem 2.2.

3. CODIMENSION-ONE RIEMANN SOLUTIONS

In order to consider conveniently codimension-one Riemann solutions, the definitions of rarefaction and shock waves in Section 2 must be generalized somewhat.

For the purposes of this paper, a *generalized rarefaction wave* of type R_i has the same definition as a rarefaction of type R_i, except that we allow $a = b$ in the interval of definition $[a, b]$.

A *generalized shock wave* consists of a *left state* U_-, a *right state* U_+ (possibly equal to U_-), a *speed* s, and a sequence of connecting orbits $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots, \tilde{\Gamma}_k$ of Eq. (1.4) from $U_- = \tilde{U}_0$ to $\tilde{U}_1$, $\tilde{U}_1$ to $\tilde{U}_2, \ldots, \tilde{U}_{k-1}$ to $\tilde{U}_k = U_+$. Note that $\tilde{U}_0, \tilde{U}_1, \ldots, \tilde{U}_k$ must be equilibria of Eq. (1.4). We allow for the possibility that $\tilde{U}_{j-1} = \tilde{U}_j$, in which case we assume that $\tilde{\Gamma}_j$ is the trivial orbit $\{\tilde{U}_j\}$.

Associated with each generalized rarefaction or generalized shock wave is a speed s, defined as before, and a curve $\bar{\Gamma}$: the rarefaction curve or the closure of $\tilde{\Gamma}_1 \cup \cdots \cup \tilde{\Gamma}_k$.

A *generalized allowed wave sequence* is a sequence (2.5) of generalized rarefaction and shock waves with increasing wave speeds. If $U_0 = U_L$ and $U_n = U_R$, then associated with a generalized allowed wave sequence $(w_1, w_2, \ldots, w_n)$ is a solution $U(x, t) = \hat{U}(x/t)$ of the Riemann problem (1.1)–(1.2). Therefore we shall also refer to a generalized allowed wave sequence as a *Riemann solution*.

A generalized allowed wave sequence (2.5) is a *codimension-one Riemann solution* provided that there is a sequence of wave types $(T_1^*, \ldots, T_n^*)$ with $\sum_{i=1}^n \rho(T_i^*) = 2$, neighborhoods $\mathscr{U}_i \subseteq \mathscr{U}$ of U_i^*, $\mathscr{I}_i \subseteq I$ of s_i^*, and $\mathscr{F}$ of F^*, a compact set $\mathscr{K}$ in $\mathbb{R}^2$, and a C^1 map

$$(G, H)\colon \mathscr{U}_0 \times \mathscr{I}_1 \times \cdots \times \mathscr{I}_n \times \mathscr{U}_n \times \mathscr{F} \to \mathbb{R}^{3n-2} \times \mathbb{R} \tag{3.1}$$

with $G(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*) = 0$ and $H(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*) = 0$ such that the following conditions are satisfied. These conditions are a simplification of the list in [7] that are adequate for the present paper. The numbering of [7, 8] has been retained.

(Q1) If $G(U_0, s_1, \ldots, s_n, U_n, F) = 0$ and $H(U_0, s_1, \ldots, s_n, U_n, F) \geqslant 0$ then there is a generalized allowed wave sequence

$$(w_1, w_2, \ldots, w_n)\colon U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n$$

for $U_t + F(U)_x = 0$ with each $\bar{\Gamma}_i$ contained in Int $\mathscr{K}$;

(Q2) if $G(U_0, s_1, \ldots, s_n, U_n, F) = 0$ and $H(U_0, s_1, \ldots, s_n, U_n, F) > 0$, then $(w_1, w_2, \ldots, w_n)$ is a structurally stable Riemann solution of type $(T_1^*, \ldots, T_n^*)$ and G exhibits its structural stability.

(Q3) If $G(U_0, s_1, \ldots, s_n, U_n, F) = 0$ and $H(U_0, s_1, \ldots, s_n, U_n, F) = 0$ then $(w_1, w_2, \ldots, w_n)$ is not a structurally stable Riemann solution.

(Q7$_1$) the linear map

$$DG(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*) \text{ restricted to}$$
$$\{(\dot{U}_0, \dot{s}_1, \ldots, \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = \dot{U}_n = 0 \text{ and } \dot{F} = 0\} \quad (3.3)$$

is an isomorphism.

If this holds, then, as in the structurally stable case, the equation $G = 0$ may be solved for $(s_1, U_1, \ldots, U_{n-1}, s_n)$ in terms of (U_0, U_n, F) near $(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*)$. Let

$$\tilde{H}(U_0, U_n, F) = H(U_0, s_1(U_0, U_n, F), \ldots, s_n(U_0, U_n, F), U_n, F) \quad (3.4)$$

Then one of the following occurs:

(E2) $\tilde{H}$ is independent of U_n, and $D_{U_0}\tilde{H}(U_0^*, U_n^*, F^*) \neq 0$.

(E3) $\tilde{H}$ is independent of U_0, and $D_{U_n}\tilde{H}(U_0^*, U_n^*, F^*) \neq 0$.

(E4) $\tilde{H}$ is independent of U_0 and U_n, and $D_F\tilde{H}(U_0^*, U_n^*, F^*) \neq 0$.

Condition (Q7$_1$) together with (E2), (E3) or (E4) implies that $(G, H)^{-1}(0)$ is a graph over a codimension-one manifold $\mathscr{S}$ in $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$, and $\mathscr{M} := (G, H)^{-1}(\{0\} \times \mathbb{R}_+)$ is a manifold-with-boundary of codimension $3n-2$. We can define maps $\bar{\Gamma}_i$: $\mathscr{M} \to \text{Int } \mathscr{K}$. We require that

(Q5) $(w_1, w_2, \ldots, w_n)$ can be chosen so that each map $\bar{\Gamma}_i$ is continuous.

(G, H) is again called a *local defining map*. If (E2) holds, there is a codimension-one manifold $\tilde{\mathscr{S}}$ through (U_0^*, F^*) in (U_0, F)-space such that $\mathscr{S} = \mathscr{U}_n \times \tilde{\mathscr{S}}$. In this case the codimension-one Riemann solution is termed a *U_L-boundary*. If (E3) holds we have a *dual U_L-boundary*. If (E4) holds, there is a codimension-one manifold $\tilde{\mathscr{S}}$ through F^* in F-space such that $\mathscr{S} = \mathscr{U}_0 \times \mathscr{U}_n \times \tilde{\mathscr{S}}$. In this case the codimension-one Riemann solution is termed an *F-boundary*.

A *rarefaction of zero strength* is one whose domain has zero length. A *shock wave of zero strength* is one with $U_L = U_R$ (and hence $\Gamma = \{U_L\}$).

A generalized allowed wave sequence is *minimal* if

- there are no rarefactions or shock waves of zero strength;
- no two successive shock waves have the same speed.

Among the minimal generalized allowed wave sequences we include sequences of no waves; these are given by a single $U_0 \in \mathbb{R}^2$, and represent constant solutions of Eq. (1.1).

We *shorten* a generalized allowed wave sequence by dropping a rarefaction or shock wave of zero strength, or by amalgamating adjacent shock waves of positive strength with the same speed. Every generalized allowed wave sequence can be shortened to a unique minimal generalized allowed wave sequence. Two generalized allowed wave sequences are *equivalent* if their minimal shortenings are the same. Equivalent generalized allowed wave sequences represent the same solution $U(x, t) = \hat{U}(x/t)$ of Eq. (1.1).

Let $(U_0^*, s_1^*, U_1^*, s_2^*,..., s_n^*, U_n^*, F^*)$ be a generalized allowed wave sequence that is a codimension-one Riemann solution of type $(T_1^*,..., T_n^*)$. Let $\mathscr{M}$ denote the associated manifold-with-boundary, $\partial\mathscr{M}$ being a graph over the manifold $\mathscr{S}$. Suppose there is an equivalent generalized allowed wave sequence $(U_0^\sharp, s_1^\sharp, U_1^\sharp, s_2^\sharp,..., s_m^\sharp, U_m^\sharp, F^*)$ that is a codimension-one Riemann solution in $\partial\mathscr{N}$, $\mathscr{N} = (G^\sharp, H^\sharp)^{-1}(\{0\} \times \mathbb{R}_+)$, where Int $\mathscr{N}$ consists of structurally stable Riemann solutions of some type $(T_1^\sharp,..., T_m^\sharp)$. Suppose in addition that $\partial\mathscr{N}$ is also a graph over $\mathscr{S}$, and the points in $\partial\mathscr{M}$ and $\partial\mathscr{N}$ above the same point in $\mathscr{S}$ are equivalent. Then the codimension-one Riemann solution (2.5) (or its equivalent generalized wave sequence) is said to lie in a *join*.

$\mathscr{M}$ and $\mathscr{N}$ are each graphs over the union of one side of $\mathscr{S}$ and $\mathscr{S}$ itself. If $\mathscr{M}$ and $\mathscr{N}$ are graphs over different sides of $\mathscr{S}$, we have a *regular join*; if $\mathscr{M}$ and $\mathscr{N}$ are graphs over the same side of $\mathscr{S}$, we have a *folded join*. In the case of a folded join, we do not have local existence and uniqueness of Riemann solutions.

In the following three sections we shall consider a Riemann solution (2.5) in which a wave of type $SA \cdot RS$ is preceded or followed by a rarefaction of zero strength.

Under additional nondegeneracy conditions, we shall show that such a Riemann solution (2.5) is of codimension one and lies in a join. Our arguments will have three steps:

Step 1. We verify that (2.5) is a codimension-one Riemann solution.

Step 2. We construct a Riemann solution equivalent to (2.5) and verify that it too is a codimension-one Riemann solution.

Step 3. We show that the two types of codimension-one Riemann solutions are defined on the same codimension-one surface $\mathscr{S}$ in U_0U_nF-space; the two types of codimension-one Riemann solutions above a given point in $\mathscr{S}$ are equivalent; and the Riemann solution join that we therefore have is of a certain type (U_L-boundary or F-boundary, regular or folded join).

All three steps will make use of the local defining map (G, H). In each case, G is the map that would be used for structurally stable Riemann solutions of type $(T_1, \ldots, T_n)$. In Steps 1 and 2 we shall carefully verify $(Q7_1)$ and either (E2) or (E4), and we shall explain enough of the geometry to make the other conditions evident.

4. MISSING "STAND-ALONE" 2-RAREFACTION

Theorem 4.1. *Let (2.5) be a Riemann solution of type $(T_1, \ldots, T_n)$ for $U_t + F^*(U)_x = 0$. Assume there is an integer k such that $T_k \neq * \cdot SA$, $T_{k+1} = R_2$, $T_{k+2} = SA \cdot RS$, $T_{k+3} = R_1$. Assume:*

(1) All hypotheses of Theorem 2.2 are satisfied, except that the 2-rarefaction w^*_{k+1} has zero strength.

(2) The forward wave curve mapping $U_k(U_0, \sigma)$ is regular at (U^*_0, σ^*).

Then (2.5) is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type $(T_1, \ldots, T_k, S \cdot RS, R_1, T_{k+4}, \ldots, T_n)$ because the $S \cdot RS$ shock wave becomes an $SA \cdot RS$ shock wave. Riemann solution (2.5) (and its equivalent) lies in a regular join that is a U_L-boundary.

Let us briefly explain assumption (2). According to [6], assumption (1) implies that there exist smooth mappings $s_i(U_0, \sigma)$ and $U_i(U_0, \sigma)$, $1 \leqslant i \leqslant k$, such that

$$s_i(U^*_0, \sigma^*) = s^*_i \qquad \text{and} \qquad U_i(U^*_0, \sigma^*) = U^*_i \tag{4.1}$$

and for each (U_0, σ),

$$U_0 \xrightarrow{s_1(U_0, \sigma)} \cdots \xrightarrow{s_k(U_0, \sigma)} U_k(U_0, \sigma) \tag{4.2}$$

is an admissible wave sequence of type $(T_1, \ldots, T_k)$ for $U_t + F^*(U)_x = 0$. The mapping $U_k(U_0, \sigma)$ is a *forward wave curve mapping*. The assumption that it is regular is actually somewhat restrictive. It implies that no 2-rarefaction (equivalently, no fast transitional wave group or $SA \cdot RS$ shock wave) precedes w^*_{k+1}.

Proof. We first fix the flux function F^* and denote it throughout the proof by F. We shall assume for simplicity that $k = 1$. Then (2.5) begins with the wave sequence

$$U^*_0 \xrightarrow{s^*_1} U^*_1 \xrightarrow{s^*_2} U^*_2 \xrightarrow{s^*_3} U^*_3 \xrightarrow{s^*_4} U^*_4$$

with the first wave a one-wave, $T_2 = R_2$, $T_3 = SA \cdot RS$, $T_4 = R_1$. We have

$$s_2^* = s_3^* = \lambda_2(U_2^*) = \lambda_1(U_3^*) \qquad \text{and} \qquad U_1^* = U_2^*$$

Step 1. We note that $(U_0, s_1, ..., s_3, U_3)$ near $(U_0^*, s_1^*, ..., s_3^*, U_3^*)$ represents an admissible wave sequence of type $(T_1, R_2, SA \cdot RS)$ if and only if

$$U_1 - \eta(U_0, s_1) = 0 \tag{4.5}$$

$$U_1 - \psi_2(U_2, s_2) = 0 \tag{4.6}$$

$$F(U_3) - F(U_2) - s_3(U_3 - U_2) = 0 \tag{4.7}$$

$$\lambda_2(U_2) - s_3 = 0 \tag{4.8}$$

$$\lambda_1(U_3) - s_3 = 0 \tag{4.9}$$

$$S(U_2, s_3) = 0 \tag{4.10}$$

$$\lambda_2(U_2) - s_2 \geqslant 0$$

Here $U_1 = \eta(U_0, s_1)$ is, for fixed U_0, the one-wave curve; in general, $U_k = \eta(U_0, \sigma)$ would be a forward wave-curve mapping. The function $S(U_2, s_3)$ is the separation function for $SA \cdot RS$ shock waves defined in Section 2.

Let $G(U_0, s_1, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(T_1, R_2, SA \cdot RS, R_1, T_5, ..., T_n)$ near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, ..., s_3, U_3)$ is given by the left hand sides of Eqs. (4.5)–(4.10), and $G_2(U_3, s_4, ..., s_n, U_n)$ is the local defining map for wave sequences of type $(R_1, T_5, ..., T_n)$. From the theory of [6],

$$DG_1(U_0^*, s_1^*, ..., s_3^*, U_3^*), \text{ restricted to}$$
$$\{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_3, \dot{U}_3) : \dot{U}_0 = 0\}, \text{ is an isomorphism} \tag{4.11}$$

and

$$DG_2(U_3^*, s_4^*, ..., s_n^*, U_n^*), \text{ restricted to}$$
$$\{(\dot{U}_3, \dot{s}_4, ..., \dot{s}_n, \dot{U}_n) : \dot{U}_3 = \dot{U}_n = 0\}, \text{ is an isomorphism} \tag{4.12}$$

Therefore ($Q7_1$) holds.

From (4.11), we can solve Eqs. (4.5)–(4.10) for $(s_1, U_1, ..., s_3, U_3)$ in terms of U_0 near $(U_0^*, s_1^*, ..., s_3^*, U_3^*)$. Once U_3 is found, the remainder of the Riemann solution is obtained by solving for $(s_4, U_4, ..., U_{n-1}, s_n)$ in terms of (U_3, U_n). Since a solution of $G = 0$ represents a Riemann solution of the desired type if and only if $\lambda_2(U_2)-s_2 \geqslant 0$, we now study $\tilde{H}(U_0) := \lambda_2(U_2)-s_2$. To verify (E2), we calculate

$$D\tilde{H}(U_0^*)\,\dot{U}_0 = D\lambda_2(U_2^*)\,\dot{U}_2 - \dot{s}_2 \tag{4.13}$$

by linearizing Eqs. (4.5)–(4.10) at $(U_0^*, s_1^*,..., s_3^*, U_3^*)$, solving for $(\dot{s}_1, \dot{U}_1,..., \dot{s}_3, \dot{U}_3)$ in terms of $\dot{U}_0$, and substituting the formulas for $\dot{U}_2$ and $\dot{s}_2$ into Eq. (4.13).

Linearizing Eqs. (4.5)–(4.10) at $(U_0^*, s_1^*,..., s_3^*, U_3^*)$ yields:

$$\dot{U}_1 - D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0 \tag{4.14}$$

$$\dot{U}_1 - D\psi_2(U_2^*, s_2^*)(\dot{U}_2, \dot{s}_2) = 0 \tag{4.15}$$

$$(DF(U_3^*) - s_3^* I)\,\dot{U}_3 - (DF(U_2^*) - s_3^* I)\,\dot{U}_2 - \dot{s}_3(U_3^* - U_2^*) = 0 \tag{4.16}$$

$$D\lambda_2(U_2^*)\,\dot{U}_2 - \dot{s}_3 = 0 \tag{4.17}$$

$$D\lambda_1(U_3^*)\,\dot{U}_3 - \dot{s}_3 = 0 \tag{4.18}$$

$$DS(U_2^*, s_3^*)(\dot{U}_2, \dot{s}_3) = 0 \tag{4.19}$$

Because of the nondegeneracy conditions for $SA \cdot RS$ shock waves, which are part of assumption (1), the only solution of Eqs. (4.16)–(4.19) is $\dot{U}_2 = \dot{U}_3 = 0$ and $\dot{s}_3 = 0$. To find $(\dot{s}_1, \dot{U}_1, \dot{s}_2)$, we let

$$\dot{U}_1 = a r_1(U_1^*) + b r_2(U_1^*) \tag{4.20}$$

We then multiply Eq. (4.14) and

$$\dot{U}_1 - \frac{\partial \psi_2}{\partial s_2}(U_2^*, s_2^*)\,\dot{s}_2 = 0 \tag{4.21}$$

by $\ell_1(U_1^*)$ and $\ell_2(U_1^*)$. We obtain the system

$$a - \ell_1(U_1^*)\,D_{U_0}\eta(U_0^*, s_1^*)\,\dot{U}_0 - \ell_1(U_1^*)\,\frac{\partial \eta}{\partial s_1}(U_0^*, s_1^*)\,\dot{s}_1 = 0 \tag{4.22}$$

$$b - \ell_2(U_1^*)\,D_{U_0}\eta(U_0^*, s_1^*)\,\dot{U}_0 - \ell_2(U_1^*)\,\frac{\partial \eta}{\partial s_1}(U_0^*, s_1^*)\,\dot{s}_1 = 0 \tag{4.23}$$

$$a = 0 \tag{4.24}$$

$$b - \dot{s}_2 = 0 \tag{4.25}$$

We solve for $(\dot{s}_1, a, b, \dot{s}_2)$. Then, using Eq. (4.13),

$$D\tilde{H}(U_0^*)\,\dot{U}_0 = -\dot{s}_2$$
$$= \left(\ell_1(U_1^*)\,\frac{\partial\eta}{\partial s_1}(U_0^*, s_1^*)\right)^{-1}\left\{\ell_2(U_1^*)\,\frac{\partial\eta}{\partial s_1}(U_0^*, s_1^*)\,\ell_1(U_1^*)\right.$$
$$\left. -\ell_1(U_1^*)\,\frac{\partial\eta}{\partial s_1}(U_0^*, s_1^*)\,\ell_2(U_1^*)\right\} D_{U_0}\eta(U_0^*, s_1^*)\,\dot{U}_0 \qquad (4.26)$$

In Eq. (4.26), note that:

- $\ell_1(U_1^*)\,\frac{\partial\eta}{\partial s_1}(U_0^*, s_1^*)$ is nonzero, since the one-wave curve $\eta(U_0^*, s_1)$ is transverse to the 2-rarefaction curve at U_1^* by assumption (1).
- The bracketed row vector is nonzero and is orthogonal to $\frac{\partial\eta}{\partial s_1}(U_0^*, s_1^*)$.
- The range of $D_{U_0}\eta(U_0^*, s_1^*)$ includes vectors that are linearly independent of $\frac{\partial\eta}{\partial s_1}(U_0^*, s_1^*)$, by assumption (2).

Thus $D\tilde{H}(U_0^*)$ is a nonzero vector, so that (E2) holds. Therefore $\mathscr{C} = \{U_0 : \tilde{H}(U_0) = 0\}$ is a smooth curve near U_0^*, and for (U_0, U_n) near (U_0^*, U_n^*), a solution of type $(T_1, R_2, SA\cdot RS, R_1, T_5, \ldots, T_n)$ exists provided U_0 is on the side of $\mathscr{C}$ to which this vector points.

Step 2. Next we consider the point $(U_0^*, s_1^*, U_1^*, s_3^*, U_3^*, s_4^*, \ldots, s_n^*, U_n^*)$ in $\mathbb{R}^{3n-1}$. We shall investigate the existence of nearby points $(U_0, s_1, U_1, s, U, s_4, \ldots, s_n, U_n)$ that represent Riemann solutions of type $(T_1, S\cdot RS, R_1, T_5, \ldots, T_n)$.

We note that if (U_0, s_1, U_1, s, U) near $(U_0^*, s_1^*, U_1^*, s_3^*, U_3^*)$ represents an admissible wave sequence of type $(T_1, S\cdot RS)$, then we must have

$$U_1 - \eta(U_0, s_1) = 0 \qquad (4.27)$$

$$F(U) - F(U_1) - s(U - U_1) = 0 \qquad (4.28)$$

$$\lambda_1(U) - s = 0 \qquad (4.29)$$

$$S(U_1, s) = 0 \qquad (4.30)$$

$$\lambda_2(U_1) - s \geqslant 0 \qquad (4.31)$$

(Of course, if $\lambda_2(U_1) - s = 0$, the $S\cdot RS$ wave has become an $SA\cdot RS$ wave.) We shall see that the linearization of Eqs. (4.27)–(4.30) at $(U_0^*, s_1^*, U_1^*, s_3^*, U_3^*)$, restricted to $\{(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}, \dot{U}) : \dot{U}_0 = 0\}$, is an isomorphism. As in Step 1, it follows that $(Q7_1)$ holds.

It also follows that Eqs. (4.27)–(4.30) can be solved for (s_1, U_1, s, U) in terms of U_0 near $(U_0^*, s_1^*, U_1^*, s_3^*, U_3^*)$. Once U is found, the remainder of the Riemann solution is obtained by solving for $(s_4, U_4, \ldots, U_{n-1}, s_n)$ in terms of (U, U_n).

We have a Riemann solution of the desired type if and only if the function $\tilde{H}(U_0) := \lambda_2(U_1) - s \geqslant 0$. To calculate

$$D\tilde{H}(U_0^*)\,\dot{U}_0 = D\lambda_2(U_1^*)\,\dot{U}_1 - \dot{s} \tag{4.32}$$

we linearize Eqs. (4.27)–(4.30) at $(U_0^*, s_1^*, U_1^*, s_3^*, U_3^*)$, solve the linearized equations for $(\dot{s}_1, \dot{U}_1, \dot{s}, \dot{U})$ in terms of $\dot{U}_0$, and substitute the formulas for $\dot{U}_1$ and $\dot{s}$ into Eq. (4.32).

The linearization of Eqs. (4.27)–(4.30) at $(U_0^*, s_1^*, U_1^*, s_3^*, U_3^*)$ is

$$\dot{U}_1 - D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0 \tag{4.33}$$

$$(DF(U_3^*) - s_3^* I)\,\dot{U} - (DF(U_1^*) - s_3^* I)\,\dot{U}_1 - \dot{s}(U_3^* - U_1^*) = 0 \tag{4.34}$$

$$D\lambda_1(U_3^*)\,\dot{U} - \dot{s} = 0 \tag{4.35}$$

$$DS(U_1^*, s_3^*)(\dot{U}_1, \dot{s}) = 0 \tag{4.36}$$

We make the substitution (4.20). Multiplying Eq. (4.34) by $\ell_1(U_3^*)$ yields

$$-a(\lambda_1(U_1^*) - \lambda_2(U_1^*))\,\ell_1(U_3^*)\,r_1(U_1^*) - \dot{s}\ell_1(U_3^*)(U_3^* - U_1^*) = 0 \tag{4.37}$$

Also, Eq. (4.36) simplifies to

$$-a(\lambda_1(U_1^*) - \lambda_2(U_1^*))\left(\int_{-\infty}^{\infty} \phi(\xi)\,d\xi\right) r_1(U_1^*) - \dot{s}\int_{-\infty}^{\infty} \phi(\xi)(U^*(\xi) - U_1^*)\,d\xi = 0 \tag{4.38}$$

By the nondegeneracy condition (T4) for $SA \cdot RS$ shock waves, which is part of assumption (1), Eqs. (4.37)–(4.38) imply

$$a = \dot{s} = 0$$

Multiplying Eq. (4.33) by $\ell_1(U_1^*)$ and $\ell_2(U_1^*)$ yields Eqs. (4.22)–(4.23), which can now be solved to yield b and $\dot{s}_1$; the solution is the same as in step 1. Equation (4.35) and the equation obtained by multiplying Eq. (4.34) by $\ell_2(U_3^*)$ can now be solved to yield $\dot{U}$.

Substituting $a = \dot{s} = 0$ into Eq. (4.32) yields

$$D\tilde{H}(U_0^*)\,\dot{U}_0 = b \tag{4.40}$$

which from Eq. (4.25) is the opposite of Eq. (4.26).

Thus $D\tilde{H}(U_0^*)$ is a nonzero vector, so that (E2) holds. Therefore $\mathscr{C} = \{U_0 : \tilde{H}(U_0) = 0\}$ is a smooth curve near U_0^*, and for (U_0, U_n) near (U_0^*, U_n^*), a solution of type $(T_1, S \cdot RS, R_1, T_5, \ldots, T_n)$ exists provided U_0 is on the side of $\mathscr{C}$ to which this vector points.

Step 3. It is easy to see that the curves $\mathscr{C}$ defined in Steps 1 and 2 coincide. The join is regular because the vectors $D\tilde{H}(U_0^*)$ in Steps 1 and 2 point in opposite directions. □

5. MISSING RAREFACTION IN A 2-WAVE GROUP CONSISTING OF TWO WAVES

Theorem 5.1. *Let (2.5) be a Riemann solution of type $(T_1, \ldots, T_n)$ for $U_t + F^*(U)_x = 0$. Assume there is an integer k such that $T_{k+1} = S \cdot SA$, $T_{k+2} = R_2$, $T_{k+3} = SA \cdot RS$, $T_{k+4} = R_1$. Assume:*

(1) All hypotheses of Theorem 2.2 are satisfied, except that the 2-rarefaction w_{k+2}^* has zero strength.

(2) The forward wave curve mapping $U_k(U_0, \sigma)$ is regular at (U_0^*, σ^*).

(3) Let $\tilde{U}(\xi)$ be the connection of

$$\dot{U} = F^*(U) - F^*(U_{k+2}^*) - s_{k+3}^*(U - U_{k+2}^*)$$

from U_{k+2}^* to U_{k+3}^*, and let $\phi(\xi)$ be a nontrivial bounded solution of

$$\dot{\phi} + \phi[DF^*(\tilde{U}(\xi)) - s_{k+3}^* I] = 0$$

Then the vectors

$$\begin{pmatrix} \ell_1(U_{k+3}^*) \\ \int_{-\infty}^{\infty} \phi(\xi)\, d\xi \end{pmatrix} (DF^*(U_k^*) - s_{k+3}^* I) \frac{\partial U_k}{\partial \sigma} (U_0^*, \sigma^*)$$

and

$$\begin{pmatrix} \ell_1(U_{k+3}^*)(U_{k+3}^* - U_k^*) \\ \int_{-\infty}^{\infty} \phi(\xi)(U(\xi) - U_k^*)\, d\xi \end{pmatrix}$$

are linearly independent.

(4) The expression $\mathscr{D}$ defined by Eq. (5.72) below is nonzero.

Then (2.5) is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type $(T_1,..., T_k, S \cdot RS, R_1, T_{k+5},..., T_n)$ *because an equilibrium appears that breaks the connection of the* $S \cdot RS$ *shock wave. Riemann solution (2.5) (and its equivalent) lies in a join that is a* U_L*-boundary. If* $k=1$*, the join is regular or folded according to whether the expression*

$$\frac{(BE-AF)((KL-IN)\,\eta_{13}+(KM-JN)\,\eta_{23})}{\mathscr{D}((AG-DE)\,\eta_{13}+(BG-DF)\,\eta_{23})} \tag{5.4}$$

defined below is negative or positive. If $k>1$*, an analagous condition holds.*

Proof. We first fix the flux function F^* and denote it throughout the proof by F. We shall assume for simplicity that $k=1$. Then (2.5) begins with the wave sequence

$$U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} U_2^* \xrightarrow{s_3^*} U_3^* \xrightarrow{s_4^*} U_4^* \xrightarrow{s_5^*} U_5^*$$

with the first wave a one-wave, $T_2 = S \cdot SA$, $T_3 = R_2$, $T_4 = SA \cdot RS$, $T_5 = R_1$. We have

$$s_2^* = s_3^* = s_4^* = \lambda_2(U_3^*) = \lambda_1(U_4^*) \qquad \text{and} \qquad U_2^* = U_3^*$$

Step 1. We note that $(U_0, s_1,..., s_4, U_4)$ near $(U_0^*, s_1^*,..., s_4^*, U_4^*)$ represents an admissible wave sequence of type $(T_1, S \cdot SA, R_2, SA \cdot RS)$ if and only if

$$U_1 - \eta(U_0, s_1) = 0 \tag{5.7}$$

$$F(U_2) - F(U_1) - s_2(U_2 - U_1) = 0 \tag{5.8}$$

$$\lambda_2(U_2) - s_2 = 0 \tag{5.9}$$

$$U_2 - \psi_2(U_3, s_3) = 0 \tag{5.10}$$

$$F(U_4) - F(U_3) - s_4(U_4 - U_3) = 0 \tag{5.11}$$

$$\lambda_2(U_3) - s_4 = 0 \tag{5.12}$$

$$\lambda_1(U_4) - s_4 = 0 \tag{5.13}$$

$$S(U_3, s_4) = 0 \tag{5.14}$$

$$\lambda_2(U_3) - s_3 \geqslant 0$$

Here $\eta(U_0, s_1)$ and $S(U_3, s_4)$ are as in Section 4.

Let $G(U_0, s_1,..., s_n, U_n)$ be the local defining map for wave sequences of type $(T_1, S \cdot SA, R_2, SA \cdot RS, R_1, T_6,..., T_n)$ near $(U_0^*, s_1^*,..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1,..., s_4, U_4)$ is given by the left hand sides of Eqs. (5.7)–(5.14), and $G_2(U_4, s_5,..., s_n, U_n)$ is the local defining map for wave sequences of type $(R_1, T_6,..., T_n)$. As in Section 4,

$$DG_1(U_0^*, s_1^*,..., s_4^*, U_4^*), \text{ restricted to } \{(\dot{U}_0, \dot{s}_1,..., \dot{s}_4, \dot{U}_4) : \dot{U}_0 = 0\}, \text{ is an isomorphism} \tag{5.15}$$

and

$$DG_2(U_4^*, s_5^*,..., s_n^*, U_n^*), \text{ restricted to } \{(\dot{U}_4, \dot{s}_5,..., \dot{s}_n, \dot{U}_n) : \dot{U}_4 = \dot{U}_n = 0\}, \text{ is an isomorphism} \tag{5.16}$$

Therefore $(\mathrm{Q7}_1)$ holds.

As in Section 4, we can solve Eqs. (5.7)–(5.14) for $(s_1, U_1,..., s_4, U_4)$ in terms of U_0 near $(U_0^*, s_1^*,..., s_3^*, U_3^*)$. Once U_4 is found, the remainder of the Riemann solution is obtained by solving for $(s_5, U_5,..., U_{n-1}, s_n)$ in terms of (U_4, U_n). Since a solution of $G = 0$ represents a Riemann solution of the desired type if and only if $\lambda_2(U_3) - s_3 \geqslant 0$, we now study $\tilde{H}(U_0) := \lambda_2(U_3) - s_3$. To verify (E2), we calculate

$$D\tilde{H}(U_0^*)\,\dot{U}_0 = D\lambda_2(U_3^*)\,\dot{U}_3 - \dot{s}_3 \tag{5.17}$$

by linearizing Eqs. (5.7)–(5.14) at $(U_0^*, s_1^*,..., s_4^*, U_4^*)$, solving for $(\dot{s}_1, \dot{U}_1,..., \dot{s}_4, \dot{U}_4)$ in terms of $\dot{U}_0$, and substituting the formulas for $\dot{U}_3$ and $\dot{s}_3$ into Eq. (5.17).

Linearizing Eqs. (5.7)–(5.14) at $(U_0^*, s_1^*,..., s_4^*, U_4^*)$ yields:

$$\dot{U}_1 - D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0 \tag{5.18}$$

$$(DF(U_2^*) - s_2^* I)\,\dot{U}_2 - (DF(U_1^*) - s_2^* I)\,\dot{U}_1 - \dot{s}_2(U_2^* - U_1^*) = 0 \tag{5.19}$$

$$D\lambda_2(U_2^*)\,\dot{U}_2 - \dot{s}_2 = 0 \tag{5.20}$$

$$\dot{U}_2 - D\psi_2(U_3^*, s_3^*)(\dot{U}_3, \dot{s}_3) = 0 \tag{5.21}$$

$$(DF(U_4^*) - s_4^* I)\,\dot{U}_4 - (DF(U_3^*) - s_4^* I)\,\dot{U}_3 - \dot{s}_4(U_4^* - U_3^*) = 0 \tag{5.22}$$

$$D\lambda_2(U_3^*)\,\dot{U}_3 - \dot{s}_4 = 0 \tag{5.23}$$

$$D\lambda_1(U_4^*)\,\dot{U}_4 - \dot{s}_4 = 0 \tag{5.24}$$

$$DS(U_3^*, s_4^*)(\dot{U}_3, \dot{s}_4) = 0 \tag{5.25}$$

As in Section 4, Eqs. (5.22)–(5.25) yield $\dot{U}_3 = \dot{U}_4 = 0$ and $\dot{s}_4 = 0$. We set

$$\dot{U}_0 = ir_1(U_0^*) + jr_2(U_0^*) \tag{5.26}$$

$$\dot{U}_1 = ar_1(U_1^*) + br_2(U_1^*) \tag{5.27}$$

$$\dot{U}_2 = cr_1(U_2^*) + dr_2(U_2^*) \tag{5.28}$$

We multiply Eq. (5.18) by $\ell_1(U_1^*)$ and $\ell_2(U_1^*)$, and we multiply Eqs. (5.19) and (5.21) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. We get:

$$a - \ell_1(U_1^*)\, D\eta(U_0^*, s_1^*)(ir_1(U_0^*) + jr_2(U_0^*), \dot{s}_1) = 0 \tag{5.29}$$

$$b - \ell_2(U_1^*)\, D\eta(U_0^*, s_1^*)(ir_1(U_0^*) + jr_2(U_0^*), \dot{s}_1) = 0 \tag{5.30}$$

$$\begin{aligned} &c(\lambda_1(U_2^*) - s_2^*) - a(\lambda_1(U_1^*) - s_2^*)\, \ell_1(U_2^*)\, r_1(U_1^*) \\ &\quad - b(\lambda_2(U_1^*) - s_2^*)\, \ell_1(U_2^*)\, r_2(U_1^*) - \dot{s}_2 \ell_1(U_2^*)(U_2^* - U_1^*) = 0 \end{aligned} \tag{5.31}$$

$$\begin{aligned} &-a(\lambda_1(U_1^*) - s_2^*)\, \ell_2(U_2^*)\, r_1(U_1^*) - b(\lambda_2(U_1^*) - s_2^*)\, \ell_2(U_2^*)\, r_2(U_1^*) \\ &\quad - \dot{s}_2 \ell_2(U_2^*)(U_2^* - U_1^*) = 0 \end{aligned} \tag{5.32}$$

$$D\lambda_2(U_2^*)(cr_1(U_2^*) + dr_2(U_2^*)) - \dot{s}_2 = 0 \tag{5.33}$$

$$c = 0 \tag{5.34}$$

$$d - \dot{s}_3 = 0 \tag{5.35}$$

Simplifying the notation, Eqs. (5.29)–(5.35) become

$$a - (\eta_{11} i + \eta_{12} j + \eta_{13} \dot{s}_1) = 0 \tag{5.36}$$

$$b - (\eta_{21} i + \eta_{22} j + \eta_{23} \dot{s}_1) = 0 \tag{5.37}$$

$$Cc - Aa - Bb - D\dot{s}_2 = 0 \tag{5.38}$$

$$-Ea - Fb - G\dot{s}_2 = 0 \tag{5.39}$$

$$Hc + d - \dot{s}_2 = 0 \tag{5.40}$$

$$c = 0 \tag{5.41}$$

$$d - \dot{s}_3 = 0 \tag{5.42}$$

Here the capital letters have the obvious meanings. This system can be solved for $(\dot{s}_1, a, b, \dot{s}_2, c, d, \dot{s}_3)$ in terms of (i, j) provided

$$(AG - DE)\, \eta_{13} + (BG - DF)\, \eta_{23} \neq 0 \tag{5.43}$$

Equation (5.43) expresses transversality, at U_1^*, of the one-wave curve $U_1 = \eta(U_0^*, s_1)$ to the curve of left states of $S \cdot SA$ shock waves; this transversality is part of assumption (1). We then find that

$$D\tilde{H}(U_0^*)(ir_1(U_0^*) + jr_2(U_0^*))$$
$$= -\dot{s}_3 = \frac{(BE - AF)((\eta_{11}\eta_{23} - \eta_{13}\eta_{21})\, i + (\eta_{12}\eta_{23} - \eta_{13}\eta_{22})\, j)}{(AG - DE)\, \eta_{13} + (BG - DF)\, \eta_{23}} \tag{5.44}$$

Note that:

- $BE - AF$ is clearly nonzero.
- Since $D\eta(U_0^*, s_1^*)$ is surjective by assumption (1), either $\eta_{11}\eta_{23} - \eta_{13}\eta_{21}$ or $\eta_{12}\eta_{23} - \eta_{13}\eta_{22}$ is nonzero. Thus there exist (i, j) such that the second factor in the numerator of expression (5.44) is nonzero.
- The denominator of Eq. (5.44) is nonzero by Eq. (5.43).

Thus $D\tilde{H}(U_0^*)$ is a nonzero vector, so that (E2) holds. Therefore $\mathscr{C} = \{U_0 : \tilde{H}(U_0) = 0\}$ is a smooth curve near U_0^*, and for (U_0, U_n) near (U_0^*, U_n^*), a solution of type $(T_1, S \cdot SA, R_2, SA \cdot RS, R_1, T_6, \ldots, T_n)$ exists provided U_0 is on the side of $\mathscr{C}$ to which this vector points.

Step 2. Next we consider the point $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*, s_5^*, \ldots, s_n^*, U_n^*)$ in $\mathbb{R}^{3n-4}$. We shall investigate the existence of nearby points $(U_0, s_1, U_1, s, U, s_5, \ldots, s_n, U_n)$ that represent Riemann solutions of type $(T_1, S \cdot RS, R_1, T_6, \ldots, T_n)$.

We begin by considering the three-parameter family of differential equations

$$\dot{U} = F(U) - F(U_1) - s(U - U_1) \tag{5.45}$$

near $(U_1, s, U) = (U_1^*, s_2^*, U_2^*)$. From [9] we have

Lemma 5.2. *There is a function $\gamma(U_1, s)$, defined near $(U_1, s) = (U_1^*, s_2^*)$, such that (5.45) undergoes a saddle-node bifurcation near U_2^* when the surface $\gamma = 0$ is crossed. For $\gamma > 0$ (resp. $= 0$, < 0) there are no (resp. 1, 2) equilibria of (5.45) near U_2^*. We may take*

$$D\gamma(U_1^*, s_2^*)(\dot{U}_1, \dot{s}) = -\ell_1(U_2^*)(DF(U_1^*) - s_2^* I)\, \dot{U}_1 - \dot{s}\ell_1(U_2^*)(U_2^* - U_1^*) \tag{5.46}$$

Next we construct a separation function $\tilde{S}(U_1, s)$, (U_1, s) near (U_1^*, s_4^*), that can be used to study connections of

$$\dot{U} = F(U) - F(U_1) - s(U - U_1) \tag{5.47}$$

from U_1 to equilibria near U_4^*. For $(U_1, s) = (U_1^*, s_4^*)$, (5.47) has a heteroclinic solution $\tilde{U}(\xi)$ from the saddle-attractor U_3^* to the repeller-saddle U_4^*. Let Σ be a line segment through $\tilde{U}(0)$ transverse to $\dot{\tilde{U}}(0)$, in the direction V. The center manifolds of (5.47), with $(U_1, s) = (U_1^*, s_4^*)$, at U_3^* and U_4^* perturb to invariant manifolds that meet Σ at $\bar{U}_-(U_1, s)$ and $\bar{U}_+(U_1, s)$ respectively. Using $\bar{U}_\pm(U_1, s)$ we can define a separation function $S(U_1, s)$. We have

$$\frac{\partial S}{\partial s}(U_1^*, s_4^*) = -\int_{-\infty}^{\infty} \phi(\xi)(\tilde{U}(\xi) - U_1^*)\, d\xi \tag{5.48}$$

$$D_{U_1} S(U_1^*, s_4^*) = -\left(\int_{-\infty}^{\infty} \phi(\xi), d\xi\right) \{DF(U_1^*) - s_4^* I\} \tag{5.49}$$

If (5.47) has equilibria near U_3^* we define

$$\tilde{U}_-(U_1, s) = \bar{U}_-(U_1, s)$$

If it has no equilibria near U_3^*, we define $\tilde{U}_+(U_1, s)$ to be the intersection of the unstable manifold of the saddle U_1 with Σ. We then define

$$\tilde{U}_-(U_1, s) - \bar{U}_+(U_1, s) = \tilde{S}(U_1, s)\, V$$

See Fig. 3. If the perturbation of the center manifold at U_3^* is chosen appropriately, $\tilde{S}(U_1, s)$ agrees with $S(U_1, s)$. Thus the partial derivatives of $\tilde{S}(U_1^*, s_4^*)$ are also given by Eqs. (5.48)–(5.49).

In order that (U_0, s_1, U_1, s, U) near $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$ represent an admissible wave sequence of type $(T_1, S \cdot RS)$, we must have

$$U_1 - \eta(U_0, s_1) = 0 \tag{5.52}$$

$$F(U) - F(U_1) - s(U - U_1) = 0 \tag{5.53}$$

$$\lambda_1(U) - s = 0 \tag{5.54}$$

$$\tilde{S}(U_1, s) = 0 \tag{5.55}$$

$$\gamma(U_1, s) \geqslant 0 \tag{5.56}$$

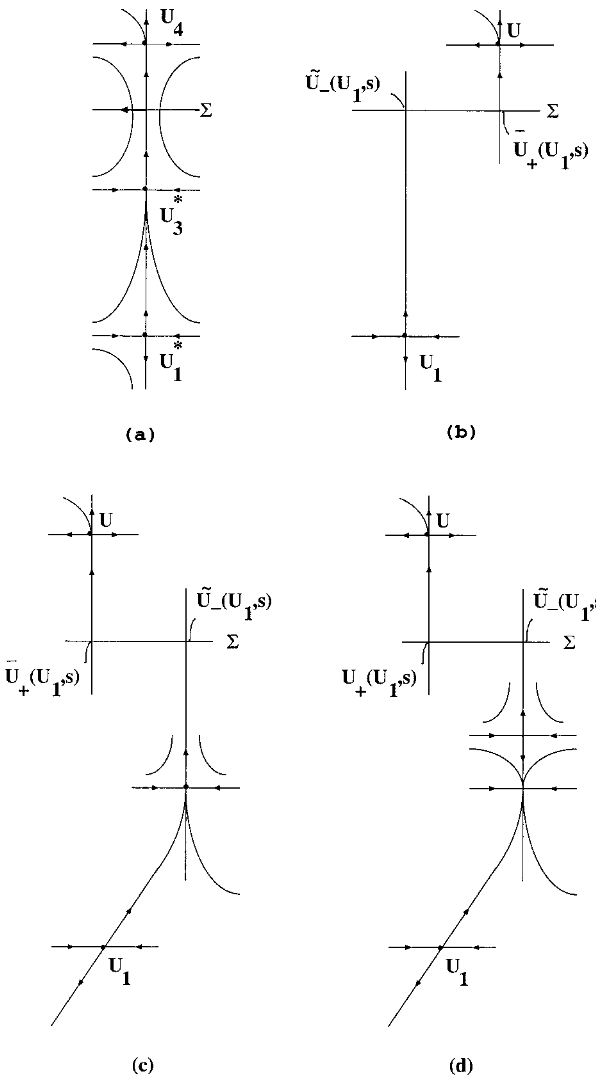


Fig. 3. Geometry of the separation function in Section 5. (a) Phase portrait of $\dot{U} = F(U) - F(U_1^*) - s_4^*(U - U_1^*)$. (b) Phase portrait of $\dot{U} = F(U) - F(U_1) - s(U - U_1)$ for a value of (U_1, s) for which $\gamma(U_1, s) > 0$, i.e., the equilibrium at U_2^* has disappeared, and for which S is positive. (c) Phase portrait for a value of (U_1, s) for which $\gamma(U_1, s) = 0$, i.e., there is a saddle-attractor near U_2^*, and for which S is negative. (d) Phase portrait for a value of (U_1, s) for which $\gamma(U_1, s) < 0$, i.e., the equilibrium at U_2^* has split into a saddle and a repeller, and for which S is negative.

We shall see that the linearization of Eqs. (5.52)–(5.55) at $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$, restricted to $\{(\dot U_0, \dot s_1, \dot U_1, \dot s, \dot U) : \dot U_0 = 0\}$, is an isomorphism. As in Step 1, it follows that $(Q7_1)$ holds.

It also follows that Eqs. (5.52)–(5.56) can be solved for (s_1, U_1, s, U) in terms of U_0 near $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$. Once U is found, the remainder of the Riemann solution is obtained by solving for $(s_5, U_5, \ldots, U_{n-1}, s_n)$ in terms of (U, U_n).

We have a Riemann solution of the desired type if and only if the function $\tilde H(U_0) := \gamma(U_1, s) \geqslant 0$. To verify (E2), we calculate

$$D\tilde H(U_0^*)\, \dot U_0 = D\gamma(U_1^*, s_4^*)(\dot U_1, \dot s) \tag{5.57}$$

by linearizing Eqs. (5.52)–(5.56) at $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$, solving the linearized equations for $(\dot s_1, \dot U_1, \dot s, \dot U)$ in terms of $\dot U_0$, and substituting the formulas for $\dot U_1$ and $\dot s$ into Eq. (5.57).

The linearization of Eqs. (5.52)–(5.55) at $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$ is:

$$\dot U_1 - D\eta(U_0^*, s_1^*)(\dot U_0, \dot s_1) = 0 \tag{5.58}$$

$$(DF(U_4^*) - s_4^* I)\, \dot U - (DF(U_1^*) - s_4^* I)\, \dot U_1 - \dot s(U_4^* - U_1^*) = 0 \tag{5.59}$$

$$D\lambda_1(U_4^*)\, \dot U - \dot s = 0 \tag{5.60}$$

$$D\tilde S(U_1^*, s_4^*)(\dot U_1, \dot s) = 0 \tag{5.61}$$

We make the substitutions (5.26), (5.27) and

$$\dot U = x r_1(U_4^*) + y r_2(U_4^*) \tag{5.62}$$

We multiply Eq. (5.58) by $\ell_1(U_1^*)$ and $\ell_2(U_1^*)$, and we multiply Eq. (5.59) by $\ell_1(U_4^*)$ and $\ell_2(U_4^*)$. We get Eqs. (5.29)–(5.30) together with:

$$\begin{aligned} &-a(\lambda_1(U_1^*) - s_4^*)\, \ell_1(U_4^*)\, r_1(U_1^*) - b(\lambda_2(U_1^*) - s_4^*)\, \ell_1(U_4^*)\, r_2(U_1^*) \\ &\quad - \dot s \ell_1(U_4^*)(U_4^* - U_1^*) = 0 \end{aligned} \tag{5.63}$$

$$\begin{aligned} &(\lambda_2(U_4^*) - s_4^*)\, y - a(\lambda_1(U_1^*) - s_4^*)\, \ell_2(U_4^*)\, r_1(U_1^*) \\ &\quad - b(\lambda_2(U_1^*) - s_4^*)\, \ell_2(U_4^*)\, r_2(U_1^*) - \dot s \ell_2(U_4^*)(U_4^* - U_1^*) = 0 \end{aligned} \tag{5.64}$$

$$D\lambda_1(U_4^*)(x r_1(U_4^*) + y r_2(U_4^*)) - \dot s = 0 \tag{5.65}$$

$$D\tilde S(U_1^*, s_4^*)(\dot U_1, \dot s_1) = 0 \tag{5.66}$$

Simplifying the notation, this system becomes Eqs. (5.36)–(5.37) together with:

$$-La-Mb-N\dot{s}=0 \tag{5.67}$$

$$Py-Qa-Rb-S\dot{s}=0 \tag{5.68}$$

$$x+Zy-\dot{s}=0 \tag{5.69}$$

$$Ia+Jb+K\dot{s}=0 \tag{5.70}$$

where the capital letters have the obvious meanings. From Eqs. (5.46), (5.57), and (5.27), and the definitions of A, B, and D in Step 1, we have

$$D\tilde{H}(U_0^*)\,\dot{U}_0=-Aa-Bb-D\dot{s} \tag{5.71}$$

Assumption (3) implies that we can solve Eqs. (5.36)–(5.37) and (5.67)–(5.70) for $(\dot{s}_1, a, b, \dot{s}, x, y)$ in terms of (i, j). We do this and and substitute into Eq. (5.71). Let

$$\mathscr{D}=A(JN-KM)+B(KL-IN)+D(IM-LJ) \tag{5.72}$$

Then we obtain

$$D\tilde{H}(U_0^*)\,\dot{U}_0=\frac{\mathscr{D}((\eta_{11}\eta_{23}-\eta_{13}\eta_{21})\,i+(\eta_{12}\eta_{23}-\eta_{13}\eta_{22})\,j)}{(KL-IN)\,\eta_{13}+(KM-JN)\,\eta_{23}} \tag{5.73}$$

We note that $\mathscr{D}$ is nonzero by assumption (4), and the second factor in the numerator is nonzero for certain (i, j) as in Step 1. The denominator is nonzero by assumption (3).

Thus $D\tilde{H}(U_0^*)$ is a nonzero vector, so that (E2) holds. Therefore $\mathscr{C}=\{U_0 : \tilde{H}(U_0)=0\}$ is a smooth curve near U_0^*, and for (U_0, U_n) near (U_0^*, U_n^*), a solution of type $(T_1, S\cdot RS, R_1, T_6, \ldots, T_n)$ exists provided U_0 is on the side of $\mathscr{C}$ to which this vector points.

Step 3. It is easy to see that the curves $\mathscr{C}$ defined in Steps 1 and 2 coincide. The last conclusion of the theorem follows from comparing Eqs. (5.44) and (5.73). □

Remark. Assumption (3) says that the generalized shock wave $U_k^* \xrightarrow{s_{k+3}^*} U_{k+3}^*$ satisfies the appropriate analogue of nondegeneracy condition (T2) for $S\cdot RS$ shock waves, as well as a wave group interaction condition. This assumption implies that the triples (U_1, s, U) that satisfy Eqs. (5.53)–(5.55) form a smooth curve $\mathscr{C}$ through (U_1^*, s_4^*, U_4^*). $\mathscr{C}$ projects to curves $\mathscr{C}_1$ through U_1^* in U_1-space and $\mathscr{C}_2$ through (U_1^*, s_4^*) in $U_1 s$-space. From Eqs. (5.67)–(5.70) it follows that, using (a, b) as coordinates of $\dot{U}_1$, the tangent vectors to $\mathscr{C}_1$ at U_1^* and to $\mathscr{C}_2$ at (U_1^*, s_4^*) are respectively $(KM-JN, IN-KL)$ and $(KM-JN, IN-KL, JL-IM)$. From the first of these formulas we see that the fact that the denominator in Eq. (5.73) is nonzero is equivalent to transversality of the one-wave curve to $\mathscr{C}_1$ at U_1^*. Now let $\mathscr{S}$ be the surface through (U_1^*, s_4^*) of points (U_1, s) such that the differential equation (5.47) has a degenerate equilibrium near U_2^*. From the second of our tangent vector formulas and Eq. (5.71), we see that the fact that $\mathscr{D}$ is nonzero is equivalent to transversality of $\mathscr{C}_2$ and $\mathscr{S}$ at (U_1^*, s_4^*). This transversality is a natural assumption to make.

6. MISSING RAREFACTION IN A 1-WAVE GROUP CONSISTING OF MORE THAN TWO WAVES

Theorem 6.1. *Let (2.5) be a Riemann solution of type* $(T_1,..., T_n)$ *for* $U_t + F^*(U)_x = 0$. *Assume there is an integer* k *such that* $T_{k+1} = R_2$, $T_{k+2} = SA \cdot RS$, $T_{k+3} = R_1$, $T_{k+4} = RS \cdot RS$, $T_{k+5} = R_1$. *Assume:*

(1) All hypotheses of Theorem 2.2 are satisfied, except that the 1-rarefaction w_{k+3}^* *has zero strength.*

(2) Let $\tilde{U}(\xi)$ *be the connection of*

$$\dot{U} = F^*(U) - F^*(U_{k+1}^*) - s_{k+2}^*(U - U_{k+1}^*)$$

from U_{k+1}^* *to* U_{k+2}^*, *and let* $\phi(\xi)$ *be a nontrivial bounded solution of*

$$\dot{\phi} + \phi[DF(\tilde{U}(\xi)) - s_{k+2}^* I] = 0$$

Then the matrix

$$\begin{pmatrix} \ell_1(U_{k+4}^*)\, r_1(U_{k+1}^*) & \ell_1(U_{k+4}^*)(U_{k+4}^* - U^*{}_{k+1}) \\ \left(\int_{-\infty}^{\infty} \phi(\xi)\, d\xi\right) r_1(U_{k+1}^*) & \int_{-\infty}^{\infty} \phi(\xi)(U(\xi) - U_{k+1}^*)\, d\xi \end{pmatrix}$$

is invertible.

Then (2.5) is a codimension-one Riemann solution that is an F-boundary. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type $(T_1,..., T_k, R_2, SA \cdot RS,\ R_1,$

$T_{k+6},\ldots, T_n$) because an equilibrium appears that breaks the connection of the $SA \cdot RS$ shock wave. If $k=1$, the join is regular or folded according to whether the expression

$$\mathscr{F}(IA-BH)(KN-LO) \tag{6.4}$$

defined below is positive or negative. If $k>1$, an analagous condition holds.

Proof. We shall assume for simplicity that $k=1$. Then the third through fifth waves of (2.5) comprise the wave sequence

$$U_2^* \xrightarrow{s_3^*} U_3^* \xrightarrow{s_4^*} U_4^* \xrightarrow{s_5^*} U_5^*$$

with $T_3 = SA \cdot RS$, $T_4 = R_1$, $T_5 = RS \cdot RS$. We have

$$s_3^* = s_4^* = s_5^* = \lambda_2(U_2^*) = \lambda_1(U_3^*) = \lambda_1(U_5^*) \qquad \text{and} \qquad U_3^* = U_4^*$$

Step 1. Let $F(U, \varepsilon)$ be a one-parameter perturbation of the flux function F^*, so that $F(U, 0) = F^*(U)$. Then $(U_2, s_3,\ldots, s_5, U_5, \varepsilon)$ near $(U_2^*, s_3^*,\ldots, s_5^*, U_5^*, 0)$ represents an admissible wave sequence of type $(SA \cdot RS, R_1, RS \cdot RS)$ for the flux function $F(U, \varepsilon)$ if and only if

$$F(U_3, \varepsilon) - F(U_2, \varepsilon) - s_3(U_3 - U_2) = 0 \tag{6.7}$$

$$\lambda_2(U_2, \varepsilon) - s_3 = 0 \tag{6.8}$$

$$\lambda_1(U_3, \varepsilon) - s_3 = 0 \tag{6.9}$$

$$S(U_2, s_3, \varepsilon) = 0 \tag{6.10}$$

$$U_4 - \psi_1(U_3, s_4, \varepsilon) = 0 \tag{6.11}$$

$$F(U_5, \varepsilon) - F(U_4, \varepsilon) - s_5(U_5 - U_4) = 0 \tag{6.12}$$

$$\lambda_1(U_4, \varepsilon) - s_5 = 0 \tag{6.13}$$

$$\lambda_1(U_5, \varepsilon) - s_5 = 0 \tag{6.14}$$

$$s_4 - \lambda_1(U_3, \varepsilon) \geqslant 0$$

By assumption (1), the linearization of Eqs. (6.7)–(6.14) at $(U_2^*, s_3^*,\ldots, s_5^*, U_5^*, 0)$, restricted to $\{(\dot{U}_2, \dot{s}_3,\ldots, \dot{s}_5, \dot{U}_5, \dot{\varepsilon}) : \dot{\varepsilon} = 0\}$, is an isomorphism. It follows that $(Q7_1)$ holds.

It also follows that we can solve Eqs. (6.7)–(6.14) for $(U_2, s_3,\ldots, s_5, U_5)$ in terms of ε near $(U_2^*, s_3^*,\ldots, s_5^*, U_5^*, 0)$. Once U_2 and U_5 are found, we can then solve for (s_1, U_1, s_2) in terms of (U_0, U_2) and $(s_6, U_6,\ldots, U_{n-1}, s_n)$ in terms of (U_5, U_n). We have a Riemann solution of the desired type if and only if $s_4 - \lambda_1(U_3, \varepsilon) \geqslant 0$. We therefore study $\tilde{H}(\varepsilon) := s_4 - \lambda_1(U_3, \varepsilon)$. To verify (E4), we calculate

$$\tilde{H}'(0)\,\dot{\varepsilon} = \dot{s}_4 - D_U\lambda_1(U_3^*, 0)\,\dot{U}_3 - \frac{\partial \lambda_1}{\partial \varepsilon}(U_3^*, 0)\,\dot{\varepsilon} \tag{6.15}$$

by linearizing Eqs. (6.7)–(6.14) at $(U_2^*, s_3^*, ..., s_5^*, U_5^*, 0)$, solving for $(\dot{U}_2, \dot{s}_2, ..., \dot{s}_5, \dot{U}_5)$ in terms of $\dot{\varepsilon}$, and substituting the formulas for $\dot{U}_3$ and $\dot{s}_4$ into Eq. (6.15).

Linearizing Eqs. (6.7)–(6.14) at $(U_2^*, s_3^*, ..., s_5^*, U_5^*, 0)$ yields:

$$(D_U F(U_3^*, 0) - s_3^* I)\,\dot{U}_3 - (D_U F(U_2^*, 0) - s_3^* I)\,\dot{U}_2 - \dot{s}_3(U_3^* - U_2^*) = -\left(\frac{\partial F}{\partial \varepsilon}(U_3^*, 0) - \frac{\partial F}{\partial \varepsilon}(U_2^*, 0)\right)\dot{\varepsilon} \tag{6.16}$$

$$D_U\lambda_2(U_2^*, 0)\,\dot{U}_2 - \dot{s}_3 = -\frac{\partial \lambda_2}{\partial \varepsilon}(U_2^*, 0)\,\dot{\varepsilon} \tag{6.17}$$

$$D_U\lambda_1(U_3^*, 0)\,\dot{U}_3 - \dot{s}_3 = -\frac{\partial \lambda_1}{\partial \varepsilon}(U_3^*, 0)\,\dot{\varepsilon} \tag{6.18}$$

$$D_U S(U_2^*, s_3^*, 0)\,\dot{U}_2 + \frac{\partial S}{\partial s}(U_2^*, s_3^*, 0)\,\dot{s}_3 = -\frac{\partial S}{\partial \varepsilon}(U_2^*, s_3^*, 0)\,\dot{\varepsilon} \tag{6.19}$$

$$\dot{U}_4 - D_U\psi_1(U_3^*, s_4^*, 0)\,\dot{U}_3 - \frac{\partial \psi_1}{\partial s}(U_3^*, s_4^*, 0)\,\dot{s}_4 = \frac{\partial \psi_1}{\partial \varepsilon}(U_3^*, s_4^*, 0)\,\dot{\varepsilon} \tag{6.20}$$

$$(D_U F(U_5^*, 0) - s_5^* I)\,\dot{U}_5 - (D_U F(U_4^*, 0) - s_5^* I)\,\dot{U}_4 - \dot{s}_5(U_5^* - U_4^*) = -\left(\frac{\partial F}{\partial \varepsilon}(U_5^*, 0) - \frac{\partial F}{\partial \varepsilon}(U_4^*, 0)\right)\dot{\varepsilon} \tag{6.21}$$

$$D_U\lambda_1(U_4^*, 0)\,\dot{U}_4 - \dot{s}_5 = -\frac{\partial \lambda_1}{\partial \varepsilon}(U_4^*, 0)\,\dot{\varepsilon} \tag{6.22}$$

$$D_U\lambda_1(U_5^*, 0)\,\dot{U}_5 - \dot{s}_5 = -\frac{\partial \lambda_1}{\partial \varepsilon}(U_5^*, 0)\,\dot{\varepsilon} \tag{6.23}$$

We set

$$\dot{U}_2 = a r_1(U_2^*) + b r_2(U_2^*) \tag{6.24}$$

$$\dot{U}_3 = c r_1(U_3^*) + d r_2(U_3^*) \tag{6.25}$$

$$\dot{U}_4 = e r_1(U_4^*) + f r_2(U_4^*) \tag{6.26}$$

$$\dot{U}_5 = g r_1(U_5^*) + h r_2(U_5^*) \tag{6.27}$$

We multiply Eqs. (6.16), (6.20), and (6.21) by $\ell_1(U_3^*)$ and $\ell_2(U_3^*)$. We get:

$$a(\lambda_1(U_2^*)-s_3^*)\,\ell_1(U_3^*)\,r_1(U_2^*)+\dot{s}_3\ell_1(U_3^*)(U_3^*-U_2^*)$$
$$=\ell_1(U_3^*)\left(\frac{\partial F}{\partial \varepsilon}(U_3^*,0)-\frac{\partial F}{\partial \varepsilon}(U_2^*,0)\right)\dot{\varepsilon} \tag{6.28}$$

$$-(\lambda_2(U_3^*)-s_3^*)\,d+a(\lambda_1(U_2^*)-s_3^*)\,\ell_2(U_3^*)\,r_1(U_2^*)+\dot{s}_3\ell_2(U_3^*)(U_3^*-U_2^*)$$
$$=\ell_2(U_3^*)\left(\frac{\partial F}{\partial \varepsilon}(U_3^*,0)-\frac{\partial F}{\partial \varepsilon}(U_2^*,0)\right)\dot{\varepsilon} \tag{6.29}$$

$$aD_U\lambda_2(U_2^*,0)\,r_1(U_2^*)+b-\dot{s}_3=-\frac{\partial \lambda_2}{\partial \varepsilon}(U_2^*,0)\,\dot{\varepsilon} \tag{6.30}$$

$$c+dD_U\lambda_1(U_3^*,0)\,r_2(U_3^*)-\dot{s}_3=-\frac{\partial \lambda_1}{\partial \varepsilon}(U_3^*,0)\,\dot{\varepsilon} \tag{6.31}$$

$$aD_US(U_2^*,s_3^*,0)\,r_1(U_2^*)+\frac{\partial S}{\partial s}(U_2^*,s_3^*,0)\,\dot{s}_3=-\frac{\partial S}{\partial \varepsilon}(U_2^*,s_3^*,0)\,\dot{\varepsilon} \tag{6.32}$$

$$e-(\dot{s}_4-dD_U\lambda_1(U_3^*,0)\,r_2(U_3^*))=\ell_1(U_3^*)\,\frac{\partial \psi_1}{\partial \varepsilon}(U_3^*,s_4^*,0)\,\dot{\varepsilon} \tag{6.33}$$

$$f-d=\ell_2(U_3^*)\,\frac{\partial \psi_1}{\partial \varepsilon}(U_3^*,s_4^*,0)\,\dot{\varepsilon} \tag{6.34}$$

$$-h(\lambda_2(U_5^*)-s_5^*)\,\ell_1(U_4^*)\,r_2(U_5^*)+\dot{s}_5\ell_1(U_4^*)(U_5^*-U_4^*)$$
$$=\ell_1(U_4^*)\left(\frac{\partial F}{\partial \varepsilon}(U_5^*,0)-\frac{\partial F}{\partial \varepsilon}(U_4^*,0)\right)\dot{\varepsilon} \tag{6.35}$$

$$-h(\lambda_2(U_5^*)-s_5^*)\,\ell_2(U_4^*)\,r_2(U_5^*)+(\lambda_2(U_4^*)-s_5^*)\,f+\dot{s}_5\ell_2(U_4^*)(U_5^*-U_4^*)$$
$$=\ell_2(U_4^*)\left(\frac{\partial F}{\partial \varepsilon}(U_5^*,0)-\frac{\partial F}{\partial \varepsilon}(U_4^*,0)\right)\dot{\varepsilon} \tag{6.36}$$

$$e+fD_U\lambda_1(U_4^*,0)\,r_2(U_4^*)-\dot{s}_5=-\frac{\partial \lambda_1}{\partial \varepsilon}(U_4^*,0)\,\dot{\varepsilon} \tag{6.37}$$

$$g+hD_U\lambda_1(U_5^*,0)\,r_2(U_5^*)-\dot{s}_5=-\frac{\partial \lambda_1}{\partial \varepsilon}(U_5^*,0)\,\dot{\varepsilon} \tag{6.38}$$

In deriving Eq. (6.32) we used the fact that $D_US(U_2^*,s_3^*,0)\,r_2(U_2^*)=0$ by Eq. (5.49), and in deriving Eqs. (6.33)–(6.34) we used Lemma 2.2 of [8].

In Eqs. (6.33)–(6.34) we note that by Lemma 6.2, which we put off to the end of this section,

$$\ell_1(U_3^*)\,\frac{\partial\psi_1}{\partial\varepsilon}\,(U_3^*, s_4^*, 0) = -\frac{\partial\lambda_1}{\partial\varepsilon}\,(U_3^*, 0) \tag{6.39}$$

$$\ell_2(U_3^*)\,\frac{\partial\psi_1}{\partial\varepsilon}\,(U_3^*, s_4^*, 0) = 0 \tag{6.40}$$

We denote the common value of the left and right sides of Eq. (6.39) by X.

Simplifying the notation, Eqs. (6.28)–(6.38) become

$$Aa + B\dot{s}_3 = U\dot{\varepsilon} \tag{6.41}$$

$$-Ed + Ca + D\dot{s}_3 = V\dot{\varepsilon} \tag{6.42}$$

$$Fa + b - \dot{s}_3 = W\dot{\varepsilon} \tag{6.43}$$

$$c + Gd - \dot{s}_3 = X\dot{\varepsilon} \tag{6.44}$$

$$Ha + I\dot{s}_3 = Y\dot{\varepsilon} \tag{6.45}$$

$$e - \dot{s}_4 + Gd = X\dot{\varepsilon} \tag{6.46}$$

$$f - d = 0 \tag{6.47}$$

$$-Nh + O\dot{s}_5 = R\dot{\varepsilon} \tag{6.48}$$

$$-Lh + Ef + K\dot{s}_5 = S\dot{\varepsilon} \tag{6.49}$$

$$e + Gf - \dot{s}_5 = X\dot{\varepsilon} \tag{6.50}$$

$$g + Mh - \dot{s}_5 = T\dot{\varepsilon} \tag{6.51}$$

where the capital letters have the obvious meanings. By assumption (1), this system can be solved for $(a, b, \dot{s}_3, c, d, \dot{s}_4, e, f, \dot{s}_5, g, h)$ in terms of $\dot{\varepsilon}$. In fact, the determinant of the left-hand side is $E(IA - BH)(KN - LO)$. E is clearly nonzero, $IA - BH$ is nonzero by nondegeneracy condition (T4) for $SA \cdot RS$ shock waves, and $KN - LO$ is nonzero by the nondegeneracy conditions for $RS \cdot RS$ shock waves. (Equations (6.48)–(6.51) for $RS \cdot RS$ shock waves are not in our usual form since we have multiplied by $\ell_i(U_4^*)$ instead of by $\ell_i(U_5^*)$, but the principle that the nondegeneracy conditions imply that the system is regular still holds.) Let

$$\begin{aligned}\mathscr{E} = {} & (IA - BH)(NR - LS + NV) \\ & + U(HDN + HKN - HLO - ICN) + Y(ALO + BNC - AKN - ADN)\end{aligned}$$

Then using Eq. (6.15) we find that

$$\tilde{H}'(0)\,\dot{\varepsilon} = \dot{s}_4 - (c + Gd) + X\dot{\varepsilon} = \frac{\mathscr{E}}{(IA - BH)(KN - LO)}\,\dot{\varepsilon} \tag{6.52}$$

To verify (E4) we must show that the perturbation $F(U, \varepsilon)$ of $F(U, 0)$ can be chosen so that $\mathscr{E} \neq 0$. We can choose $F(U, \varepsilon)$ such that (1) $F(U, \varepsilon) = F(U, 0)$ except on a small neighborhood $\mathscr{U}_5$ of U_5^*, and (2) $\frac{\partial F}{\partial \varepsilon}(U_5^*, 0)$ and $U_5^* - U_4^*$ are linearly independent. Then $NR - LS \neq 0$, and if $\mathscr{U}_5$ is sufficiently small, $U = V = Y = 0$. Therefore

$$\mathscr{E} = (IA - BH)(NR - LS) \neq 0.$$

If $\mathscr{E} \neq 0$, then for ε near 0, a Riemann solution of type (T_1, R_2, $SA \cdot RS$, R_1, $RS \cdot RS$, R_1, T_7,..., T_n) exists if and only if ε has the same sign as $\frac{\mathscr{E}}{(IA - BH)(KN - LO)}$.

Step 2. Next we consider the point (U_0^*, s_1^*, U_1^*, s_2^*, U_2^*, s_5^*, U_5^*, s_6^*,..., s_n^*, U_n^*, 0) in $\mathbb{R}^{3n-2}$. We shall investigate the existence of nearby points (U_0, s_1, U_1, s_2, U_2, s, U, s_6,..., s_n, U_n, ε) that represent Riemann solutions of type (T_1, R_2, $SA \cdot RS$, R_1, T_7,..., T_n) for the flux function $F(U, \varepsilon)$.

Consider the four-parameter family of differential equations

$$\dot{U} = F(U, \varepsilon) - F(U_2, \varepsilon) - s(U - U_2) \tag{6.54}$$

There is a function $\gamma(U_2, s, \varepsilon)$, defined near $(U_2, s, 0) = (U_2^*, s_3^*, 0)$, such that (6.54) undergoes a saddle-node bifurcation near U_3^* when the surface $\gamma = 0$ is crossed. For $\gamma > 0$ (resp. $= 0$, < 0) there are no (resp. 1, 2) equilibria of (6.54) near U_3^*. By Lemma 6.3, which we put off to the end of this section, we may take

$$\begin{aligned} D\gamma(U_2^*, s_3^*, 0)(\dot{U}_2, \dot{s}, \dot{\varepsilon}) &\\ = -\ell_1(U_3^*)(DF(U_2^*) - s_3^* I)\,\dot{U}_2 &\\ - \dot{s}\ell_1(U_3^*)(U_3^* - U_2^*) + \ell_1(U_3^*)&\left(\frac{\partial F}{\partial \varepsilon}(U_3^*, 0) - \frac{\partial F}{\partial \varepsilon}(U_2^*, 0)\right)\dot{\varepsilon} \end{aligned} \tag{6.55}$$

Next we construct a separation function $\tilde{S}(U_2, s, \varepsilon)$, (U_2, s, ε) near $(U_2^*, s_5^*, 0)$, that can be used to study connections of

$$\dot{U} = F(U, \varepsilon) - F(U_2, \varepsilon) - s(U - U_2) \tag{6.56}$$

from U_2 to equilibria near U_5^*. The construction is similar to that in Section 5, except that we use the center manifold at U_2. We find that

$$\frac{\partial \tilde{S}}{\partial s}(U_2^*, s_5^*) = -\int_{-\infty}^{\infty} \phi(\xi)(\tilde{U}(\xi) - U_2^*)\, d\xi \tag{6.57}$$

$$D_{U_2}(S)(U_2^*, s_5^*) = -\left(\int_{-\infty}^{\infty} \phi(\xi)\, d\xi\right)\{DF(U_2^*) - s_5^* I\} \tag{6.58}$$

In order that (U_2, s, U, ε) near $(U_2^*, s_5^*, U_5^*, 0)$ represent a shock wave of type $SA \cdot RS$ for the flux function $F(U, \varepsilon)$, we must have

$$F(U, \varepsilon) - F(U_2, \varepsilon) - s(U - U_2) = 0 \tag{6.59}$$

$$\lambda_2(U_2, \varepsilon) - s = 0 \tag{6.60}$$

$$\lambda_1(U, \varepsilon) - s = 0 \tag{6.61}$$

$$\tilde{S}(U_2, s, \varepsilon) = 0 \tag{6.62}$$

$$\gamma(U_2, s, \varepsilon) \geqslant 0 \tag{6.63}$$

We shall see that the linearization of Eqs. (6.59)–(6.62) at $(U_2^*, s_5^*, U_5^*, 0)$, restricted to $\{(\dot{U}_2, \dot{s}, \dot{U}, \dot{\varepsilon}) : \dot{\varepsilon} = 0\}$, is an isomorphism. It follows that $(Q7_1)$ holds.

It also follows that we can solve Eqs. (6.59)–(6.63) for (U_2, s, U) in terms of ε near $(U_2^*, s_5^*, U_5^*, 0)$. Once U_2 and U are found, we can then solve for (s_1, U_1, s_2) in terms of (U_0, U_2) and $(s_6, U_6, \ldots, U_{n-1}, s_n)$ in terms of (U, U_n). We have a Riemann solution of the desired type if and only if $\gamma(U_2, s, \varepsilon) \geqslant 0$. We therefore study $\tilde{H}(\varepsilon) := \gamma(U_2, s, \varepsilon)$. To verify (E4) we calculate

$$D\tilde{H}(0)\,\dot{\varepsilon} = D\gamma(U_2^*, s_3^*, 0)(\dot{U}_2, \dot{s}, \dot{\varepsilon}) \tag{6.64}$$

by linearizing Eqs. (6.59)–(6.63) at $(U_2^*, s_5^*, U_5^*, 0)$, solving the linearized equations for $(\dot{U}_2, \dot{s}, \dot{U})$ in terms of $\dot{\varepsilon}$, and substituting the formulas for $\dot{U}_2$ and $\dot{s}$ into Eq. (6.64).

The linearization of Eqs. (6.59)–(6.62) at $(U_2^*, s_5^*, U_5^*, 0)$ is:

$$\begin{aligned}(D_U F(U_5^*, 0) - s_5^* I)\,\dot{U}_5 - (D_U F(U_2^*, 0) - s_5^* I)\,\dot{U}_2 - \dot{s}(U_5^* - U_2^*)\\ = -\left(\frac{\partial F}{\partial \varepsilon}(U_5^*, 0) - \frac{\partial F}{\partial \varepsilon}(U_2^*, 0)\right)\dot{\varepsilon}\end{aligned} \tag{6.65}$$

$$D_U \lambda_2(U_2^*, 0)\,\dot{U}_2 - \dot{s} = -\frac{\partial \lambda_2}{\partial \varepsilon}(U_2^*, 0)\,\dot{\varepsilon} \tag{6.66}$$

$$D_U\lambda_1(U_5^*,0)\,\dot{U}-\dot{s}=-\frac{\partial\lambda_1}{\partial\varepsilon}(U_5^*,0)\,\dot{\varepsilon} \tag{6.67}$$

$$D_U\tilde{S}(U_2^*,0)\,\dot{U}_2+\frac{\partial\tilde{S}}{\partial s}(U_2^*,0)\,\dot{s}=-\frac{\partial\tilde{S}}{\partial\varepsilon}(U_2^*,0)\,\dot{\varepsilon} \tag{6.68}$$

We make the substitutions (6.24) and

$$\dot{U}=xr_1(U_5^*)+yr_2(U_5^*) \tag{6.69}$$

We multiply Eq. (6.65) by $\ell_1(U_3^*)$ and $\ell_2(U_3^*)$, which yields

$$\begin{aligned}
&-y(\lambda_2(U_5^*)-s_5^*)\,\ell_1(U_3^*)\,r_2(U_5^*)+a(\lambda_1(U_2^*)-s_5^*)\,\ell_1(U_3^*)\,r_1(U_2^*)\\
&\quad+\dot{s}\ell_1(U_3^*)(U_5^*-U_2^*)\\
&\quad=\ell_1(U_3^*)\left(\frac{\partial F}{\partial\varepsilon}(U_5^*,0)-\frac{\partial F}{\partial\varepsilon}(U_2^*,0)\right)\dot{\varepsilon}
\end{aligned} \tag{6.70}$$

$$\begin{aligned}
&-y(\lambda_2(U_5^*)-s_5^*)\,\ell_2(U_3^*)\,r_2(U_5^*)+a(\lambda_1(U_2^*)-s_5^*)\,\ell_2(U_3^*)\,r_1(U_2^*)\\
&\quad+\dot{s}\ell_2(U_3^*)(U_5^*-U_2^*)\\
&\quad=\ell_2(U_3^*)\left(\frac{\partial F}{\partial\varepsilon}(U_5^*,0)-\frac{\partial F}{\partial\varepsilon}(U_2^*,0)\right)\dot{\varepsilon}
\end{aligned} \tag{6.71}$$

Simplifying the notation, the system Eqs. (6.70)–(6.71), (6.66)–(6.68) becomes

$$-Ny+Aa+(O+B)\,\dot{s}=(R+U)\,\dot{\varepsilon} \tag{6.72}$$

$$-Ly+Ca+(K+D)\,\dot{s}=(S+V)\,\dot{\varepsilon} \tag{6.73}$$

$$Fa+b-\dot{s}=W\dot{\varepsilon} \tag{6.74}$$

$$-\dot{s}+x+My=P\dot{\varepsilon} \tag{6.75}$$

$$Ha+I\dot{s}=Y\dot{\varepsilon} \tag{6.76}$$

From Eqs. (6.55), (6.64), and (6.24), and the definitions of A, B, and U in Step 1, we have

$$D\tilde{H}(0)\,\dot{\varepsilon}=-Aa-B\dot{s}+U\dot{\varepsilon} \tag{6.77}$$

Let

$$\mathcal{F}=ICN-KHN-NDH-LIA+LHO+BHL$$

the determinant of the left-hand side of Eqs. (6.72)–(6.76). Assumption (2) implies that the left-hand sides of Eqs. (6.65)–(6.68) constitute an invertible system; in fact, assumption (2) says that the generalized $SA \cdot RS$ shock wave $U^*_{k+1} \xrightarrow{s^*_{k+4}} U^*_{k+4}$ satisfies the appropriate analogue of nondegeneracy condition (T4). Therefore $\mathscr{F} \neq 0$. Thus we can solve Eqs. (6.72)–(6.76) for $(a, b, \dot{s}, x, y)$ in terms of $\dot{\varepsilon}$. We do this and and substitute into Eq. (6.77). We obtain

$$D\tilde{H}(0)\,\dot{\varepsilon} = -\frac{\mathscr{E}}{\mathscr{F}}\,\dot{\varepsilon} \tag{6.79}$$

If $\mathscr{E} \neq 0$, then for ε near 0, a Riemann solution of type $(T_1, R_2, SA \cdot RS, R_1, T_7, \ldots, T_n)$ exists if and only if ε has the same sign as $-\frac{\mathscr{E}}{\mathscr{F}}$.

Step 3. The last conclusion of the theorem follows from comparing Eqs. (6.52) and (6.79). □

Lemma 6.2. *Let $\psi_1(U_-, s, \varepsilon)$ be the 1-rarefaction mapping of $\dot{U} = F(u, \varepsilon)$, so that ψ_1 satisfies*

$$\frac{\partial \psi_1}{\partial s}(U_-, s, \varepsilon) = r_1(\psi_1(U_-, s, \varepsilon))$$

$$\psi_1(U_-, \lambda_1(U_-, \varepsilon), \varepsilon) = U_-$$

Then

$$\ell_1(U_-)\,\frac{\partial \psi_1}{\partial \varepsilon}(U_-, \lambda_1(U_-, 0), 0) = -\frac{\partial \lambda_1}{\partial \varepsilon}(U_-, 0)$$

and

$$\ell_2(U_-)\,\frac{\partial \psi_1}{\partial \varepsilon}(U_-, \lambda_1(U_-, 0), 0) = 0$$

Proof. Let $\chi(U_-, t, \varepsilon)$ be the flow of $\dot{U} = r_1(U, \varepsilon)$, so that

$$\frac{\partial \chi}{\partial t}(U_-, t, \varepsilon) = r_1(\chi(U_-, t, \varepsilon), \varepsilon)$$

$$\chi(U_-, 0, \varepsilon) = U_-$$

Then

$$\psi_1(U_-, s, \varepsilon) = \chi(U_-, s - \lambda_1(U_-, \varepsilon), \varepsilon)$$

Therefore

$$\frac{\partial \psi_1}{\partial \varepsilon}(U_-, \lambda_1(U_-, \varepsilon), \varepsilon) = -\frac{\partial \chi}{\partial t}(U_-, 0, \varepsilon)\frac{\partial \lambda_1}{\partial \varepsilon}(U_-, \varepsilon) + \frac{\partial \chi}{\partial \varepsilon}(U_-, 0, \varepsilon)$$

$$= -r_1(U_-, \varepsilon)\frac{\partial \lambda_1}{\partial \varepsilon}(U_-, \varepsilon)$$

The result follows. □

Lemma 6.3. *Formula (6.55) for $D\gamma(U_2^*, s_3^*, 0)$ holds.*

Proof. According to center manifold reduction, the differential equation (6.54), restricted to its center manifold at $(U, U_2, s, \varepsilon) = (U_3^*, U_2^*, s_3^*, 0)$, is

$$\dot{u} = au^2 + W(U_2 - U_2^*) + b(s - s_3^*) + c\varepsilon + \cdots \tag{6.84}$$

where

$$a = \frac{1}{2}\ell_1(U_2^*)\, D^2F(U_2^*, 0)(r_1(U_2^*), r_1(U_2^*)) = \frac{1}{2}$$

$$W = -\ell_1(U_3^*)\, D_U F(U_2^*) - s_3^* I$$

$$b = -\ell_1(U_3^*)(U_3^* - U_2^*)$$

$$c = -\ell_1(U_3^*)\left(\frac{\partial F}{\partial \varepsilon}(U_3^*, 0) - \frac{\partial F}{\partial \varepsilon}(U_2^*, 0)\right)$$

The result follows. □

7. EXAMPLES

7.1. A Class of Examples

Consider the system of conservation laws

$$u_t = f(u)_x \tag{7.1}$$

$$v_t = g(u, v)_x \tag{7.2}$$

with $f(u)$ to be specified and

$$g(u, v) = 6v(2u+v) \tag{7.3}$$

Let $F(u, v) = (f(u), g(u, v))$, so that

$$DF(u, v) = \begin{pmatrix} f'(u) & 0 \\ 12v & 12(u+v) \end{pmatrix} \tag{7.4}$$

Strict hyperbolicity fails if $f'(u) = 12(u+v)$, or

$$v = \tfrac{1}{12} f'(u) - u \tag{7.5}$$

Off the curve (7.5), eigenvalues and eigenvectors of (7.4) are

$$\mu_1(u, v) = f'(u), \qquad \tilde{r}_1(u, v) = (f'(u) - 12(u+v), 12v)$$

and

$$\mu_2(u, v) = 12(u+v), \qquad \tilde{r}_2(u, v) = (0, 1)$$

Notice that

$$D\mu_1(u, v)\, \tilde{r}_1(u, v) = f''(u)(f'(u) - 12(u+v))$$
$$D\mu_2(u, v)\, \tilde{r}_2(u, v) = 12$$

Thus genuine nonlinearity for $\mu_1(u, v)$ fails along the curve (7.5) and the lines $u = a$, where $f''(a) = 0$.

7.2. A Structurally Stable Riemann Solution with a Doubly Sonic Transitional Wave

Consider the system of conservation laws (7.1)–(7.2) with

$$f(u) = (u+1)^2 (u-1)^2 \tag{7.8}$$

and $g(u, v)$ given by Eq. (7.3). Then $f''(u) = 0$ if and only if $u = \pm\frac{1}{\sqrt{3}}$. The curves (7.5) and $u = \pm\frac{1}{\sqrt{3}}$ are shown in Fig. 4.

Let

$$\mathcal{U}_- = \left\{ (u, v) : u < -\frac{1}{\sqrt{3}}, v < \frac{1}{3} u(u^2 - 4) \right\}$$

$$\mathcal{U}_+ = \left\{ (u, v) : u > \frac{1}{\sqrt{3}}, v > \frac{1}{3} u(u^2 - 4) \right\}$$

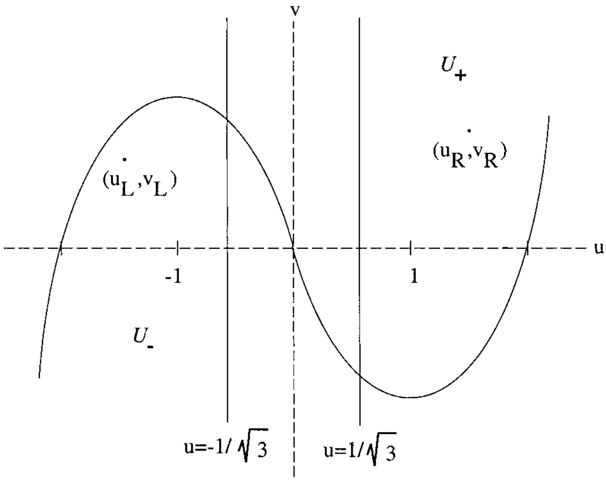


Fig. 4. Regions $\mathscr{U}_-$ and $\mathscr{U}_+$ and left and right states for a structurally stable Riemann solution containing a shock of type $SA \cdot RS$.

See Fig. 4. Strict hyperbolicity and genuine nonlinearity hold in $\mathscr{U}_-$ and $\mathscr{U}_+$. In $\mathscr{U}_-$,

$$12(u+v) = \lambda_1(u, v) < \lambda_2(u, v) = 4u(u^2-1)$$

$$r_1(u, v) = \left(0, \frac{1}{12}\right), \qquad r_2(u, v) = \frac{(u^3-4u-3v, 3v)}{4(3u^2-1)(u^3-4u-3v)}$$

In $\mathscr{U}_+$,

$$4u(u^2-1) = \lambda_1(u, v) < \lambda_2(u, v) = 12(u+v)$$

$$r_1(u, v) = \frac{(u^3-4u-3v, 3v)}{4(3u^2-1)(u^3-4u-3v)}, \qquad r_2(u, v) = \left(0, \frac{1}{12}\right)$$

Thus in $\mathscr{U}_-$, 1-rarefactions are vertical lines, with λ_1 increasing in the upward direction; 2-rarefactions are horizontal curves (but note that the u-axis is a 2-rarefaction), with λ_2 increasing to the right. In $\mathscr{U}_+$, 1-rarefactions are horizontal curves (but note that the u-axis is a 1-rarefaction), with λ_1 increasing to the right; 2-rarefactions are vertical lines, with λ_2 increasing in the upward direction. In $\mathscr{U}_-$, 1-shock curves are vertical lines. There is a 1-shock from (u_-, v_-) to (u_-, v_+) if $v_- > v_+$; the speed is $s = 6(2u_- + v_- + v_+)$. In $\mathscr{U}_+$ the same holds for 2-shock curves.

Let us consider shocks with left state $(-1, 0)$ and speed 0. Since $F(-1, 0) = (0, 0)$, the differential equation for connecting orbits is

$$\dot{u} = (u+1)^2 (u-1)^2$$
$$\dot{v} = 6v(2u+v)$$

The phase portrait near the u-axis is shown in Fig. 1, with the equilibria at $(-1, 0)$ and $(1, 0)$. There is thus an $SA \cdot RS$ shock wave from $(-1, 0)$ to $(1, 0)$ with speed 0. The nondegeneracy conditions are satisfied. To check (T4) let $U^*(\xi) = (u(\xi), 0)$ be the connection from $(-1, 0)$ to $(1, 0)$. A bounded solution to the adjoint equation is

$$\phi(\xi) = \exp\left(-\int_0^\xi \operatorname{div} F(u(\sigma), 0)\, d\sigma\right)(0, \dot{u}(\xi))^T = (0, a(\xi))^T, \qquad a(\xi) > 0 \tag{7.9}$$

Therefore

$$\int_{-\infty}^{\infty} \phi(\xi)\, d\xi = (0, a)^T, \qquad a > 0 \tag{7.10}$$

Now $r_1(-1, 0) = (0, \frac{1}{12})$, $r_2(-1, 0) = (\frac{1}{8}, 0)$, $r_1(1, 0) = (\frac{1}{8}, 0)$, $r_2(1, 0) = (0, \frac{1}{12})$. Therefore

$$\ell_1(1, 0)\, r_1(-1, 0) = (8, 0)^T \left(0, \frac{1}{12}\right) = 0$$

$$\ell_1(1, 0)((1, 0) - (-1, 0)) = (8, 0)^T\, (2, 0) = 16$$

$$\left(\int_{-\infty}^{\infty} \phi(\xi)\, d\xi\right) r_1(-1, 0) = (0, a)^T \left(0, \frac{1}{12}\right) = \frac{a}{12}$$

$$\int_{-\infty}^{\infty} \phi(\xi)(U(\xi) - U_-)\, d\xi = \int_{-\infty}^{\infty} (0, a(\xi))^T\, (\dot{u}(\xi) + 1, 0)\, d\xi = 0$$

Thus (T4) is satisfied.

Consider the Riemann problem with

$$U_L = (u_L, v_L), \qquad -2 < u_L < -1, \qquad 0 < v_L < \tfrac{1}{3} u_L(u_L^2 - 4) \tag{7.11}$$

$$U_R = (u_R, v_R), \qquad 1 < u_R < 2, \qquad 0 < v_R \tag{7.12}$$

One can construct a Riemann solution consisting of a 1-shock from (u_L, v_L) to $(u_L, 0)$, a 2-rarefaction from $(u_L, 0)$ to $(-1, 0)$, an $SA \cdot RS$ shock wave from $(-1, 0)$ to $(1, 0)$, a 1-rarefaction from $(1, 0)$ to $(u_R, 0)$, and a 2-rarefaction from $(u_R, 0)$ to (u_R, v_R). Figure 5 shows a solution to this Riemann problem computed numerically by the upwind method. Its structure agrees exactly with that predicted by the theory.

7.3. A Codimension-One Riemann Solution Illustrating Theorem 4.1

We consider the Riemann problem for Eqs. (7.1)–(7.2), (7.8), and (7.3), with

$$U_L = (-1, v_L), \qquad 0 < v_L < 1 \tag{7.13}$$

and U_R given by (7.12). The Riemann solution consists of a 1-shock from $(-1, v_L)$ to $(-1, 0)$, an $SA \cdot RS$ shock wave from $(-1, 0)$ to $(1, 0)$, and two rarefactions as above. This is a "missing rarefaction" solution; the line of U_L that satisfy (7.13) is a U_L-boundary. The previous subsection illustrates Riemann solutions with U_L to the left of this line. Theorem 4.1 predicts that for U_L to the right of this line and the same U_R, there should be a Riemann solution consisting of a 1-shock from U_L to a state near $(-1, 0)$, an $S \cdot RS$ shock wave from this state to one near $(1, 0)$, and two rarefactions. In fact, let U_L be a point in $\mathscr{U}_-$ with $-1 < u_L$ and $0 < v_L$. There is a 1-shock from (u_L, v_L), to $(u_L, 0)$ with speed $s = 6(2u_- + v_- + v_+)$. Now a shock with left state $(u_L, 0)$ and speed s has a connection that satisfies the differential equation

$$\dot{u} = (u+1)^2 (u-1)^2 - (u_L+1)^2 (u_L-1)^2 - s(u-u_L) \tag{7.14}$$

$$\dot{v} = 6v(2u+v) - sv \tag{7.15}$$

There is a connection along the u-axis from $(u_L, 0)$ to a second equilibrium $(u_+, 0)$ provided u_+ is an equilibrium of (7.14) and there is a solution of (7.14) that goes from u_L to u_+. To understand the phase portrait of (7.14), we graph the quartic function $(u+1)^2 (u-1)^2$ and the linear function $(u_L+1)^2 (u_L-1)^2 + s(u-u_L)$ on the same axes. For appropriate s the graph is shown in Fig. 6; the line is tangent to the curve at $u = u_+$ near $u = 1$. This implies that there is an $S \cdot RS$ shock wave from $(u_L, 0)$ to $(u_+, 0)$ with speed s. The Riemann solution concludes with two rarefactions.

7.4. A Codimension-One Riemann Solution Illustrating Theorem 5.1

Consider the system of conservation laws (7.1)–(7.2) with

$$f(u) = (u+3)(u+1)^2 (u-1)^2 \tag{7.16}$$

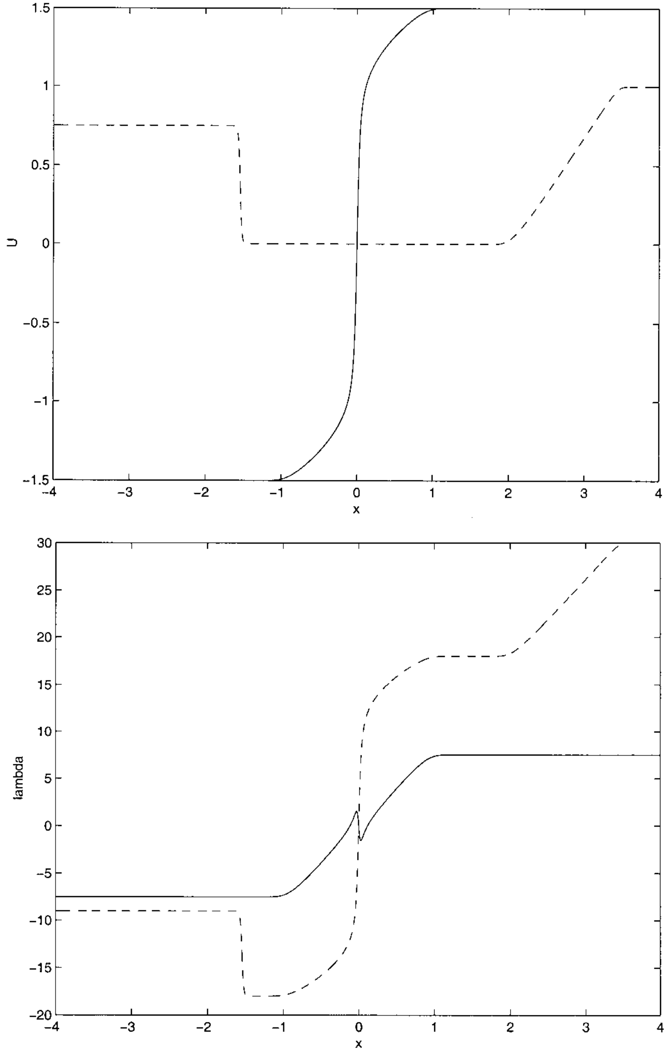


Fig. 5. Riemann solution computed by the upwind method. The system of conservation laws is Eqs. (7.1)–(7.3), (7.8); the Riemann data are $U_L = (-1.5, 0.75)$, $U_R = (1.5, 1.0)$. The first graph shows the solution on $-4 \leqslant x \leqslant 4$ at $t = 4/35$. A shift $x \to x - 35t$ has been used in the course of the computation to permit use of the upwind method. The solid curve is $u(x)$, the dashed curve is $v(x)$. The $SA \cdot RS$ shock wave is the abrupt change in u near $x = 0$. The smooth changes in u that precede and follow it are 1- and 2-rarefactions respectively. The abrupt change in v at the left is a 1-shock; the smooth change in v at the right is a 2-shock. The second graph shows the eigenvalues of $DF(U(x))$ for this solution. The eigenvalue $4u(u^2 - 1)$ is the solid curve, and the eigenvalue $12(u + v)$ is the dashed curve. Along the rarefactions, the corresponding eigenvalues change linearly.

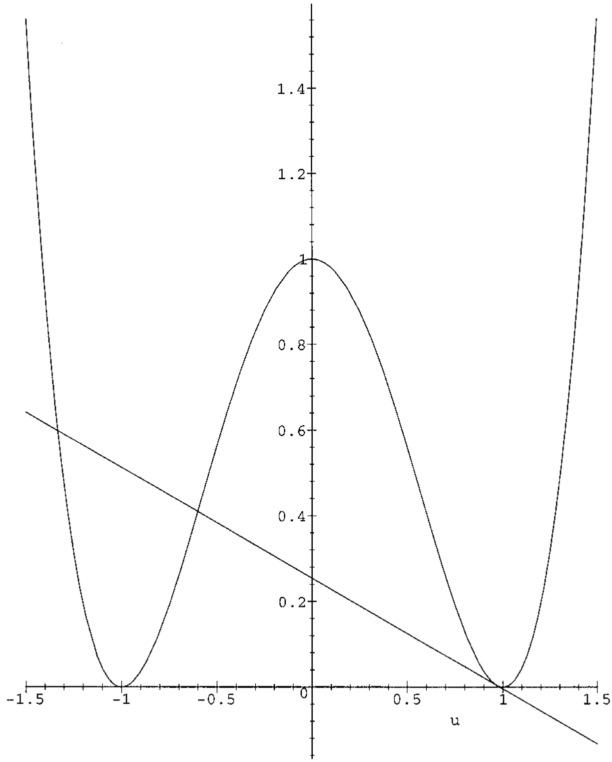


Fig. 6. Graphs of the quartic function $(u+1)^2(u-1)^2$ and the linear function $(u_L+1)^2(u_L-1)^2+s(u-u_L)$, with $u_L = -0.6$ and $s = -0.2587$.

and $g(u, v)$ given by Eq. (7.3). Then $f''(u) = 0$ if and only if $u = a_i$, $i = 1$–3, where $a_1 \simeq -1.95$, $a_2 \simeq -0.485$, $a_3 \simeq 0.635$. The curve (7.5) and the lines $u = a_i$ are shown in Fig. 7.

Let

$$\mathscr{U}_1 = \{(u, v) : u < a_1, v < \tfrac{1}{12} f'(u) - u\}$$
$$\mathscr{U}_2 = \{(u, v) : a_1 < u < a_2, v < \tfrac{1}{12} f'(u) - u\}$$
$$\mathscr{U}_3 = \{(u, v) : a_3 < u, \tfrac{1}{12} f'(u) - u < v\}$$

Strict hyperbolicity and genuine nonlinearity hold in each $\mathscr{U}_i$. In $\mathscr{U}_1$ and $\mathscr{U}_2$,

$$12(u+v) = \lambda_1(u, v) < \lambda_2(u, v) = f'(u)$$

$$r_1(u, v) = \left(0, \frac{1}{12}\right), \qquad r_2(u, v) = \frac{(f'(u) - 12(u+v), 12v)}{f''(u)(f'(u) - 12(u+v))}$$

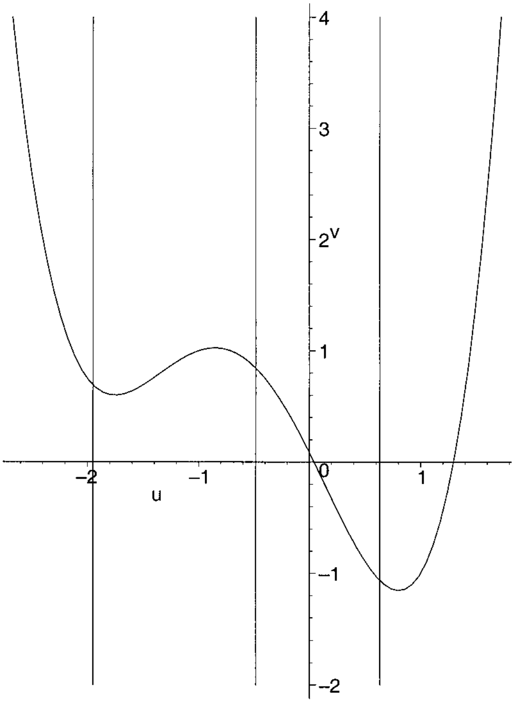


Fig. 7. Graph of $v = \frac{1}{12} f'(u) - u$, with $f(u) = (u+3)(u+1)^2 (u-1)^2$, and the lines $u = a_i$, $i = 1$–3.

In $\mathcal{U}_3$,

$$f'(u) = \lambda_1(u, v) < \lambda_2(u, v) = 12(u+v)$$

$$r_1(u, v) = \frac{(f'(u) - 12(u+v), 12v)}{f''(u)(f'(u) - 12(u+v))}, \qquad r_2(u, v) = \left(0, \frac{1}{12}\right)$$

The vector $\frac{(f'(u)-12(u+v), 12v)}{f''(u)(f'(u)-12(u+v))}$ points to the right if $f''(u)$ is positive, which occurs in $\mathcal{U}_2$ and $\mathcal{U}_3$, and to the left if $f''(u)$ is negative, which occurs in $\mathcal{U}_1$.

Let us consider shocks with left state $(-3, 0)$ or $(-1, 0)$ and speed 0. Since $F(-3, 0) = F(-1, 0) = (0, 0)$, the differential equation for connecting orbits is

$$\dot{u} = f(u)$$

$$\dot{v} = g(u, v)$$

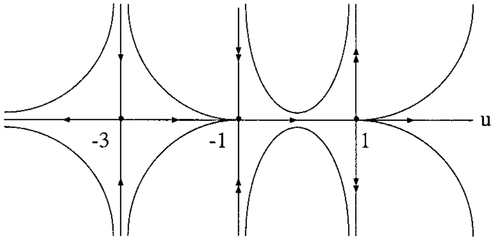


Fig. 8. Phase portrait of $\dot{u} = (u+3)(u+1)^2\,(u-1)^2$, $\dot{v} = 12(u+v)$.

The phase portrait near the u-axis is shown in Fig. 8. There is thus an $S \cdot SA$ shock wave from $(-3, 0)$ to $(-1, 0)$ and an $SA \cdot RS$ shock wave from $(-1, 0)$ to $(1, 0)$, both with speed 0.

Let b be the largest solution of $\frac{1}{12} f'(u) - u = 0$, $b \simeq 1.295$. We consider the Riemann problem with

$$U_L = (-3, v_L), \qquad v_L < 0 \tag{7.17}$$

$$U_R = (u_R, v_R), \qquad 1 < u_R < b, \qquad 0 < v_R \tag{7.18}$$

both in the strictly hyperbolic region. There is a Riemann solution consisting of a 1-rarefaction from $(-3, v_L)$ to $(-3, 0)$, an $S \cdot SA$ shock wave from $(-3, 0)$ to $(-1, 0)$, an $SA \cdot RS$ shock wave from $(-1, 0)$ to $(1, 0)$, and two rarefactions. This is a "missing rarefaction" solution; the line of U_L that satisfy (7.17) is a U_L-boundary.

One can check that the $SA \cdot RS$ shock wave satisfies nondegeneracy condition (T4) as in Section 7.2. To check that assumption (3) of Theorem 5.1 is satisfied, let $\tilde{U}(\xi) = (u(\xi), 0)$ be the connection from $(-1, 0)$ to $(1, 0)$. A bounded solution to the adjoint equation is given by (7.9), so that (7.10) holds. Now the 1-wave curve at $(-3, 0)$ has tangent vector $r_1(-3, 0) = (0, \frac{1}{12})$. Thus

$$\begin{pmatrix} \ell_1(1, 0) \\ \int_{-\infty}^{\infty} \phi(\xi)\, d\xi \end{pmatrix} (DF(-3, 0) - 0I)\, r_1(-3, 0)$$

$$= \begin{pmatrix} 32 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} f'(-3) & 0 \\ 0 & -36 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{12} \end{pmatrix} = \begin{pmatrix} 0 \\ -3a \end{pmatrix}$$

and

$$\begin{pmatrix} \ell_1(1,0)((1,0)-(-3,0)) \\ \int_{-\infty}^{\infty} \phi(\xi)(U(\xi)-(-3,0)\,d\xi \end{pmatrix} = \begin{pmatrix} (32,0)^T\,(4,0) \\ \int_{-\infty}^{\infty} (0,a(\xi))^T\,(\dot{u}(\xi)+3,0)\,d\xi \end{pmatrix} = \begin{pmatrix} 128 \\ 0 \end{pmatrix}$$

are linearly independent.

To check that assumption (4) of Theorem 5.1 is satisfied, we recall that $\mathscr{D}$ is defined by Eq. (5.72), where the meanings of the letters are deduced from Eqs. (5.31), (5.38), (5.63), (5.66), (5.67), and (5.70). It is easy to see that $M=B=D=0$, so $\mathscr{D}=AJN$. It is also easy to see that A, J, and N are nonzero. Therefore $\mathscr{D}$ is nonzero.

The expression (5.4), which we shall not compute, is positive. However, one can see that the join is regular as follows. A shock with left state $(u_L, 0)$ and speed s has a connection that satisfies the differential equation

$$\dot{u} = f(u) - f(u_L) - s(u-u_L) \tag{7.21}$$

$$\dot{v} = 6v(2u+v) - sv \tag{7.22}$$

For u_L a little less than -3 and appropriate s, the graphs of $f(u)$ and $f(u_L)+s(u-u_L)$ are tangent at a point near $(1, 0)$ with no intermediate crossings; see Fig. 9. For -3 a little less than u_L and appropriate s, the graphs of $f(u)$ and $f(u_L)+s(u-u_L)$ are tangent at a point near and to the left of $(-1, 0)$ with no intermediate crossings; see Fig. 9. Thus for $U_L = (u_L, v_L)$ with u_L a little less than -3 and $v_L < 0$, and U_R given by (7.18), there is a Riemann solution consisting of a 1-rarefaction from (u_L, v_L) to $(u_L, 0)$, an $S \cdot RS$ shock wave from $(u_L, 0)$ to a point on the u-axis to the right of $(1, 0)$, a 1-rarefaction from there to $(u_R, 0)$, and a 2-rarefaction from there to (u_R, v_R). For $U_L = (u_L, v_L)$ with u_L a little bigger than -3, and U_R given by (7.18), there is a Riemann solution consisting of a 1-rarefaction from (u_L, v_L) to $(u_L, 0)$, an $S \cdot SA$ shock wave from $(u_L, 0)$ to a point on the u-axis to the left of $(-1, 0)$, a 2-rarefaction from there to $(-1, 0)$, an $SA \cdot RS$ shock wave from $(-1, 0)$ to $(1, 0)$, and two more rarefactions.

7.5. A Codimension-One Riemann Solution Illustrating Theorem 6.1

Consider the system of conservation laws (7.1)–(7.2) with

$$f(u) = (u+1)^2\,(u-1)^2\,(u-3)^2 \tag{7.23}$$

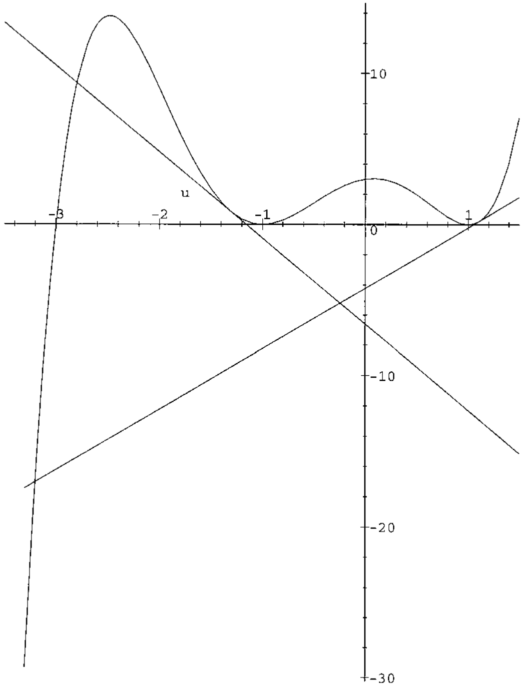


Fig. 9. Graphs of the quintic function $(u+3)(u+1)^2(u-1)^2$ and the linear function $(u_L+1)^2(u_L-1)^2+s(u-u_L)$, with (1) $u_L=-3.2$ and $s=4.0130$, and (2) $u_L=-2.8$ and $s=-5.6986$.

and $g(u, v)$ given by Eq. (7.3). Then $f''(u)=0$ if and only if $u=a_i$, $i=1$–4, where $a_1 \simeq -0.680$, $a_2 \simeq 0.385$, $a_3 \simeq 1.61$, $a_4 \simeq 2.680$. The curve (7.5) and the lines $u=a_i$ are shown in Fig. 10.

Let

$$\mathcal{U}_1 = \{(u, v) : u < a_1, v < \tfrac{1}{12} f'(u) - u\}$$
$$\mathcal{U}_2 = \{(u, v) : a_2 < u < a_3, \tfrac{1}{12} f'(u) - u < v\}$$
$$\mathcal{U}_3 = \{(u, v) : a_4 < u, \tfrac{1}{12} f'(u) - u < v\}$$

Strict hyperbolicity and genuine nonlinearity hold in each $\mathcal{U}_i$. In $\mathcal{U}_1$,

$$12(u+v) = \lambda_1(u, v) < \lambda_2(u, v) = f'(u)$$

$$r_1(u, v) = \left(0, \frac{1}{12}\right), \qquad r_2(u, v) = \frac{(f'(u)-12(u+v), 12v)}{f''(u)(f'(u)-12(u+v))}$$

In $\mathscr{U}_2$ and $\mathscr{U}_3$,

$$f'(u) = \lambda_1(u, v) < \lambda_2(u, v) = 12(u+v)$$

$$r_1(u, v) = \frac{(f'(u) - 12(u+v), 12v)}{f''(u)(f'(u) - 12(u+v))}, \qquad r_2(u, v) = \left(0, \frac{1}{12}\right)$$

The vector $\frac{(f'(u)-12(u+v), 12v)}{f''(u)(f'(u)-12(u+v))}$ points to the right if $f''(u)$ is positive, which occurs in all three $\mathscr{U}_i$.

Let us consider shocks with left state $(-1, 0)$ or $(1, 0)$ and speed 0. Since $F(-1, 0) = F(1, 0) = (0, 0)$, the differential equation for connecting orbits is

$$\dot{u} = f(u)$$

$$\dot{v} = g(u, v)$$

The phase portrait near the u-axis is shown in Fig. 11. There is thus an $SA \cdot RS$ shock wave from $(-1, 0)$ to $(1, 0)$ and an $RS \cdot RS$ shock wave from $(1, 0)$ to $(3, 0)$, both with speed 0.

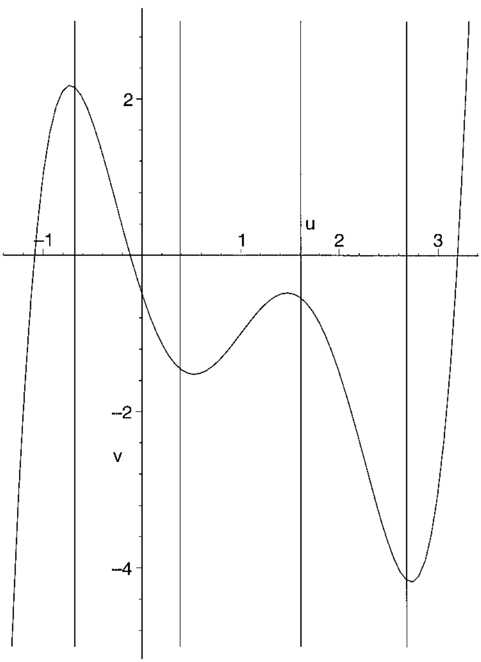


Fig. 10. Graph of $v = \frac{1}{12} f'(u) - u$, with $f(u) = (u+1)^2 (u-1)^2 (u-3)^2$, and the lines $u = a_i$, $i = 1$–4.

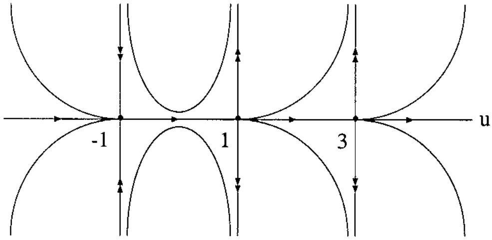


Fig. 11. Phase portrait of $\dot{u} = (u+1)^2 (u-1)^2 (u-3)^2$, $\dot{v} = 12(u+v)$.

Let b_1 and b_3 be the smallest and largest solutions of $\frac{1}{12} f'(u) - u = 0$, $b_1 \simeq -1.085$, $b_3 \simeq 3.198$. We consider the Riemann problem with

$$U_L = (u_L, v_L), \qquad b_1 < u_L < -1, \qquad v_L < 0 \tag{7.24}$$

$$U_R = (u_R, v_R), \qquad 3 < u_R < b_3, \qquad 0 < v_R \tag{7.25}$$

both in the strictly hyperbolic region. There is a Riemann solution consisting of a 1-rarefaction from (u_L, v_L) to $(u_L, 0)$, a 2-rarefaction from there to $(-1, 0)$, an $SA \cdot RS$ shock wave from $(-1, 0)$ to $(1, 0)$, an $RS \cdot RS$ shock wave from $(1, 0)$ to $(3, 0)$, and two more rarefactions. This is a "missing rarefaction" solution. From its description we see that it is stable to perturbation of U_L and U_R. To create a structurally stable Riemann solution, we must change F.

As in Section 7.2 we can check that the $SA \cdot RS$ shock wave satisfies nondegeneracy condition (T4). The check that assumption (2) of Theorem 6.1 holds is a computation similar to one in the previous section.

The expression (6.4), which we shall not compute, is positive. Thus we have an F-boundary that is a regular join. Let us give a concrete perturbation of the flux function that exhibits this fact. Consider the family of conservation laws

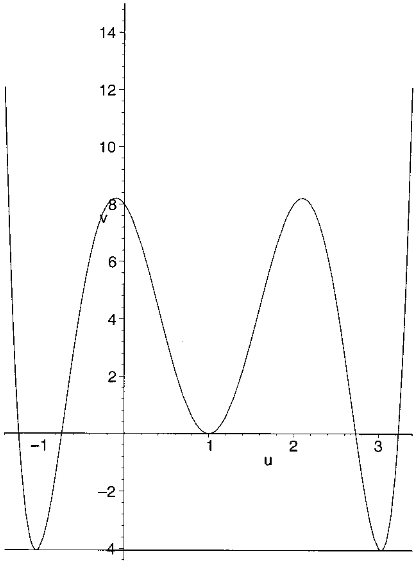
$$u_t = f(u, \varepsilon)_x$$

$$v_t = g(u, v)_x$$

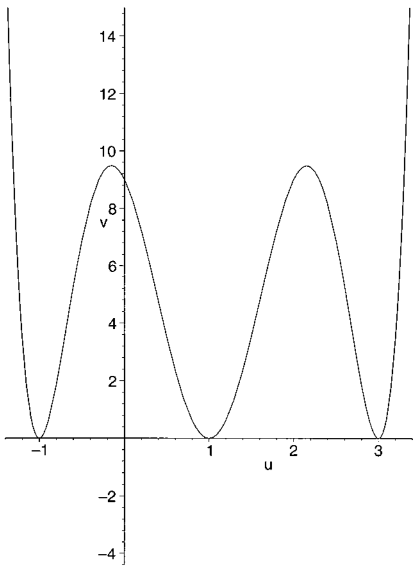
with

$$f(u.\varepsilon) = (u+1)^2 (u-1)^2 (u-3)^2 + \varepsilon(u-1)^2$$

and $g(u, v)$ given by Eq. (7.3). Graphs of $f(u, -1)$, $f(u, 0)$ (previously called f), and $f(u, 1)$ are shown in Fig. 12.

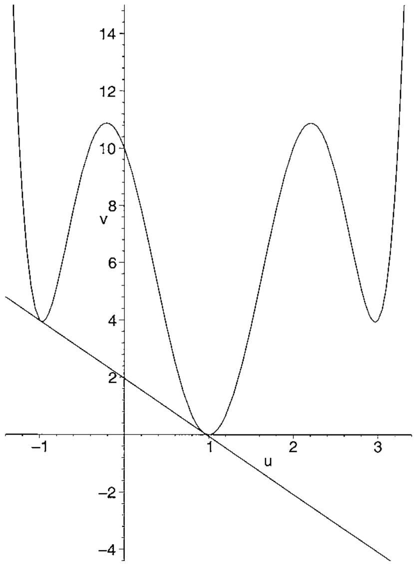


a



b

Fig. 12. Graphs of (a) $(u+1)^2\,(u-1)^2\,(u-3)^2-(u-1)^2$ and a linear function (b) $(u+1)^2\,(u-1)^2\,(u-3)^2$, and (c) $(u+1)^2\,(u-1)^2\,(u-3)^2+(u-1)^2$ and a linear function.



c

Fig. 12. (*Continued*).

For small $\varepsilon < 0$ and Riemann data as before, the Riemann solution is of type $(R_1, R_2, SA \cdot RS, R_1, R_2)$. To see this for $\varepsilon = -1$, consider Fig. 12a. As shown in the figure, for $u_L \simeq -1.030$ and $s = 0$, the graphs of $f(u, -1)$ and $f(u_L, -1) + s(u - u_L)$ are tangent at $u = u_L$ and again at $u \simeq 3.030$, and the graph of $f(u, -1)$ lies above the graph of the line. The existence of the desired Riemann solution follows easily.

On the other hand, for small $\varepsilon > 0$ and Riemann data as before, the Riemann solution is of type $(R_1, R_2, SA \cdot RS, R_2, RS \cdot RS, R_1, R_2)$. To see this for $\varepsilon = 1$, consider Fig. 12c. As shown in the figure, for $u_L \simeq -0.984$ and $s \simeq -2.022$, the graphs of $f(u, 1)$ and $f(u_L, 1) + s(u - u_L)$ are tangent at $u = u_L$ and again at $u \simeq .940$, and the graph of $f(u, 1)$, lies above the graph of the line. The symmetric line about $u = 1$ has the symmetric property: for $u_L \simeq 1.060$ and $s \simeq 2.022$, the graphs of $f(u, 1)$ and $f(u_L, 1) + s(u - u_L)$ are tangent at $u = u_L$ and again at $u \simeq 2.984$, and the graph of $f(u, 1)$, lies above the graph of the line. Again the existence of the desired Riemann solution follows easily.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grants DMS-9205535 and DMS-9501255.

REFERENCES

1. Čanić, S., and Plohr, B. J. (1995). Shock wave admissibility for quadratic conservation laws. *J. Differential Equations* **118**, 293–335.
2. Čanić, S. (1997). Quadratic systems of conservation laws with generic behavior at infinity. *J. Dynamics and Differential Equations* **9**, 401–426.
3. Dumortier, F., El Morsalini, M., and Rousseau, C. (1996). Hilbert's 16th problem for quadratic systems and cyclicity of elementary graphics. *Nonlinearity* **9**, 1209–1261.
4. Isaacson, E. L., Marchesin, D., Plohr, B. J., and Temple, J. B. (1992). Multiphase flow models with singular Riemann problems. *Mat. Aplic. Comp.* **11**, 147–166.
5. Marchesin, D., Plohr, B. J., and Schecter, S. (1997). An organizing center for wave bifurcation in multiphase flow models. *SIAM. J. Appl. Math.* **57**, 1189–1215.
6. Schecter, S., Marchesin, D., and Plohr, B. J. (1996). Structurally stable Riemann solutions. *J. Differential Equations* **126**, 303–354.
7. Schecter, S., Plohr, B. J., and Marchesin, D. (2001). Classification of codimension-one Riemann solutions, *J. Dynamics and Differential Equations* **13**, 523–588.
8. Schecter, S. (1999). Codimension-one Riemann solutions: classical missing rarefaction cases. *J. Differential Equations* **157**, 247–318.
9. Schecter, S. (2000). Codimension-one Riemann solutions: missing rarefactions in transitional wave groups, *Adv. Differential Equations* **5**, 929–975.
10. de Souza, A. (1992). Stability of singular fundamental solutions under perturbations for flow in porous media. *Mat. Aplic. Comp.* **11**, 73–115.