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STABILITY OF SELF-SIMILAR SOLUTIONS OF THE DAFERMOS REGULARIZATION OF A SYSTEM OF CONSERVATION LAWS*

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Abstract. In contrast to a viscous regularization of a system of n conservation laws, a Dafermos regularization admits many self-similar solutions of the form $u = u(\frac{X}{T})$. In particular, it is known in many cases that Riemann solutions of a system of conservation laws have nearby self-similar smooth solutions of an associated Dafermos regularization. We refer to these smooth solutions as *Riemann–Dafermos solutions*. In the coordinates $x = \frac{X}{T}$, $t = \ln T$, Riemann–Dafermos solutions become stationary, and their time-asymptotic stability as solutions of the Dafermos regularization can be studied by linearization. We study the stability of Riemann–Dafermos solutions near Riemann solutions consisting of n Lax shock waves. We show, by studying the essential spectrum of the linearized system in a weighted function space, that stability is determined by eigenvalues only. We then use asymptotic methods to study the eigenvalues and eigenfunctions. We find there are fast eigenvalues of order $\frac{1}{\epsilon}$ and slow eigenvalues of order 1. The fast eigenvalues correspond to eigenvalues of the viscous profiles for the individual shock waves in the Riemann solution; these have been studied by other authors using Evans function methods. The slow eigenvalues are related to inviscid stability conditions that have been obtained by various authors for the underlying Riemann solution.

Key words. conservation law, Riemann problem, Dafermos regularization, stability, spectrum, singular perturbation

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1. Introduction.

Consider a system of *viscous conservation laws* in one space dimension, i.e., a partial differential equation of the form

$$u_T + f(u)_X = (B(u)u_X)_X, \tag{1.1}$$

where $X \in \mathbb{R}$, $T \in [0, \infty)$, $u \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, and $B(u)$ is an $n \times n$ matrix for which all eigenvalues have positive real part. We are interested in the behavior, as $T \to \infty$, of solutions of (1.1) that satisfy the constant boundary conditions

$$u(-\infty, T) = u^\ell, \quad u(+\infty, T) = u^r, \quad 0 \le T < \infty, \tag{1.2}$$

and some initial condition $u(X, 0) = u^0(X)$. Our interest is not in the solution for any particular initial condition, but in the possible asymptotic behavior of solutions as $T \to \infty$.

It is believed that as $T \to \infty$, solutions of such initial-boundary-value problems typically approach Riemann solutions for the system of conservation laws

$$u_T + f(u)_X = 0 \tag{1.3}$$

obtained from (1.1) by dropping the viscous term. In numerical simulations, the convergence is seen when the solution is viewed in the rescaled spatial variable $x = \frac{X}{T}$; the rescaling counteracts the tendency of the solution to spread as time increases. The

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shock waves in the observed Riemann solution satisfy the viscous profile criterion for the viscosity $B(u)$. Speaking very roughly, Riemann solutions are believed to play the same role for (1.1)–(1.2) that constant solutions (equilibria) play for ordinary differential equations (ODEs): they are the simplest asymptotic states. An important difference, however, is that Riemann solutions are not solutions of (1.1) but only of the related equation (1.3). We recall that a *shock wave* is a weak solution with a jump discontinuity of the system of conservation laws (1.3). The simplest such solutions are

$$u(X,T) = \begin{cases} u^- & \text{for } X < sT, \\ u^+ & \text{for } X > sT. \end{cases} \tag{1.4}$$

For (1.4) to be a weak solution of (1.3), the triple (u^-, s, u^+) must satisfy the *Rankine–Hugoniot condition*

$$f(u^+) - f(u^-) - s(u^+ - u^-) = 0. \tag{1.5}$$

A shock wave (1.4) satisfies the *viscous profile criterion* for the viscosity $B(u)$, provided (1.1) has a traveling wave solution $u(X - sT)$ that satisfies the boundary conditions

$$u(-\infty) = u^-, \quad u(+\infty) = u^+. \tag{1.6}$$

A traveling wave solution of (1.1) that satisfies these boundary conditions exists if and only if the *traveling wave ODE*

$$\dot{u} = B(u)^{-1}(f(u) - f(u^-) - s(u - u^-)) \tag{1.7}$$

has an equilibrium at u^+ (it automatically has one at u^-) and a connecting orbit from u^- to u^+. The condition that (1.7) have an equilibrium at u^+ is just the Rankine–Hugoniot condition (1.5).

A *Riemann problem* for the system of conservation laws (1.3) is an initial value problem of the form

$$u(X,0) = \begin{cases} u^\ell & \text{for } X < 0, \\ u^r & \text{for } X > 0. \end{cases} \tag{1.8}$$

Since (1.3), (1.8) is invariant under the transformations $(X,T) \to (aX, aT)$, to avoid one-parameter families of solutions, a solution $u(X,T)$ of (1.3), (1.8) should have the form $u(X,T) = \hat{u}(x)$, $x = \frac{X}{T}$. Then $\hat{u}(x)$ satisfies

$$(Df(u) - xI)u_x = 0, \quad -\infty < x < \infty; \qquad u(-\infty) = u^\ell, \quad u(\infty) = u^r. \tag{1.9}$$

Notice that even though a Riemann problem in the form (1.3), (1.8) is an initial value problem, in the form (1.9) it is a boundary value problem.

Normally one looks for a solution of (1.9) consisting of constant parts, continuously changing parts (*rarefaction waves*), and jump discontinuities (shock waves). Shock waves occur when

$$\lim_{x \to s-} \hat{u}(x) = u^- \neq u^+ = \lim_{x \to s+} \hat{u}(x).$$

We shall require that each such triple (u^-, s, u^+) satisfy the viscous profile criterion for a given $B(u)$.

It is known that even with the viscous profile criterion, Riemann problems can have multiple solutions. This is disconcerting if the Riemann problem is regarded as an initial value problem. There is no such difficulty, however, when Riemann problems are regarded as boundary value problems whose solutions represent asymptotic states of (1.1)–(1.2). Indeed, in this context, multiple solutions of a Riemann problem represent multiple asymptotic states of (1.1)–(1.2), which are approached for different initial conditions $u^0(X)$. For a model initial-boundary-value problem (1.1)–(1.2) whose associated Riemann problem has three solutions, Azevedo et al. [2] have done careful numerical work that indicates that this is in fact the case. Two of the Riemann solutions appear to be attractors, while the third appears to attract a codimension-one set of initial conditions.

The study of the stability of Riemann solutions as asymptotic states of (1.1)–(1.2) is not easy. If the Riemann solution is a single shock wave, then it corresponds to a traveling wave solution of (1.1), and one can use a moving coordinate system to convert the traveling wave solution to a steady state solution. One can then study stability by studying the spectrum of the linearization at this solution. There is always a zero eigenvalue, which corresponds to shifts of the traveling wave. An additional difficulty is that the continuous spectrum touches the imaginary axis. For a single conservation law, Sattinger [39] dealt with this difficulty by using an exponentially weighted norm, which shifts the continuous spectrum to the left. For systems, the gap lemma of Gardner and Zumbrun [14] (see also [19]) allows one to study eigenvalues of the linearization near the origin despite the continuous spectrum. A series of papers by Liu, Zumbrun, and Howard justifies the passage from linear to nonlinear stability [28], [29], [27], [50].

Alternatively, one can study stability of viscous shock waves by energy methods [34], [15]. A relation between the two approaches is that energy methods can be used to verify that the spectrum of the linearization is contained in the left half plane.

Riemann solutions other than a single shock wave do not correspond to traveling wave solutions of (1.1). Thus one cannot determine their stability by finding the spectrum of a linear operator. In some situations one can construct an approximate solution of (1.1)–(1.2) near the Riemann solution and show that solutions of (1.1)–(1.2) that start near the approximate solution approach it. See [26] for Riemann solutions consisting of weak Lax shock waves and [45] for Riemann solutions consisting of a single rarefaction.

Riemann solutions are functions of $\frac{X}{T}$ only, and it is in the variables (x, T) with $x = \frac{X}{T}$ that the convergence of solutions of (1.1)–(1.2) to Riemann solutions is observed. With this motivation, in (1.1) we make the change of variables

$$x = \frac{X}{T}, \quad t = \ln T. \tag{1.10}$$

(The substitution $t = \ln T$ is simply for convenience. Decay that is algebraic in T becomes exponential in t.) We obtain

$$u_t + (Df(u) - xI)u_x = e^{-t}(B(u)u_x)_x. \tag{1.11}$$

Thus in the (x, t) variables, which are natural for the study of the large-time behavior of solutions of (1.1), (1.1) becomes a system that is both spatially dependent and nonautonomous. In studying nonautonomous systems, it is natural to first freeze the time variable and study the resulting autonomous system. In this case one sets $\epsilon = e^{-t}$; for large t, ϵ is small. One obtains

$$u_t + (Df(u) - xI)u_x = \epsilon(B(u)u_x)_x. \tag{1.12}$$

Returning to (X, T) variables, (1.12) becomes

$$u_T + f(u)_X = \epsilon T(B(u)u_X)_X. \tag{1.13}$$

Equation (1.13) is the *Dafermos regularization* of the system of conservation laws (1.3) associated with the viscosity $B(u)$ ([8]; see also [46], [47]). It is usually regarded as an artificial, nonphysical equation because of the factor T in the viscous term. As we have seen, however, if one is interested in the behavior of solutions of (1.1)–(1.2) for large T and uses the appropriate variables (1.10) for large T, the Dafermos regularization is actually a natural simplification of the physical equations. Like the Riemann problem, but unlike (1.1), (1.13) has many solutions of the form $u(X,T) = \hat{u}(x)$, $x = \frac{X}{T}$. (This is why it was originally introduced.) They satisfy a *Dafermos ODE*

$$(Df(u) - xI)u_x = \epsilon(B(u)u_x)_x. \tag{1.14}$$

Corresponding to the Riemann data (1.8) we have the boundary conditions

$$u(-\infty) = u^\ell, \quad u(+\infty) = u^r. \tag{1.15}$$

We shall refer to a solution $u_\epsilon(x)$ of (1.14)–(1.15) as a *Riemann–Dafermos solution* of (1.13) for the boundary data (u^ℓ, u^r). A Riemann–Dafermos solution of (1.13) is just a stationary solution of (1.12). The boundary value problem (1.14)–(1.15) is a viscous regularization of the Riemann boundary value problem (1.9).

Actually, Dafermos always used $B(u) \equiv I$. For this case, he conjectured that Riemann–Dafermos solutions of the boundary value problem (1.14)–(1.15) converge to a corresponding Riemann solution as $\epsilon \to 0$. This conjecture has been proved for u^r close to u^ℓ by Tzavaras [48]. His proof relies on showing that the Riemann–Dafermos solutions are of uniformly bounded variation and oscillation.

Recently, Szmolyan [44] studied the boundary value problem (1.14)–(1.15) with $B(u) \equiv I$ using geometric singular perturbation theory [18]. The idea is to think of a Riemann solution, with shock waves that satisfy the viscous profile criterion for $B(u) \equiv I$, as a singular solution ($\epsilon = 0$), and then show by geometric singular perturbation theory that, for small $\epsilon > 0$, there is a nearby Riemann–Dafermos solution.

A Riemann solution is *structurally stable* if the number and types of its waves do not change when the flux function or boundary data are varied slightly [40]. (This use of the term “structurally stable” is consistent with its use in dynamical systems theory, but differs from Majda’s use of the term in [32].) For $B(u) \equiv I$, Szmolyan proved that, for small $\epsilon > 0$, structurally stable classical Riemann solutions, which consist of n rarefactions and Lax shock waves, have Riemann–Dafermos solutions of (1.14)–(1.15) nearby. There is no requirement that u^ℓ and u^r be close.

A valuable feature of the Dafermos regularization is that it works equally well for general $B(u)$. Schecter [41] makes this point explicit and shows that any structurally stable Riemann solution consisting entirely of shock waves that satisfy the viscous profile criterion for a given $B(u)$ has, for small $\epsilon > 0$, a Riemann–Dafermos solution of (1.14)–(1.15) nearby. Undercompressive shock waves, whose existence and location are very dependent on $B(u)$, are explicitly allowed.

It is likely that any structurally stable Riemann solution whose shock waves satisfy the viscous profile criterion for a given $B(u)$ has Riemann–Dafermos solutions of the corresponding Dafermos regularization nearby. Some nonstructurally stable Riemann solutions are treated in [30].

In this paper we shall study the Dafermos system (1.13) in the transformed form (1.12), with boundary conditions

$$u(-\infty,t)=u^\ell,\quad u(\infty,t)=u^r,\quad 0\le t<\infty. \tag{1.16}$$

Our goal is to begin the study of the asymptotic stability of Riemann–Dafermos solutions (i.e., steady state solutions) of (1.12), (1.16). We will consider (1.12), (1.16) on the time interval $t\ge 0$, which corresponds to considering (1.13) on $T\ge 1$.

The possible usefulness of this study for the study of the stability of Riemann solutions as asymptotic states of (1.1)–(1.2) is as follows. Let

$$u(x,t)=u_\epsilon(x) \text{ with } \epsilon=e^{-t},$$

where the $u_\epsilon(x)$ are Riemann–Dafermos solutions of (1.12) that converge, as $\epsilon\to 0$, to a Riemann solution $\hat u(x)$ of (1.3), (1.8). Then for large t, $u(x,t)$ is almost a solution of (1.11) and converges as $t\to\infty$ to $\hat u(x)$. With a good enough understanding of the stability of the $u_\epsilon(x)$ as solutions of (1.12), one can perhaps show that near $u(x,t)$ is a true solution of (1.11) with the same stability that $u_\epsilon(x)$ has as a solution of (1.12) for small ϵ.

Tzavaras [48] gives a different argument for the relevance of the Dafermos regularization to understanding Riemann solutions as asymptotic states of (1.1). We now preview the remainder of the paper. For simplicity, we shall take $B(u)\equiv I$. Then (1.12) becomes

$$u_t+(Df(u)-xI)u_x=\epsilon u_{xx}. \tag{1.17}$$

We consider a structurally stable Riemann solution of (1.3) that consists of exactly n Lax shock waves with speeds $\bar s^1<\bar s^2<\cdots<\bar s^n$. We assume that each Lax shock wave satisfies the viscous profile criterion for $B(u)=I$. Precise definitions are given in section 2. We do not assume that u^ℓ and u^r are close.

We write the Riemann solution as a piecewise constant function $u_0(x)$ that is undefined at $x=\bar s^i$, $i=1,\dots,n$, where $u_0(x)$ has jumps. From [44] or [41], near it there is, for small $\epsilon>0$, a Riemann–Dafermos solution $u_\epsilon(x)$ of (1.17). It has sharp transition layers near $x=\bar s^i$, $i=1,\dots,n$.

In section 3, we construct an asymptotic expansion of $u_\epsilon(x)$ in powers of ϵ. In the *regular layer*, which is $\mathbb{R}$ with $\bar s^i$, $i=1,\dots,n$, removed, $u_\epsilon(x)$ has an expansion of the form

$$u^R_\epsilon(x)=\sum_{j=0}^\infty \epsilon^j u^R_j(x),$$

in which $u^R_0(x)$ is just the piecewise constant Riemann solution $u_0(x)$.

We shall refer to a small neighborhood of $\bar s^i$ as the *ith singular layer* and denote it S^i, $i=1,\dots,n$. The Riemann–Dafermos solution $u_\epsilon(x)$ has sharp transition layers at

$$x^i(\epsilon)=\sum_{j=0}^\infty \epsilon^j x^i_j,\quad i=1,\dots,n,$$

with $x^i(0) = \bar{s}^i$. Near $x^i(\epsilon)$ we use the stretched variable $\xi = \frac{x - x^i(\epsilon)}{\epsilon}$. In terms of this variable, the solution has an expansion

$$u^i_\epsilon(\xi) = \sum_{j=0}^{\infty} \epsilon^j u^i_j(\xi) \quad \text{in the singular layer } S^i.$$

It turns out that $u^i_0(\xi)$ is a traveling wave of (1.1) with speed $\bar{s}^i$.

This description of $u_\epsilon(x)$ is consistent with its construction by geometric singular perturbation theory.

Let $C(\gamma, \mathbb{R}_x)$, $\gamma \geq 0$, be the Banach space of uniformly continuous functions $U(x)$ such that the weighted norm $|U|_\gamma := \sup_x |U(x)|e^{\gamma|x|} < \infty$. Let

$$C^2(\gamma, \mathbb{R}_x) := \{U : U, U', U'' \in C(\gamma, \mathbb{R}_x)\}.$$

On $C^2(\gamma, \mathbb{R}_x)$ we will use the equivalent norms $|U|_{2,\gamma,\epsilon} := |U|_\gamma + \epsilon|U'|_\gamma + \epsilon^2|U''|_\gamma$, where $\epsilon > 0$ is the small parameter in (1.17). This family of norms was used by Fife [12]; the ϵ scales out when the stretched variable $\xi = \frac{x - x^i(\epsilon)}{\epsilon}$ is used instead of x. An advantage of this family of norms is that one can have a family of functions $U_\epsilon(x)$ for which $\sup_x |U'_\epsilon(x)| = O(\frac{1}{\epsilon})$ and $\sup_x |U''_\epsilon(x)| = O(\frac{1}{\epsilon^2})$ but $|U_\epsilon|_{2,\gamma,\epsilon}$ remains bounded as $\epsilon \to 0$.

Let X_γ denote the affine space of functions $u(x) = u_\epsilon(x) + U(x)$ with $U \in C^2(\gamma, \mathbb{R}_x)$. This function space includes the most important perturbations of $u_\epsilon(x)$. We shall study (1.17) together with the boundary conditions (1.16) in the space X_γ. In section 4 we show that for $\gamma \geq 0$, (1.17), (1.16) is well-posed in a neighborhood of $u_\epsilon(x)$ in X_γ. The size of the neighborhood is uniform in the norm $|\cdot|_{2,\gamma,\epsilon}$ as $\epsilon \to 0$. Thus, for small $\epsilon > 0$, perturbations with large derivatives are allowed.

An argument like that of Evans [10] shows that linearized stability of $u_\epsilon(x)$ in X_γ implies nonlinear stability in X_γ. Therefore we consider the linearized stability of $u_\epsilon(x)$ in X_γ.

In section 5 we show that for γ sufficiently large, using the exponentially weighted norm moves the essential spectrum of the linearization of (1.17) about $u_\epsilon(x)$ to the left of the imaginary axis, as in [39], [38]. Thus linearized stability of $u_\epsilon(x)$ in X_γ is determined by the eigenvalues.

In sections 6 and 7 we study eigenvalues for $\gamma > 0$ using asymptotic expansions in ϵ. We assume the eigenvalues have asymptotic expansions of the form

$$\lambda = \sum_{j=-1}^{\infty} \epsilon^j \lambda_j$$

and the corresponding eigenfunctions have similar expansions. Section 6 is devoted to eigenvalues with $\lambda_{-1} \neq 0$. The corresponding eigenfunctions are *local*; i.e., their expansions are nonzero only in singular layers. These eigenvalues reflect the fast convergence of the solution to traveling waves in the singular layers. Section 7 is devoted to eigenvalues with $\lambda_{-1} = 0$, which we discuss in more detail below. The fact that there are both $O(\frac{1}{\epsilon})$ and $O(1)$ eigenvalues is consistent with the description of solutions at the beginning of section 6.

The fast eigenvalues $\lambda = \frac{\lambda_{-1}}{\epsilon} + O(1)$, with $\lambda_{-1} \neq 0$, correspond to the nonzero eigenvalues λ_{-1} of the individual traveling waves that are found by Evans function methods [14], [3]. However, a nondegeneracy condition is needed to ensure that a zero of the Evans function can be continued to a fast eigenvalue $\lambda = \frac{\lambda_{-1}}{\epsilon} + O(1)$; see section 6. Thus, roughly speaking, a necessary condition for stability of the Riemann–Dafermos solution is that the Evans function for each individual viscous shock wave in the Riemann solution have no zero with positive real part. Slow eigenvalues have the form $\lambda = \lambda_0 + O(\epsilon)$. It turns out that $\lambda_0 = 0$ is never an eigenvalue, while $\lambda_0 = -1$ is always among the $O(1)$ eigenvalues. Its multiplicity is n. The corresponding eigenfunctions are local. To lowest order they are just the derivatives of the individual traveling waves in the n singular layers and correspond to shifts of the traveling waves.

Other $O(1)$ eigenvalues are nonlocal: The corresponding eigenfunctions asymptotically satisfy a piecewise continuous system of ODEs in x, along with jump conditions at $x = \bar{s}^i$, $i = 1, \ldots, n$. To lowest order, these $O(1)$ eigenvalues and eigenfunctions can be interpreted as eigenvalues and eigenfunctions for a system of first-order hyperbolic equations. This system has been used by many authors to study perturbations of Riemann solutions of the inviscid equation (1.3) that contain only shock waves. There are two types of treatment of this equation of which we are aware: (1) One can show that if a nondegeneracy condition (Majda's stability condition) holds for each shock wave, the system can be solved by characteristics for all time [32]. (2) Assuming the same nondegeneracy condition, one can interpret the system as describing the scattering of incoming small shock waves by the large shock waves that comprise the original Riemann solution, and one can find sufficient conditions that guarantee that, in some norm, the total weight of the scattered shocks is smaller than the total weight of the incoming shocks [42], [4], [5], [49], [22], [21]. A condition of this type can then be used in Glimm's scheme to show the existence of solutions of (1.3) for initial data close to the original Riemann data. For a Riemann solution with $n = 2$ that consists of two Lax shocks, this approach yields a simple computable inviscid stability condition.

The system that determines the $O(1)$ eigenvalues to lowest order is also related to the SLEP system used by Nishiura and Fujii [35] for reaction-diffusion equations to study the stability of solutions with several sharp layers.

In this paper we study only the possible values of λ_0 for slow eigenvalues. The study of conditions under which λ_0 can actually be continued to a slow eigenvalue $\lambda = \lambda_0 + O(\epsilon)$ of the Riemann–Dafermos solution $u_\epsilon(x)$ is deferred to a later paper.

A necessary condition for stability of the Riemann–Dafermos solution is that no slow eigenvalue have positive real part. For $n = 2$, we show that to lowest order in ϵ, the $O(1)$ eigenvalues, other than -1, of a Riemann–Dafermos solution with two Lax shock waves all have the same real part. They are evenly spaced along a line in the complex plane. We compute the real part of these eigenvalues; the condition that it be negative turns out to be the $n = 2$ inviscid stability condition mentioned above. For $n > 2$, the relationship between the $O(1)$ eigenvalues and the known sufficient conditions for inviscid stability remains to be determined.

In section 9 we calculate slow eigenvalues other than -1 for Riemann solutions of the p-system that consist of two Lax shocks. They all have real part -2, independent of the Riemann solution. The calculation is essentially the same as the calculation of the inviscid stability criterion for these Riemann solutions in [4].

Thus, for Riemann–Dafermos solutions whose underlying Riemann solution consists of n Lax shock waves, our analysis suggests that they should be asymptotically stable if (1) each viscous shock wave is linearly stable, a matter that is determined by the wave's Evans function, and (2) the Riemann solution is stable in the inviscid sense, sufficient conditions for which have been determined by studying the scattering of small shock waves by large ones. The stability analysis of Riemann–Dafermos solutions thus unites two distinct lines of research. These relationships are explored in a little more detail in section 8.

A shortcoming of our analysis is that we do not address the possible existence of eigenvalues intermediate between fast and slow. This issue is discussed at the end of section 6. Its resolution may well involve Majda's stability condition, which is known to be related to the derivative of the Evans function at the origin [14], [3].

It should not be difficult to extend the results of this paper to more general diffusion matrices $B(u)$ or to general structurally stable Riemann solutions consisting entirely of shock waves, including undercompressive shock waves. However, we do not see how to deal with rarefactions, for which the asymptotic expansions are much more difficult due to loss of normal hyperbolicity in the underlying geometric singular perturbation problem [44].

2. Riemann solutions. In this section we define precisely the notion of a structurally stable Riemann solution consisting of Lax shock waves. A *Lax i-shock* for (1.3) that satisfies the viscous profile criterion for $B(u) \equiv I$ is a function

$$u(x) = \begin{cases} u^- & \text{for } x < s, \\ u^+ & \text{for } x > s, \end{cases} \tag{2.1}$$

with $x = \frac{X}{T}$, together with a solution $q(\xi)$ of the traveling wave ODE

$$\dot{u} = f(u) - f(u^-) - s(u - u^-), \tag{2.2}$$

such that the following hold:

(L1) $f(u^+) - f(u^-) - s(u^+ - u^-) = 0$.

(L2) The eigenvalues $\nu_1^- < \cdots < \nu_n^-$ of $Df(u^-)$ are real and distinct and satisfy $\nu_{i-1}^- < s < \nu_i^-$.

(L3) The eigenvalues $\nu_1^+ < \cdots < \nu_n^+$ of $Df(u^+)$ are real and distinct and satisfy $\nu_i^+ < s < \nu_{i+1}^+$.

(L4) $q(\xi)$ approaches u^- as $\xi \to -\infty$ and u^+ as $\xi \to \infty$.

Notice that (L1), (L2), and (L3) imply that for (2.2), $u^\pm$ are hyperbolic equilibria, the unstable manifold of u^- has dimension $n - i + 1$, and the stable manifold of u^+ has dimension i. Assumption (L4) says that these manifolds intersect. Because of the dimensions of the manifolds, generically, if they intersect, they do so in curves.

Remark 2.1. The function $q(\xi)$ is also a solution of

$$(Df(u) - sI)u_\xi = u_{\xi\xi} \tag{2.3}$$

and satisfies the boundary conditions (1.6).

A solution of the Riemann problem (1.3), (1.8) that consists of n Lax shock waves, each satisfying the viscous profile criterion for $B(u) \equiv I$, is a piecewise constant function

$$u_0(x) = \begin{cases} \bar{u}^0 & \text{for } x < \bar{s}^1, \\ \bar{u}^i & \text{for } \bar{s}^i < x < \bar{s}^{i+1}, \quad i = 1, \dots, n-1, \\ \bar{u}^n & \text{for } x > \bar{s}^n, \end{cases} \tag{2.4}$$

with $x = \frac{X}{T}$, together with $\mathbb{R}^n$-valued functions $q^i(\xi)$, $i = 1, \dots, n$, such that

(R1) $\bar{u}^0 = u^\ell$ and $\bar{u}^n = u^r$;

(R2) for each $i = 1, \dots, n$, the triple $(\bar{u}_{i-1}, \bar{s}_i, \bar{u}_i)$, together with the function $q^i(\xi)$, defines a Lax i-shock.

Define a mapping $G : \mathbb{R}^{n^2+2n} \to \mathbb{R}^{n^2}$ by

$$\begin{aligned} &G(u^0, s^1, u^1, \dots, u^{n-1}, s^n, u^n) \\ &\qquad = (f(u^1) - f(u^0) - s^1(u^1 - u^0), \dots, f(u^n) - f(u^{n-1}) - s^n(u^n - u^{n-1})). \end{aligned}$$

Notice that

$$G(\bar{u}^0, \bar{s}^1, \bar{u}^1, \dots, \bar{u}^{n-1}, \bar{s}^n, \bar{u}^n) = 0. \tag{2.5}$$

The Riemann solution just defined is *structurally stable*, provided

(S1) $DG(\bar{u}^0, \bar{s}^1, \bar{u}^1, \dots, \bar{u}^{n-1}, \bar{s}^n, \bar{u}^n)$, restricted to the n^2-dimensional space of vectors $(U^0, S^1, U^1, \dots, U^{n-1}, S^n, U^n)$ with $U^0 = U^n = 0$, is invertible;

(S2) for each $i = 1, \dots, n$, the unstable manifold of $\bar{u}^{i-1}$ and the stable manifold of $\bar{u}^i$ for the traveling wave ODE $\dot{u} = f(u) - f(\bar{u}^{i-1}) - \bar{s}^i(u - \bar{u}^{i-1})$ meet transversally along $q^i(\xi)$.

If (S1) and (S2) are satisfied, then for each set of Riemann data (u^0, u^n) near $(\bar{u}^0, \bar{u}^n)$, there is a Riemann solution near the original one. Condition (S1) can be restated as follows:

(S1′) If we set $(U^0, U^n) = (0, 0)$, then the system of linear equations

$$(Df(\bar{u}^i) - \bar{s}^i I)U^i - (Df(\bar{u}^{i-1}) - \bar{s}^i I)U^{i-1} - S^i(\bar{u}^i - \bar{u}^{i-1}) = 0, \quad i = 1, \dots, n,$$

has only the trivial solution

$$(S^1, U^1, \dots, U^{n-1}, S^n) = (0, 0, \dots, 0, 0).$$

A condition equivalent to (S2) is the following:

(S2′) For each $i = 1, \dots, n$, the linear differential equation

$$(Df(q^i(\xi)) - \bar{s}^i I)U_\xi = U_{\xi\xi}$$

has, up to scalar multiplication, a unique solution that approaches zero as $\xi \to \pm\infty$. It is $q^i_\xi(\xi)$.

3. Stationary solutions. Consider the Riemann problem (1.3), (1.8). Assume that it has a solution (2.4) that consists of n Lax shock waves and is structurally stable. We shall study (1.17) together with the boundary conditions

$$u(-\infty, t) = u^\ell, \quad u(\infty, t) = u^r, \quad 0 \le t < \infty. \tag{3.1}$$

Stationary solutions $u_\epsilon(x)$ of (1.17), (3.1) satisfy

$$(Df(u) - xI)u_x = \epsilon u_{xx} \tag{3.2}$$

and the boundary conditions

$$u(-\infty) = u^\ell, \quad u(\infty) = u^r. \tag{3.3}$$

We shall look for $u_\epsilon(x)$ that lie near the given structurally stable Riemann solution (2.4). Such stationary solutions are known to exist, and to approach 0 exponentially as $x \to \pm\infty$, from the geometric singular perturbation arguments of [44].

In the regular layer, which is $\mathbb{R}$ with $\bar{s}^i$, $i = 1, \dots, n$, removed, $u_\epsilon(x)$ has an expansion of the form

$$u^R_\epsilon(x) \sim \sum \epsilon^j u^R_j(x), \tag{3.4}$$

in which $u_0^R(x)$ is just the piecewise constant Riemann solution (2.4). The regular layer is divided by the points $\bar{s}^i$ into $n+1$ connected sublayers

$$\begin{aligned} R^0 &= (-\infty, \bar{s}^1), \\ R^i &= (\bar{s}^i, \bar{s}^{i+1}), \quad i = 1, \dots, n-1, \\ R^n &= (\bar{s}^n, \infty). \end{aligned}$$

Each $u_j^R(x)$ is defined and piecewise C^∞ in the regular layer. At the jump points $\bar{s}^i$, we assume that each $u_j^R(x)$ has one-sided limits $u_j^R(x_0^i\pm) := \lim_{x\to x_0^i\pm} u_j^R(x)$. We assume that the same is true for all derivatives of the $u_j^R(x)$.

As explained in the introduction, we shall refer to a small neighborhood of $\bar{s}^i$ as the *ith singular layer* and denote it by S^i, $i = 1, \dots, n$. The Riemann–Dafermos solution $u_\epsilon(x)$ has sharp transition layers at

$$x^i(\epsilon) = \sum \epsilon^j x_j^i, \quad i = 1, \dots, n, \tag{3.5}$$

with $x^i(0) = \bar{s}^i$. Near $x^i(\epsilon)$ we use the stretched variable $\xi = \frac{x - x^i(\epsilon)}{\epsilon}$. In terms of this variable, the solution has an expansion

$$u_\epsilon^i(\xi) = \sum \epsilon^j u_j^i(\xi) \quad \text{in the singular layer } S^i. \tag{3.6}$$

The expansions $u_\epsilon^R(x)$ and $u_\epsilon^i(\xi)$ satisfy, respectively,

$$(Df(u^R) - xI)u_x^R = \epsilon u_{xx}^R, \tag{3.7}$$

$$(Df(u^i) - xI)u_\xi^i = u_{\xi\xi}^i, \quad x = x^i(\epsilon) + \epsilon\xi. \tag{3.8}$$

We first consider the regular layer. We substitute (3.4) into (3.7) and expand in powers of ϵ. At order ϵ^0 we obtain

$$(Df(u_0^R(x)) - xI)u_{0x}^R = 0.$$

We shall set $u_0^R(x)$ equal to the Riemann solution (2.4), which satisfies this equation.

In the regular layer, at order ϵ^1,

$$(Df(u_0^R(x)) - xI)u_{1x}^R = u_{0xx}^R = 0.$$

Thus $u_1^R(x)$ is constant on each regular sublayer. By induction, we can show that $u_j^R(x)$ is constant on each regular sublayer for all j.

We denote the constant value of $u_j^R(x)$ in R^i by $\bar{u}_j^i$. Thus

$$\bar{u}_0^i = \bar{u}^i, \quad i = 0, \dots, n.$$

From the boundary condition (3.3),

$$\bar{u}_j^0 = 0 \text{ for } j = 1, \dots, \infty, \quad \bar{u}_j^n = 0 \text{ for } j = 1, \dots, \infty. \tag{3.9}$$

Next, we consider the ith singular layer S^i. We substitute (3.6) and (3.5) into (3.8) and expand in powers of ϵ. At order ϵ^0, we obtain

$$(Df(u_0^i) - x_0^i I)u_{0\xi}^i = u_{0\xi\xi}^i. \tag{3.10}$$

To match the solutions at order ϵ^0 in regular and singular layers, we must have

$$u_0^i(-\infty) = \bar{u}_0^{i-1} = \bar{u}^{i-1} \quad \text{and} \quad u_0^i(\infty) = \bar{u}_0^i = \bar{u}^i. \tag{3.11}$$

We set

$$x_0^i = \bar{s}_i, \quad i = 1, \dots, n.$$

Then by (S2′) in section 2, (3.10), (3.11) has the solution $u_0^i(\xi) = q^i(\xi)$. As $\xi \to \pm\infty$, $q^i(\xi)$ approaches the limits exponentially fast. By (S2), the solution q^i is locally unique up to a shift in ξ.

In S^i, at order ϵ^1, we have

$$u^i_{1\xi\xi} - ((Df(q^i) - \bar{s}^i I)u_1^i)_\xi = -(x_1^i + \xi)q_\xi^i. \tag{3.12}$$

We look for u_1^i that satisfies the matching conditions

$$u_1^i(-\infty) = \bar{u}_1^{i-1} \quad \text{and} \quad u_1^i(\infty) = \bar{u}_1^i. \tag{3.13}$$

By (3.9), $\bar{u}_1^0 = \bar{u}_1^n = 0$. The other $\bar{u}_1^i$ and the x_1^i are to be determined.

Integrating (3.12) from $\xi = -\infty$ to $\xi = \infty$, we obtain

$$\begin{aligned}(Df(\bar{u}_0^i) - \bar{s}^i I)\bar{u}_1^i - (Df(\bar{u}_0^{i-1}) - \bar{s}^i I)\bar{u}_1^{i-1} - x_1^i(\bar{u}_0^i - \bar{u}_0^{i-1}) \\ = \int_{-\infty}^{\infty} \xi q_\xi^i d\xi, \quad i = 1, \dots, n.\end{aligned} \tag{3.14}$$

After making the substitutions $\bar{u}_1^0 = \bar{u}_1^n = 0$, (3.14) becomes a system of n^2 linear equations in the n^2 unknowns x_1^i, $i = 1, \dots, n$, and $\bar{u}_1^i$, $i = 1, \dots, n-1$. By (S1) there is a unique solution.

The space of bounded solutions of the adjoint system to the homogeneous part of (3.12), $\psi_{\xi\xi} + (Df(q^i) - \bar{s}^i I)\psi_\xi = 0$, is n-dimensional and consists of constant solutions. Therefore, using lemmas from [6], [24], condition (3.14) is necessary and sufficient for the existence of solutions $u_1^i(\xi)$ to (3.12) that satisfy the boundary conditions (3.13). For completeness, we state this fact as a lemma and present a simpler proof, taking advantage of the fact that (3.12) is in conservation form.

LEMMA 3.1. *Consider the system*

$$U_{\xi\xi} - ((Df(q^i(\xi)) - \bar{s}^i I)U)_\xi = g(\xi), \tag{3.15}$$

where $g(\xi)$ approaches zero exponentially as $\xi \to \pm\infty$. There is a solution U such that $U(\xi) \to U^\pm$ exponentially as $\xi \to \pm\infty$ if and only if

$$(Df(q^i(\infty)) - \bar{s}^i I)U^+ - (Df(q^i(-\infty)) - \bar{s}^i I)U^- + \int_{-\infty}^{\infty} g(s)ds = 0. \tag{3.16}$$

Proof. It is easy to see that the condition is necessary. We prove only that the condition is sufficient. The system (3.15) is equivalent to the system

$$U_\xi - (Df(q^i(\xi)) - \bar{s}^i I)U(\xi) + (Df(q^i(-\infty)) - \bar{s}^i I)U^- = G(\xi), \tag{3.17}$$

where $G(\xi) := \int_{-\infty}^{\xi} g(s)ds$ is bounded, $G(\xi) \to 0$ exponentially as $\xi \to -\infty$, and $G(\xi) \to \int_{-\infty}^{\infty} g(s)ds$ exponentially as $\xi \to \infty$.

From the definition of a Lax i-shock, $Df(q^i(\pm\infty)) - \bar{s}^i I$ is hyperpolic, so system (3.17) has exponential dichotomies [7] on $\mathbb{R}^{\pm}$. Therefore there exist nonunique bounded solutions $U_L(\xi)$ and $U_R(\xi)$ that solve (3.17) on $\mathbb{R}^-$ and $\mathbb{R}^+$, respectively.

For the dichotomy on $\mathbb{R}^-$, let $P_u(0-)$ denote projection onto the unstable subspace at $x=0$, with kernel the stable subspace. Similarly, for the dichotomy on $\mathbb{R}^+$, let $P_s(0+)$ denote projection onto the stable subspace at $x=0$, with kernel the unstable subspace. Then the definition of a Lax i-shock implies that $\mathcal{R}P_u(0-)+\mathcal{R}P_s(0+)=\mathbb{R}^n$. Therefore there exists a (nonunique) pair (ϕ_u,ϕ_s) such that

$$\begin{aligned}&U_L(0-)+\phi_u = U_R(0+)+\phi_s,\\ &\phi_u \in \mathcal{R}P_u(0-), \quad \phi_s \in \mathcal{R}P_s(0+).\end{aligned}$$

Let $\Phi(\xi,\zeta)$ be the principle matrix solution to (3.17). The solution $U(\xi), \xi \in \mathbb{R}$, can be obtained by letting

$$\begin{aligned}U(\xi) &= U_L(\xi) + \Phi(\xi,0)\phi_u, \quad \xi \le 0,\\ U(\xi) &= U_R(\xi) + \Phi(\xi,0)\phi_s, \quad \xi \ge 0.\end{aligned}$$

From (3.17) and (3.16), using the limits of $G(\xi)$ as $\xi \to \pm\infty$, it is easy to show that $U(\xi) \to U^-$ as $\xi \to -\infty$ and $U(\xi) \to U^+$ as $\xi \to \infty$. ∎

Proceeding inductively, we can solve for all x^i_j and $\bar{u}^i_j$.

Our asymptotic expansions are justified by the fact that $u_\epsilon(x)$ is known to exist from the geometric singular perturbation theory arguments of [44]. Alternatively, a proof of existence of the exact stationary solutions $u_\epsilon(x)$ can be based on the existence of the formal asymptotic expansions (3.4)–(3.5). For this approach to singular perturbation theory, see [25]. The same assumptions (S1) and (S2) are used in both types of arguments.

We summarize the results about stationary solutions in the following.

PROPOSITION 3.2. *In the regular layer, to all orders of ϵ, $u^R_\epsilon(x)$ is piecewise constant with jumps at $x^i_0(\epsilon)$, $i=1,\dots,n$, only. At lowest order, $u^R_0(x)$ is the Riemann solution* (2.4). *In the ith singular layer, at lowest order, $u^i_0(\xi)=q^i(\xi)$, a heteroclinic solution connecting the states $\bar{u}^{i-1}_0$ and $\bar{u}^i_0$. Higher order terms $u^R_j(x)$, $u^i_j(\xi)$, and x^i_j can be obtained recursively, using the matching of regular and singular layers and Lemma* 3.1.

4. Well-posedness. To show the well-posedness of initial value problems with initial conditions near a Riemann–Dafermos solution, it is convenient to use the stretched variables $\xi = \frac{x}{\epsilon}$ and $\tau = \frac{t}{\epsilon}$. We shall translate the results back to (x,t) variables at the end of the section.

Using the stretched variables, (1.17) becomes

$$u_\tau + (Df(u) - \epsilon\xi I)u_\xi = u_{\xi\xi}. \tag{4.1}$$

Let $u_\epsilon(x)$ be a stationary solution of (1.17), (3.1). Then $u_\epsilon(\epsilon\xi)$ is a stationary solution of (4.1). A solution of (4.1) near $u_\epsilon(\epsilon\xi)$ can be expressed as $u_\epsilon(\epsilon\xi)+U(\xi,\tau)$ with U satisfying

$$U_\tau + (Df(u_\epsilon+U) - \epsilon\xi I)U_\xi + (Df(u_\epsilon+U) - Df(u_\epsilon))u_{\epsilon\,\xi} = U_{\xi\xi}. \tag{4.2}$$

For any $\rho \ge 0$, let $C(\rho,\mathbb{R}_\xi)$ be the Banach space of uniformly continuous functions $U(\xi)$, $\xi \in \mathbb{R}$, such that the weighted norm $|U|_\rho := \sup_\xi |U(\xi)|e^{\rho|\xi|} < \infty$. Let $C^k(\rho,\mathbb{R}_\xi)$

be the space of functions $U(\xi)$ such that $U, U', \ldots, U^{(k)} \in C(\rho, \mathbb{R}_\xi)$. On $C^k(\rho, \mathbb{R}_\xi)$ we use the norm $|U|_{k,\rho} := |U|_\rho + |U'|_\rho + \cdots + |U^{(k)}|_\rho$. One can define $C(\rho, \mathbb{R}_\xi^\pm)$ and $C^k(\rho, \mathbb{R}_\xi^\pm)$ similarly.

We shall show that for any $\rho \geq 0$, (4.2) is well-posed for small initial data in $C^2(\rho, \mathbb{R}_\xi)$. The intuitive reason is that for the underlying Riemann problem, the characteristics on the two unbounded regular layers head inward. This keeps a space of exponentially decaying profiles invariant.

Stronger results can be obtained using fractional powers of Banach spaces or intermediate spaces [13], [17], [36], [9], [31], [23]. We choose to use $C^2(\rho, \mathbb{R}_\xi)$ for simplicity.

We rewrite (4.2) as

$$U_\tau + (Df(u_\epsilon) - \epsilon\xi I)U_\xi + D^2 f(u_\epsilon)u_{\epsilon\,\xi}U + g_\epsilon(U, U_\xi, \xi) = U_{\xi\xi}, \tag{4.3}$$

with

$$\begin{aligned} &g_\epsilon(U, U_\xi, \xi) \\ &\quad = (Df(u_\epsilon + U) - Df(u_\epsilon))U_\xi + (Df(u_\epsilon + U) - Df(u_\epsilon) - D^2 f(u_\epsilon)U)u_{\epsilon\,\xi}. \end{aligned} \tag{4.4}$$

Note that because of the dependence on U_ξ in (4.4), if $U \in C^2(\rho, \mathbb{R}_\xi)$, then $g_\epsilon \in C^1(\rho, \mathbb{R}_\xi)$. Moreover, we have

$$\begin{aligned} |g_\epsilon(U)|_{1,\rho} &\leq C|U|_{2,\rho}^2, \\ |g_\epsilon(U_1) - g_\epsilon(U_2)|_{1,\rho} &\leq C\max\{|U_1|_{2,\rho}, |U_2|_{2,\rho}\}|U_1 - U_2|_{2,\rho}. \end{aligned} \tag{4.5}$$

We first consider the inhomogeneous linear system

$$U_\tau + (Df(u_\epsilon) - \epsilon\xi I)U_\xi + D^2 f(u_\epsilon)u_{\epsilon\,\xi}U + h_\epsilon(\xi, \tau) = U_{\xi\xi}. \tag{4.6}$$

The hypotheses on h in the following lemma are motivated by the observations just made about g.

PROPOSITION 4.1. *Let $\tau_0 > 0$, $\epsilon_0 > 0$, and $\rho \geq 0$. Assume that*

(1) *for each $0 < \epsilon \leq \epsilon_0$, $h_\epsilon(\cdot, \tau)$ is a continuous mapping from $0 \leq \tau \leq \tau_0$ to $C^1(\rho, \mathbb{R}_\xi)$;*

(2) *there is a constant M such that $|h_\epsilon(\cdot, \tau)|_{1,\rho} \leq M$ on $\{(\tau, \epsilon) : 0 \leq \tau \leq \tau_0, 0 < \epsilon \leq \epsilon_0\}$.*

Let

$$U(\xi, 0) = \phi(\xi), \tag{4.7}$$

with $\phi \in C^2(\rho, \mathbb{R}_\xi)$. Then there exists τ_1, $0 < \tau_1 \leq \tau_0$, such that for each $0 < \epsilon \leq \epsilon_0$, the initial value problem (4.6), (4.7) *has a solution $U(\xi, \tau)$, $0 \leq \tau \leq \tau_1$. The mapping $\tau \to U(\cdot, \tau)$ is continuous from $[0, \tau_1]$ to $C^2(\rho, \mathbb{R}_\xi)$, and there is a constant C such that, for each (τ, ϵ),*

$$|U|_{2,\rho} \leq C(|\phi|_\rho + |h|_{1,\rho}).$$

The numbers τ_1 and C depend on ϵ_0 but are independent of ρ and M.

Proof. Let $y = e^{\epsilon\tau}\xi$ and define $v(y, \tau) := U(e^{-\epsilon\tau}y, \tau)$. Then

$$v_\tau + e^{\epsilon\tau}(Df(u_\epsilon)v_y + D^2 f(u_\epsilon)u_{\epsilon\,y}v) + h = e^{2\epsilon\tau}v_{yy}.$$

Let $s = \frac{e^{2\epsilon\tau}-1}{2\epsilon}$, so that $\tau = \tau(s) = (1+2\epsilon s)^{\frac{1}{2\epsilon\tau}}$. Let $w(y,s) := v(y,\tau(s))$. Then

$$
\begin{aligned}
&w_s + \frac{1}{\sqrt{2\epsilon s+1}}(Df(u_\epsilon)w_y + D^2f(u_\epsilon)u_{\epsilon\,y}w) + \frac{1}{2\epsilon s+1}h = w_{yy},\\
&w(y,0) = \phi(y).
\end{aligned} \tag{4.8}
$$

Moreover, if τ_1 is sufficiently small, then for each $0<\epsilon\le\epsilon_0$, h_ϵ defines a continuous function from $0\le s\le s_1(\epsilon)$ to $C^1(\rho,\mathbb{R}_\xi)$, where $s_1(\epsilon) = \frac{e^{2\epsilon\tau_1}-1}{2\epsilon} \approx \tau_1$. In (4.8) the coefficients of w and w_y, and the inhomogeneous term, are bounded on

$$\{(y,s,\epsilon) : y\in\mathbb{R}, 0\le s\le s_1(\epsilon), 0<\epsilon\le\epsilon_0\}.$$

Let $\Phi(y,s) := \frac{1}{2\sqrt{\pi s}}e^{-y^2/4s}$ be the fundamental solution of the heat equation $w_s = w_{yy}$. The solution of (4.8) is the fixed point of the integral equation

$$
\begin{aligned}
\bar{w}(y_0,s_0) &= \int_{-\infty}^{\infty}\Phi(y_0-y,s_0)\phi(y)dy - \int_0^s\int_{-\infty}^{\infty}\Phi(y_0-y,s_0-s)\frac{1}{2\epsilon s+1}h_\epsilon(y,s)dyds\\
&- \int_0^s\int_{-\infty}^{\infty}\Phi(y_0-y,s_0-s)\frac{1}{\sqrt{2\epsilon s+1}}(Df(u_\epsilon)w_y(y,s) + D^2f(u_\epsilon)u_{\epsilon\,y}w(y,s))dyds.
\end{aligned}
$$

If $w(y,s)$ defines a continuous function from $0\le s\le s_1(\epsilon)$ to $C^2(\rho,\mathbb{R}_\xi)$, then it is easy to show that $\bar{w}$ defines a continuous function from $0\le s\le s_1(\epsilon)$ to $C^2(\rho,\mathbb{R}_\xi)$. Moreover, if τ_1 is sufficiently small, then, independent of ρ, the mapping $w\to\bar{w}$ is a contraction mapping in the space of continuous functions from $0\le s\le s_1(\epsilon)$ to $C^2(\rho,\mathbb{R}_\xi)$. Therefore, there exists a unique fixed point $w(y,s)$ in $C^2(\rho,\mathbb{R}_\xi)$, which is the solution of (4.8).

Then

$$|U(\xi,\tau)| = |v(y,\tau(s))| = |w(y,s)| \le C(|\phi|_\rho + |h|_{1,\rho})e^{-\rho|y|} \le C_1(|\phi|_\rho + |h|_{1,\rho})e^{-\rho|\xi|}.$$

Similar estimates for $|U_\xi|$ and $|U_{\xi\xi}|$ can also be obtained from the integral equation for w. The proof that $w:[0,\tau_1]\to C^2(\rho,\mathbb{R}_\xi)$ is continuous uses a well-known technique from the theory of evolution equations in abstract Banach spaces [17] and will be omitted. ∎

Using Proposition 4.1, the estimates (4.5), and the contraction mapping theorem in $C^2(\rho,\mathbb{R}_\xi)$, we can easily prove the following proposition.

PROPOSITION 4.2. *Consider the initial value problem* (4.2), (4.7), *with* $\phi\in C^2(\rho,\mathbb{R}_\xi)$ *and* $\rho\ge0$. *There exist positive constants* τ_1, ϵ_1, *and* δ_1, *all independent of* ρ, *such that if* $|\phi|_{2,\rho}\le\delta_1$, *then for each* $0<\epsilon\le\epsilon_1$, *the initial value problem has a unique solution* $U(\xi,\tau)$, $0\le\tau\le\tau_1$, *such that* $\tau\to U(\cdot,\tau)$ *is a continuous mapping from* $[0,\tau_1]$ *to* $C^2(\rho,\mathbb{R}_\xi)$.

We can apply Proposition 4.2 repeatedly until the maximal time interval of existence is reached.

We recall from the introduction that for a C^k function $\psi(x)$, we define

$$|\psi|_{k,\gamma,\epsilon} := |\psi|_\gamma + \epsilon|\psi'|_\gamma + \cdots + \epsilon^k|\psi^{(k)}|_\gamma.$$

LEMMA 4.3. *Let* k *be a nonnegative integer. Let* $\psi\in C^k(\gamma,R_x)$. *Define* $\phi(\xi) = \psi(\epsilon\xi)$. *Then* $\phi\in C^k(\epsilon\gamma,R_\xi)$, *and* $|\phi|_{k,\epsilon\gamma} = |\psi|_{k,\gamma,\epsilon}$.

Proof. We have

$$|\psi(x)|e^{\gamma|x|} = |\psi(\epsilon\xi)|e^{\gamma\epsilon|\xi|} = |\phi(\xi)|^{\epsilon\gamma|\xi|},$$
$$\epsilon|\psi_x(x)|e^{\gamma|x|} = \epsilon|\psi_x(\epsilon\xi)|e^{\gamma\epsilon|\xi|} = |\phi_\xi(\xi)|e^{\epsilon\gamma|\xi|}, \tag{4.9}$$

etc. The result follows. □

In the original variables $x = \epsilon\xi$ and $t = \epsilon\tau$, (4.2) becomes

$$V_t + (Df(u_\epsilon + V) - xI)V_x + (Df(u_\epsilon + V) - Df(u_\epsilon))u_{\epsilon\,x} = V_{xx}, \tag{4.10}$$

and the initial condition (4.7) becomes

$$V(x, 0) = \psi(x). \tag{4.11}$$

COROLLARY 4.4. *Consider the initial value problem* (4.10), (4.11), *with* $\psi \in C^2(\gamma, \mathbb{R}_x)$ *and* $\gamma \geq 0$. *There exist positive constants* τ_1, ϵ_1, *and* δ_1, *all independent of* γ, *such that if* $|\psi|_{2,\gamma,\epsilon} \leq \delta_1$, *then for each* $0 < \epsilon \leq \epsilon_1$, *there is a unique solution* $V(x,t)$, $0 \leq t \leq \epsilon\tau_1$, *such that* $t \to V(\cdot, t)$ *is a continuous mapping from* $[0, \epsilon\tau_1]$ *to* $C^2(\gamma, \mathbb{R}_x)$.

Proof. The constants τ_1, ϵ_1, and δ_1 are those of Proposition 4.2. Suppose $|\psi|_{2,\gamma,\epsilon} \leq \delta_1$. Let $\phi(\xi) = \psi(\epsilon\xi)$. By Lemma 4.3, $|\phi|_{2,\epsilon\gamma} = |\psi|_{2,\gamma,\epsilon}$. By Proposition 4.2, the initial value problem (4.2), (4.7) has a solution $U(\xi, \tau)$, $0 \leq \tau \leq \tau_1$. Let $V(x,t) = U(\frac{x}{\epsilon}, \frac{t}{\epsilon})$. □

As noted in the introduction, the condition $|\psi|_{2,\gamma,\epsilon} \leq \delta_1$ allows, for small $\epsilon > 0$, initial perturbations of the Riemann–Dafermos solution $u_\epsilon(x)$ with very large derivatives.

5. Essential spectrum. In the space of uniformly bounded functions, a traveling wave (viscous shock) solution of (1.1) has an essential spectrum that touches the imaginary axis. This is the main difficulty in proving stability of the traveling wave. The same difficulty occurs for a Riemann–Dafermos solution u_ϵ of the Dafermos regularization. Following an idea of Sattinger [39], we use weighted function spaces to move the essential spectrum to the left.

Let $\tilde{\delta} > 0$ be given. For sufficiently large $\gamma > 0$, we shall show that, for small $\epsilon > 0$, in the space $C^2(\gamma, \mathbb{R}_x)$, the essential spectrum of the linearization of (1.17) about a Riemann–Dafermos solution $u_\epsilon(x)$ lies in the region $\mathrm{Re}\tilde{\lambda} \leq -\tilde{\delta}$. Therefore the stability of the Riemann–Dafermos solution is determined by the eigenvalues.

Let $T(\xi, \zeta)$ be the fundamental matrix solution for a first-order system

$$W_\xi = B(\xi)W, \quad \xi \in J. \tag{5.1}$$

DEFINITION 5.1. *Let* $\beta < \alpha$ *be real numbers. System* (5.1) *has a* pseudoexponential dichotomy on J with spectral gap $\beta < \alpha$ *if there is a real number* $C \geq 0$ *and projections* $P(\xi)$, $\xi \in J$, *such that*

(1) $T(\xi, \zeta)P(\zeta) = P(\xi)T(\xi, \zeta)$;

(2) *if* $w_s \in \mathcal{R}P(\zeta)$, *and* $\xi > \zeta$ *in* J, *then*

$$|T(\xi, \zeta)w_s| \leq Ce^{\beta(\xi-\zeta)}|w_s|;$$

(3) *if* $w_u \in \mathcal{R}(I - P(\zeta))$, *and* $\xi < \zeta$ *in* J, *then*

$$|T(\xi, \zeta)w_u| \leq Ce^{\alpha(\xi-\zeta)}|w_u|;$$

(4) $P(\xi)$ *is continuous with respect to* ξ.

Notice that $P(\xi)$ is not assumed to be uniformly bounded.

The linearization of (1.17) about $u_\epsilon(x)$ is

$$U_t + (Df(u_\epsilon) - xI)U_x + D^2 f(u_\epsilon)u_{\epsilon\,x}U = \epsilon U_{xx}. \tag{5.2}$$

The complex number $\tilde{\lambda}$ is in the resolvent set of (5.2), provided the spectral equation

$$\tilde{\lambda}U + (Df(u_\epsilon) - xI)U_x + D^2 f(u_\epsilon)u_{\epsilon\,x}U + \tilde{h} = \epsilon U_{xx} \tag{5.3}$$

can be solved for U in terms of $\tilde{h}$, and the mapping $\tilde{h} \to U$ is bounded.

In (5.3) let $\lambda = \epsilon\tilde{\lambda}$, $\xi = \frac{x}{\epsilon}$, and $h = \epsilon\tilde{h}$. Then (5.3) becomes

$$\lambda U + (Df(u_\epsilon) - \epsilon\xi I)U_\xi + D^2 f(u_\epsilon)u_{\epsilon\,\xi}U + h = U_{\xi\xi}. \tag{5.4}$$

Let $\tilde{\delta} > 0$ be given. We shall show that for $\epsilon > 0$ sufficiently small and $\mathrm{Re}\lambda \geq -\epsilon\tilde{\delta}$, (5.4) with $h = 0$ has, for an appropriate $a > 0$, pseudoexponential dichotomies on the intervals $\xi \leq -\frac{a}{\epsilon}$ and $\xi \geq \frac{a}{\epsilon}$. Although the projection operators $P(\lambda, \epsilon, \xi)$ of the pseudoexponential dichotomies are not uniformly bounded, even for fixed (λ, ϵ), we will show that the restriction of $P(\lambda, \epsilon, \xi)$ to the subspace of $\mathbb{R}^{2n}$ defined by setting the first n coordinates equal to zero is uniformly bounded. Based on these results we will show that for $\epsilon > 0$ sufficiently small, the essential spectrum of (5.3) is in the region $\mathrm{Re}\tilde{\lambda} \leq -\tilde{\delta}$.

Let $W = (U, V)$ and let

$$\tilde{B}(\lambda, \epsilon, x) := \begin{pmatrix} 0 & I \\ \lambda + \epsilon D^2 f(u_\epsilon)u_{\epsilon\,x} & Df(u_\epsilon) - xI \end{pmatrix}. \tag{5.5}$$

Let

$$B(\lambda, \epsilon, \xi) := \tilde{B}(\lambda, \epsilon, \epsilon\xi) = \begin{pmatrix} 0 & I \\ \lambda + D^2 f(u_\epsilon)u_{\epsilon\,\xi} & Df(u_\epsilon) - \epsilon\xi I \end{pmatrix}. \tag{5.6}$$

Then (5.4) can be recast as

$$W_\xi = B(\lambda, \epsilon, \xi)W + (0,\ h)^\top. \tag{5.7}$$

Our proof that $W_\xi = BW$ has pseudoexponential dichotomies on the intervals $\xi \leq -\frac{a}{\epsilon}$ and $\xi \geq \frac{a}{\epsilon}$ is motivated by the proof of Coppel's Proposition 1 [7, p. 50]. This result says, roughly speaking, that if the matrices $B(\xi)$, $\xi \in J$, are uniformly bounded and uniformly hyperbolic, and vary slowly with ξ, then (5.1) has an exponential dichotomy on J. Our case differs in that the matrices $B(\lambda, \epsilon, \xi)$ are not uniformly bounded, even for fixed (λ, ϵ). In addition, they have eigenvalues near 0 for small ϵ, so we are interested in pseudoexponential dichotomies rather than exponential dichotomies.

Let

$$\tilde{A}(\lambda, x) := \begin{pmatrix} 0 & I \\ \lambda & Df(u^r) - xI \end{pmatrix}.$$

LEMMA 5.1. *For $\delta > 0$ sufficiently small, there are numbers $\beta(\delta) < \alpha(\delta) < 0$ such that if $\mathrm{Re}\lambda \geq -\delta$ and $x_0^n \leq x$, then $\tilde{A}(\lambda, x)$ has n eigenvalues with real parts less than $\beta(\delta)$ and n eigenvalues with real parts between $\alpha(\delta)$ and 0. As $\delta \to 0$, $\beta(\delta)$ approaches a negative limit, and $\alpha(\delta)$ is $O(\delta)$.*

Proof. Since (1.3) is strictly hyperbolic, the eigenvalues of $Df(u^r)$ are real and distinct. Denote them by $\nu_1 < \cdots < \nu_n$ and denote the corresponding eigenvectors by $\mathbf{r}_1, \ldots, \mathbf{r}_n$.

Let μ be an eigenvalue of $\tilde{A}(\lambda, x)$. It is easily verified that

$$\det(\lambda + \mu(Df(u^r) - (x + \mu)I)) = 0.$$

Therefore one of the following equations must hold:

$$\mu^2 + (x - \nu_j)\mu - \lambda = 0, \quad j = 1, \ldots, n.$$

Thus there are two eigenvalues of $\tilde{A}(\lambda, x)$ for each j,

$$\mu_j^{\pm} = -\frac{-x - \nu_j}{2} \pm \sqrt{\left(\frac{x - \nu_j}{2}\right)^2 + \lambda},$$

with corresponding eigenvectors

$$(\mathbf{r}_j, \mu_j^{\pm}\mathbf{r}_j)^{\top}.$$

For each x with $x_0^n \le x$, we have $\nu_n < x$. Let $p = \frac{1}{2}(x_0^n - \nu_n) > 0$. Let δ be such that $0 < \delta < p^2$. Let

$$\beta(\delta) = -p - \sqrt{p^2 - \delta}, \quad \alpha(\delta) = -p + \sqrt{p^2 - \delta} = \frac{-\delta}{p + \sqrt{p^2 - \delta}}.$$

Notice that $\beta(\delta) < \alpha(\delta) < 0$, $\lim_{\delta \to 0} \beta(\delta) = -2p < 0$, and $\alpha(\delta)$ is $O(\delta)$.

Let $1 \le j \le n$, let $\mathrm{Re}\lambda \ge -\delta$, and let $x_0^n \le x$. From Corollary 5.6 at the end of this section, with $r = p_j = \frac{1}{2}(x - \nu_j)$, $\mu_j^{\pm}$ must satisfy

$$\mathrm{Re}\mu_j^- \le -p_j - \sqrt{p_j^2 - \delta} \le \beta(\delta)$$

and

$$\mathrm{Re}\mu_j^+ \ge -p_j + \sqrt{p_j^2 - \delta} = \frac{-\delta}{p_j + \sqrt{p_j^2 - \delta}} \ge \alpha(\delta). \qquad \square \tag{5.8}$$

We shall refer to the μ_j^-, $j = 1, \ldots, n$, as *pseudostable eigenvalues* and the μ_j^+, $j = 1, \ldots, n$, as *pseudounstable eigenvalues*.

We now construct projections associated to the pseudostable and pseudounstable eigenvalues.

Let $\mathbf{R} = (\mathbf{r}_1 \ldots \mathbf{r}_n)$ and $\mathcal{M}^{\pm}(\lambda, x) = \mathrm{diag}(\mu_1^{\pm} \ldots \mu_n^{\pm})$ be $n \times n$ matrices. The eigenvectors of $\tilde{A}(\lambda, x)$ form a $2n \times 2n$ matrix

$$H(\lambda, x) := \begin{pmatrix} \mathbf{R} & 0 \\ 0 & \mathbf{R} \end{pmatrix} \begin{pmatrix} I_n & I_n \\ \mathcal{M}^- & \mathcal{M}^+ \end{pmatrix}.$$

The first n columns of H are eigenvectors $(\mathbf{r}_j, \mu_j^- \mathbf{r}_j)^{\top}$ for the corresponding μ_j^-, and the last n columns are eigenvectors $(\mathbf{r}_j, \mu_j^+ \mathbf{r}_j)^{\top}$ for the corresponding μ_j^+. Let $D(\lambda, x) = \mathcal{M}^+ - \mathcal{M}^- = \mathrm{diag}(\mu_j^+ - \mu_j^-)$. Then

$$H^{-1} = \begin{pmatrix} \mathcal{M}^+ D^{-1} & -D^{-1} \\ -\mathcal{M}^- D^{-1} & D^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{R}^{-1} & 0 \\ 0 & \mathbf{R}^{-1} \end{pmatrix}.$$

Let $\tilde{P} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$. The projection to the space spanned by the pseudostable eigenvectors is

$$P(\lambda, x) = H\tilde{P}H^{-1} = \begin{pmatrix} \mathbf{R} & 0 \\ 0 & \mathbf{R} \end{pmatrix} \begin{pmatrix} \mathcal{M}^+ D^{-1} & -D^{-1} \\ -\lambda D^{-1} & -\mathcal{M}^- D^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{R}^{-1} & 0 \\ 0 & \mathbf{R}^{-1} \end{pmatrix}.$$

Here we have used $\mathcal{M}^-\mathcal{M}^+ = -\lambda I_n$.

PROPOSITION 5.2. *Let $\tilde{\delta} > 0$. Let $a > \max_{i=1,\ldots,n} |x_0^i|$. Then for $\epsilon > 0$ sufficiently small and* $\mathrm{Re}\lambda \geq -\epsilon\tilde{\delta}$, *$W_\xi = BW$ has pseudoexponential dichotomies with n-dimensional pseudostable and pseudounstable spaces on $\xi \leq -\frac{a}{\epsilon}$ and on $\xi \geq \frac{a}{\epsilon}$. The spectral gaps are $0 < \beta_1 \epsilon < \alpha_1$ for $\xi \leq -\frac{a}{\epsilon}$ and $\beta_2 < \alpha_2 \epsilon < 0$ for $\xi \geq \frac{a}{\epsilon}$. The numbers α_j and β_j, $j = 1, 2$, are independent of λ. The constant C in the definition of pseudoexponential dichotomy is independent of (λ, ϵ).*

Proof. We will consider only the interval $\xi \geq \frac{a}{\epsilon}$, since the interval $\xi \leq -\frac{a}{\epsilon}$ can be handled similarly.

From section 3, on the interval $x \geq a$, $u_\epsilon(x) - u^r$ is 0 to any finite order in ϵ. Thus on the interval $\xi \geq \frac{a}{\epsilon}$, $W_\xi = BW$ is approximately $W_\xi = AW$, with $W = (U, V)$ and

$$A(\lambda, \epsilon, \xi) := \tilde{A}(\lambda, \epsilon\xi) = \begin{pmatrix} 0 & I \\ \lambda & Df(u^r) - \epsilon\xi I \end{pmatrix}.$$

Let $\delta = \delta(\epsilon) = \epsilon\tilde{\delta}$. Choose $\tilde{\epsilon} > 0$ such that $\tilde{\epsilon}\tilde{\delta}$ is small enough that Lemma 5.1 applies. In the following we consider only ϵ with $0 < \epsilon < \tilde{\epsilon}$.

Let $\mathcal{M}(\lambda, x) := \mathrm{diag}(\mathcal{M}^-, \mathcal{M}^+)$. Then $\tilde{A} = H\mathcal{M}H^{-1}$. Consider the (λ, x)-dependent change of variables $W = HZ$. After making the substitution $x = \epsilon\xi$, $W_\xi = AW$ becomes

$$Z_\xi = \mathcal{M}Z - H^{-1}H_\xi Z. \tag{5.9}$$

The differential equation (5.9) is a perturbation of the diagonalized system

$$Z_\xi = \mathcal{M}Z. \tag{5.10}$$

That is, $z_j' = \mu_j^- z_j$ if $1 \leq j \leq n$ and $z_j' = \mu_{j-n}^+ z_j$ if $n+1 \leq j \leq 2n$. For $0 < \epsilon < \tilde{\epsilon}$, system (5.10) has a pseudoexponential dichotomy with projection $\tilde{P}$ and spectral gap $\beta(\delta) < \alpha(\delta) < 0$, with $\delta = \epsilon\tilde{\delta}$.

It is easily verified that there is a constant C, independent of δ for δ sufficiently small, such that

$$\frac{1 + |\mu_j^-| + |\mu_j^+|}{|\sqrt{(x - \nu_j)^2 + 4\lambda}|} \leq C$$

for all (j, λ, x), with $j = 1, \ldots, n$, $\mathrm{Re}\lambda \geq -\delta$ and $x_0^n \leq x$. Therefore $|H^{-1}| \leq C$ uniformly with respect to (λ, x). Moreover, using $x = \epsilon\xi$, we have

$$\partial\mu_j^\pm / \partial\xi = \frac{-\epsilon \pm \epsilon(x - \nu_j)((x - \nu_j)^2 + 4\lambda)^{-\frac{1}{2}}}{2} = O(\epsilon)$$

for all (j, λ, x). Therefore $H^{-1}H_\xi = O(\epsilon)$. From this, one can show by an argument similar to the proof of roughness of exponential dichotomies that for sufficiently small

ϵ, (5.9) also has a pseudoexponential dichotomy on $\xi \geq \frac{a}{\epsilon}$. The projection, which we denote by $\tilde{Q}(\lambda, \epsilon, \xi)$, is $O(\epsilon)$ close to $\tilde{P}$. For appropriate negative constants α_2 and β_2, the spectral gap is $\beta_2 < \alpha_2\epsilon < 0$. The constant C in the definition of pseudoexponential dichotomy is independent of (λ, ϵ).

Because the system $W_\xi = AW$ is just (5.9) after a linear change of variables, it also has a pseudoexponential dichotomy on $\xi \geq \frac{a}{\epsilon}$ with spectral gap $\beta_2 < \alpha_2\epsilon < 0$.

The matrices A and B differ by $O(\epsilon)$ terms that are in the last n rows only. Existence of a pseudoexponential dichotomy on $\xi \geq \frac{a}{\epsilon}$ for $W_\xi = BW$ then follows by an argument similar to the proof of roughness of exponential dichotomies. $\square$

The pseudoexponential dichotomy for $W_\xi = AW$ has the projection $\bar{Q} := H\tilde{Q}H^{-1} = H(\tilde{P} + O(\epsilon))H^{-1} = O(1 + \epsilon|x| + \sqrt{|\lambda|})$, which can be large for large ξ and $|\lambda|$.

LEMMA 5.3. *Let $Q(\lambda, \xi)$ be the projection for the pseudoexponential dichotomy for $W_\xi = BW$. Then $|Q(\lambda, \xi)(I - \tilde{P})|$ is uniformly bounded for all (λ, ξ) with* $\mathrm{Re}\lambda \geq -\delta$ *and $|\xi| \geq \frac{a}{\epsilon}$.*

Proof. We will show the result for $W_\xi = AW$. The result for $W_\xi = BW$ then follows by an argument similar to the proof of roughness of exponential dichotomies.

Observe that

$$\bar{Q}(I - \tilde{P}) = H\tilde{Q}H^{-1}(I - \tilde{P}),$$
$$|\bar{Q}(I - \tilde{P})| \leq |H||\tilde{Q}||H^{-1}(I - \tilde{P})|.$$

Using the facts

$$|H| \leq C(1 + |\mathcal{M}^-| + |\mathcal{M}^+|),$$
$$|\tilde{Q}| \leq C,$$
$$|H^{-1}(I - \tilde{P})| \leq C|(\mathcal{M}^+ - \mathcal{M}^-)^{-1}|,$$

we obtain that $|\bar{Q}(I - \tilde{P})|$ is uniformly bounded with respect to (λ, ϵ, ξ) in the domain of consideration. $\square$

Let γ be a constant such that $\gamma > \max\{-\alpha_2, \beta_1\}$. We now show that in the function space $C(\gamma, \mathbb{R}_x)$, the region $\mathrm{Re}\tilde{\lambda} \geq -\tilde{\delta}$ consists of normal points only. Observe that in the ξ-coordinate, the space is $C(\epsilon\gamma, \mathbb{R}_\xi)$.

Without loss of generality, assume that $x = 0$ is between x_0^1 and x_0^n. Consider the nonhomogeneous equation (5.4), where $h \in C(\epsilon\gamma, \mathbb{R}_\xi)$. This is equivalent to the first-order system

$$W_\xi = BW + (0,\ h)^\top. \tag{5.11}$$

By Proposition 5.2, the associated homogeneous system of (5.11) has pseudoexponential dichotomies on $\xi \leq -\frac{a}{\epsilon}$ and $\xi \geq \frac{a}{\epsilon}$. These dichotomies can be extended from $(-\infty, -\frac{a}{\epsilon}]$ to $\mathbb{R}^-$ and from $[\frac{a}{\epsilon}, \infty)$ to $\mathbb{R}^+$. The constants of the extended dichotomies are ϵ dependent and may approach ∞ as $\epsilon \to 0$, but the exponents remain the same. If, for certain λ, the n-dimensional pseudounstable space at $\xi = 0-$ has a nontrivial intersection with the n-dimensional pseudostable space at $\xi = 0+$, then λ is obviously an eigenvalue.

Next assume that for some λ, the n-dimensional pseudounstable space at $\xi = 0-$ has trivial intersection with the n-dimensional pseudostable space at $\xi = 0+$, so that

$$\mathcal{R}Q(0^+) \oplus \mathcal{R}(I - Q(0^-)) = \mathbb{R}^n. \tag{5.12}$$

Let $w_s \in \mathcal{R}Q(0^+)$ and $w_u \in \mathcal{R}(I - Q(0^-))$. Then the solution of (5.11) can be expressed as

$$
\begin{aligned}
w(\xi) &= T(\xi,0)w_s + \int_0^\xi T(\xi,\zeta)Q(\zeta)(0,\ h(\zeta))^\top d\zeta \\
&\quad + \int_\infty^\xi T(\xi,\zeta)(I - Q(\zeta))(0,\ h(\zeta))^\top d\zeta, \quad \xi > 0, \\
w(\xi) &= T(\xi,0)w_u + \int_0^\xi T(\xi,\zeta)(I - Q(\zeta))(0,\ h(\zeta))^\top d\zeta \\
&\quad + \int_{-\infty}^\xi T(\xi,\zeta)Q(\zeta)(0,\ h(\zeta))^\top d\zeta, \quad \xi < 0.
\end{aligned} \tag{5.13}
$$

Using Lemma 5.3 and the fact that $(0,\ h(\zeta))^\top = (I - \tilde{P})(0,\ h(\zeta))^\top$, it is easy to show that the integrals in (5.13) are convergent and define functions in $C(\epsilon\gamma, \mathbb{R}_\xi^+)$ for $\xi > 0$ and in $C(\epsilon\gamma, \mathbb{R}_\xi^-)$ for $\xi < 0$.

It remains to find $w_s \in \mathcal{R}Q(0^+)$ and $w_u \in \mathcal{R}(I - Q(0^-))$ such that $w(0^-) = w(0^+)$. From (5.13),

$$
w_u - w_s = \int_\infty^0 T(0,\zeta)(I - Q(\zeta))(0,\ h(\zeta))^\top d\zeta - \int_{-\infty}^0 T(0,\zeta)Q(\zeta)(0,\ h(\zeta))^\top d\zeta. \tag{5.14}
$$

By (5.12), there exist unique $w_s \in \mathcal{R}Q(0^+)$ and $w_u \in \mathcal{R}(I - Q(0^-))$ such that (5.14) holds.

Thus the spectral equation (5.4) has a unique solution U for each h. From (5.13), we see that $|U|_{\epsilon\gamma} \le C_\epsilon |h|_{\epsilon\gamma}$. This shows that λ is in the resolvent of the linear partial differential equation (5.4).

We have proved the following.

THEOREM 5.4. *Let $\tilde{\delta}$ be a positive constant. Let $\gamma > \max\{-\alpha_2, \beta_1\}$. Then for $\epsilon > 0$ sufficiently small, system (5.3) on the space $C^2(\gamma, \mathbb{R}_x)$ (resp., system (5.4) on the space $C^2(\epsilon\gamma, \mathbb{R}_\xi)$) has only normal points in the region $\mathrm{Re}\tilde{\lambda} \ge -\tilde{\delta}$ (resp., in the region $\mathrm{Re}\lambda \ge -\delta := -\epsilon\tilde{\delta}$).*

We end this section by stating a lemma that will also be used in the next section and a corollary that was used in the proof of Lemma 5.1.

LEMMA 5.5. *Let $\lambda = \sigma + \omega i$ and $z = x + yi$ be complex variables, with $\sigma, \omega, x, y \in \mathbb{R}$. For a given real $r \neq 0$, consider the mapping*

$$z = \sqrt{r^2 + \lambda}$$

and its inverse

$$\lambda = z^2 - r^2.$$

(1) *For any $a > 0$, the mapping $\lambda = z^2 - r^2$ takes each vertical line $\mathrm{Re}z = \pm a$ bijectively onto the parabola*

$$\sigma = a^2 - r^2 - \frac{\omega^2}{4a^2}.$$

The regions $\mathrm{Re}z \ge a$ and $\mathrm{Re}z \le -a$ are each mapped bijectively onto the closure of the region to the right of the parabola, i.e., onto

$$\sigma \ge a^2 - r^2 - \frac{\omega^2}{4a^2}.$$

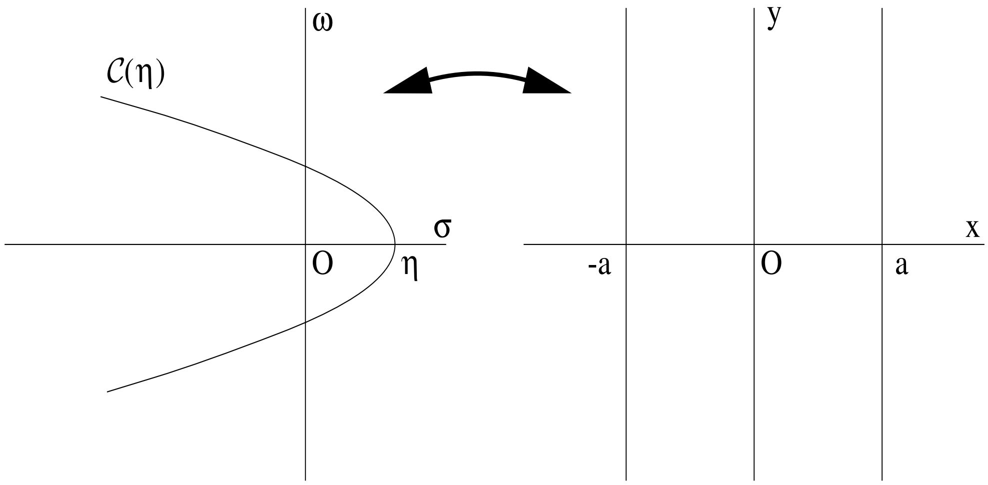

FIG. 5.1. *The mapping* $\lambda = z^2 - r^2$ *takes each vertical line* $\mathrm{Re}z = \pm a$ *bijectively onto the parabola* $\mathcal{C}(\eta)$.

(2) *For any* $\eta > -r^2$, *let*

$$\mathcal{C}(\eta) := \left\{ (\sigma, \omega) : \sigma = \eta - \frac{\omega^2}{4(r^2 + \eta)} \right\},$$

a parabola with vertex at $(\eta, 0)$ *that opens to the left. Then the mapping* $z = \sqrt{r^2 + \lambda}$ *takes* $\mathcal{C}(\eta)$ *onto the vertical lines* $\mathrm{Re}z = \pm a = \pm\sqrt{r^2 + \eta}$. *The closure of the region to the right of* $\mathcal{C}(\eta)$, *denoted* $\mathcal{R}(\eta)$, *is mapped onto* $|z| \geq a = \sqrt{r^2 + \eta}$.

(3) *If* $\eta > 0$, *then* $a > |r|$; *if* $-r^2 < \eta < 0$, *then* $0 < a < |r|$.

See Figure 5.1.

COROLLARY 5.6. *For any* $0 < \delta < r^2$, *let* $\eta = -\delta$. *Then the region* $\mathrm{Re}\lambda \geq -\eta$ *is in* $\mathcal{R}(-\delta)$ *and is mapped by* $z = \sqrt{r^2 + \lambda}$ *into* $|\mathrm{Re}z| \geq \sqrt{r^2 + \eta} = \sqrt{r^2 - \delta}$.

6. $O(\frac{1}{\epsilon})$ Eigenvalues.

Let us first consider a time-dependent solution $u_\epsilon(x, t)$ of (1.17) with initial data $u_\epsilon(x, 0) = \phi_\epsilon(x)$ near the Riemann–Dafermos solution $u_\epsilon(x)$. Thus, $\phi_\epsilon(x)$ has n sharp transition layers at $\bar{x}^i_\epsilon$, with $\bar{x}^i_\epsilon$ near $\bar{s}^i$. Then we expect that $u_\epsilon(x, t)$ has n sharp jumps near curves $\bar{x}^i_\epsilon(t)$, with $\bar{x}^i_\epsilon(0) = \bar{x}^i_\epsilon$. (If the Riemann–Dafermos solution is stable, we expect that $\bar{x}^i_\epsilon(t) \to x^i(\epsilon)$ as $t \to \infty$.) Near the curve $\bar{x}^i_\epsilon(t)$ we use the fast spatial variable $\xi = \frac{x - \bar{x}^i_\epsilon(t)}{\epsilon}$. Then (1.17) becomes

$$\epsilon u_t = u_{\xi\xi} - \left(Df(u) - \bar{x}^i_\epsilon(t) - \frac{d}{dt}\bar{x}^i_\epsilon(t) - \epsilon\xi \right) u_\xi.$$

Unless ϕ_ϵ is a stationary solution of (1.17), we have $u_t = O(\frac{1}{\epsilon})$ near $\bar{x}^i_\epsilon$; i.e., the system exhibits very fast motion near $\bar{x}^i_\epsilon$. It is common in singular perturbation problems to have an *initial layer* in which there is motion with speed of order $\frac{1}{\epsilon}$ for time of order ϵ. Thus we expect the existence of eigenvalues of order $\frac{1}{\epsilon}$, with the support of the eigenfunctions concentrated near the points $\bar{x}^i_\epsilon$.

Assume now that in the singular layers, the solution quickly converges to traveling-wave-like solutions. Then after the initial time layer, the solution behaves like convection in the regular layer coupled with traveling waves in singular layers. This motion occurs for $t > O(\epsilon)$ and has $u_t = O(1)$. Thus we expect to find eigenvalues of order 1 and related eigenfunctions.

We discuss fast eigenvalues of order $\frac{1}{\epsilon}$ in this section. Slow eigenvalues of order 1 will be studied in the next section.

We recall that the linear variational system at a Riemann–Dafermos solution $u_\epsilon(x)$ is

$$U_t + (Df(u_\epsilon) - xI)U_x + D^2 f(u_\epsilon)u_{\epsilon\,x}U = \epsilon U_{xx}.$$

We shall study this equation in the space $C^2(\gamma, \mathbb{R}_x)$, $\gamma > 0$.

Eigenvalues $\tilde{\lambda}$ and corresponding eigenfunctions $U(x)$ satisfy

$$\tilde{\lambda}U + (Df(u_\epsilon) - xI)U_x + D^2 f(u_\epsilon)u_{\epsilon\,x}U = \epsilon U_{xx}. \tag{6.1}$$

In section 3 we found an expansion for $u_\epsilon(x)$ in the regular layer. We also found expansions for the jump positions $x^i(\epsilon)$, and for $u^i_\epsilon(\xi)$ in singular layers centered around $x^i(\epsilon)$, in the stretched coordinate $\xi = \frac{x - x^i(\epsilon)}{\epsilon}$. We shall use these expansions in what follows.

We shall look for eigenvalues

$$\tilde{\lambda} = \sum_{j=-1}^{\infty} \epsilon^j \lambda_j. \tag{6.2}$$

Fast eigenvalues have $\lambda_{-1} \neq 0$; slow eigenvalues have $\lambda_{-1} = 0$. We shall look for corresponding eigenfunctions with expansions

$$U^R_\epsilon(x) = \sum_{j=0}^{\infty} \epsilon^j U^R_j(x) \qquad \text{in the regular layer,} \tag{6.3}$$

$$U^i_\epsilon(\xi) = \sum_{j=0}^{\infty} \epsilon^j U^i_j(\xi) \qquad \text{in the singular layer } S^i. \tag{6.4}$$

In this section we look for fast eigenvalues, which have the form (6.2) with $\lambda_{-1} \neq 0$.

We shall show that under certain conditions, fast eigenvalues have eigenfunctions that are localized in a single singular layer. These eigenvalues correspond to zeros of Evans functions on each singular layer.

We first consider the regular layer. We substitute (3.4), (6.2), and (6.3) into (6.1) and expand in powers of ϵ. At order ϵ^{-1} (the lowest order) we obtain

$$\lambda_{-1} U^R_0 = 0. \tag{6.5}$$

Since $\lambda_{-1} \neq 0$, $U^R_0 = 0$.

At order ϵ^0 we obtain

$$\lambda_{-1} U^R_1 = \text{terms involving } U^R_0 = 0.$$

Since $\lambda_{-1} \neq 0$, $U^R_1 = 0$. Similarly, higher-order expansions of eigenvalues and the corresponding eigenfunctions are determined by a system of algebraic equations. In particular, we find that $U^R_j = 0$ for all j.

In the ith singular layer, we rewrite (6.1) as

$$\epsilon(\tilde{\lambda} + 1)U + ((Df(u_\epsilon) - xI)U^i)_\xi = U^i_{\xi\xi}, \qquad \text{with } x = x_i(\epsilon) + \epsilon\xi. \tag{6.6}$$

We substitute (3.6), (6.2), and (6.4) into (6.6) and expand in powers of ϵ. At order ϵ^0 (the lowest order) we obtain

$$\lambda_{-1}U_0^i + ((Df(q^i) - x_0^i I)U_0^i)_\xi = U_{0\xi\xi}^i. \tag{6.7}$$

Since $U_0^R = 0$, we must have $U_0^i(\xi) \to 0$ as $\xi \to \pm\infty$. We note that (6.7) also arises in the study of the stability of the traveling wave solution $u(X,T) = q^i(X - x_0^i T)$ of the system of viscous conservation laws (1.1); it determines the eigenvalues and eigenfunctions of the linearization of (1.1) at the traveling wave. Let us assume the following:

(H1) For the complex number $\lambda_{-1} \neq 0$, there is exactly one i, $1 \le i \le n$, such that (6.7) has a nontrivial solution $U_0^i(\xi)$ that satisfies the boundary conditions $U_0^i(\xi) \to 0$ as $\xi \to \pm\infty$.

(H2) For that i, λ_{-1} is a semisimple eigenvalue [20, p. 41] of the linear differential operator

$$U_{0\xi\xi}^i - ((Df(q^i) - x_0^i I)U_0^i)_\xi \tag{6.8}$$

on the Banach space of uniformly continuous functions that approach 0 as $\xi \to \pm\infty$, with the sup norm.

Consider first the index i of assumption (H1). Let $\lambda_{-1}^i := \lambda_{-1}$. Let the multiplicity of λ_{-1}^i as an eigenvalue of (6.8) be m_i. Let $\phi_j^i(\xi)$, $j = 1, \ldots, m_i$, be a basis for the eigenspace. Then to lowest order, an eigenfunction associated to $\tilde{\lambda} = \sum_{j=-1}^{\infty} \lambda_j^i \epsilon^j$ has the form $U_0^i(\xi) = \sum_{j=1}^{m_i} c_j^i \phi_j^i(\xi)$ in the ith singular layer for some constants $\{c_j^i\}_{j=1}^{m_i}$ and is zero in the regular layer and other singular layers.

We now show how to determine the possible values of λ_0^i and $\{c_j^i\}_{j=1}^{m_i}$ using the expansions to order ϵ^1.

Later, we will show that in certain regions of λ-space, the limiting systems of (6.7) at $\xi = \pm\infty$ have exponential dichotomies with n-dimensional unstable and stable subspaces. The eigenfunction U_0^i corresponds to a nontrivial intersection of the unstable subspace at $\xi = -\infty$ and the stable subspace at $\xi = \infty$.

By [33], the adjoint system to (6.7) must also have an m_i-dimensional space of bounded solutions. Let $\{\psi_\ell^i\}_{\ell=1}^{m_i}$ be a basis for this space.

In the ith singular layer, at order ϵ^1, we have

$$\begin{aligned}(\lambda_0^i + 1)U_0^i &+ ((D^2 f(q^i)u_1^i - (x_1^i + \xi)I)U_0^i)_\xi \\ &+ \lambda_{-1}^i U_1^i + ((Df(q^i) - x_0^i I)U_1^i)_\xi = U_{1\xi\xi}^i.\end{aligned} \tag{6.9}$$

The solvability condition of (6.9) can be obtained from Fredholm's alternative [33]:

$$\langle \psi_\ell^i, (\lambda_0^i + 1)U_0^i + ((D^2 f(q^i)u_1^i - (x_1^i + \xi)I)U_0^i)_\xi \rangle = 0, \quad \ell = 1, \ldots, m_i. \tag{6.10}$$

Recall that $U_0^i = \sum_{j=1}^{m_i} c_j^i \phi_j^i(\xi)$. Since λ_{-1}^i is semisimple, without loss of generality, we assume that $\langle \psi_\ell^i, \phi_j^i \rangle = \delta_j^\ell$.

Let $\mathcal{B}^i = \{b_{\ell,j}^i\}$ be the $m_i \times m_i$ matrix whose entries are

$$b_{\ell,j}^i := \langle \psi_\ell^i, ((D^2 f(q^i)u_1^i - (x_1^i + \xi)I)\phi_j^i)_\xi \rangle.$$

The solvability condition (6.10) becomes

$$((\lambda_0^i + 1)I - \mathcal{B}^i)\mathbf{c}^i = \mathbf{0}, \tag{6.11}$$

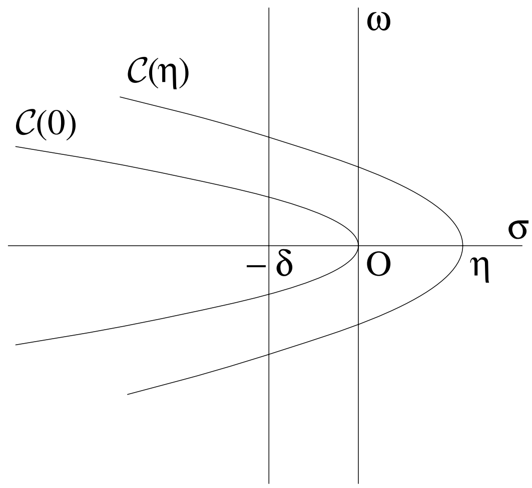

FIG. 6.1. *Eigenvalues to the right of* $\mathcal{C}(\eta)$ *and* $\mathcal{C}(0)$.

where $\mathbf{c}^i = (c_1^i, \ldots, c_{m_i}^i)$, and I is the $m_i \times m_i$ identity matrix. Therefore $\lambda_0^i + 1$ is an eigenvalue of the matrix $\mathcal{B}$ and $(c_1^i, \ldots, c_{m_i}^i)$ is the corresponding eigenvector. The algebraic system (6.11) determines the possible values of λ_0^i and the corresponding $\mathbf{c}^i$.

We assume the following:

(H3) The eigenvalues of the matrix $\mathcal{B}^i$ are distinct.

Of course, (H3) holds automatically in the most common case, $m_i = 1$.

From (H3), we have m_i distinct eigenvalues $\lambda_0^i + 1$, each with an eigenvector $\mathbf{c}^i$ corresponding to an eigenfunction $U_0^i = \sum c_j^i \phi_j^i$. Thus, for $\epsilon > 0$, λ_1^i splits into m_i distinct eigenvalues.

Assuming (H3), higher-order expansions of eigenvalues and the corresponding eigenfunctions in singular layers can be obtained by a straightforward formal procedure, which will not be presented here.

Next, we consider i other than the one specified in assumption (H1). It is clear that $U_0^i = 0$. From (6.9) we find that $U_1^i = 0$. Similarly, all $U_j^i = 0$.

We refer to the $O(\frac{1}{\epsilon})$ eigenvalues as *local eigenvalues* since the asymptotic expansions of their associated eigenfunctions are localized in a single singular layer.

Our next object is to define, for the ith singular layer, an Evans function $E^i(\lambda)$ [11] whose zeros are complex numbers λ_{-1}^i for which (6.7) has solutions that approach 0 as $\xi \to \pm\infty$. For an arbitrary $\eta > 0$, we will define a parabola $\mathcal{C}(\eta)$ that opens to the left and has its vertex at $(\eta, 0), \eta > 0$, in the complex plane. The parabolas $\mathcal{C}(\eta)$ do not intersect. As $\eta \to 0+$, they approach a parabola $\mathcal{C}(0)$ with vertex at $(0, 0)$. See Figure 6.1 Let the region to the right of $\mathcal{C}(\eta)$ be $\mathcal{R}(\eta)$. The Evans function $E^i(\lambda)$ can be defined on $\mathcal{R}(0)$. For each small $\eta > 0$, if λ_{-1}^i is a zero of the Evans function defined in $\mathcal{R}(\eta)$, then (6.7) has a nontrivial solution that satisfies $U_0^i(\xi) = O(e^{-\eta|\xi|})$.

As in section 5, let $x_0^i = \bar{s}^i$, $i = 1, \ldots, n$. Let $N > \max\{|x_0^1|, |x_0^n|\}$. Thus the compact interval $[-N, N]$ contains all the points x_0^i, $i = 1, \ldots, n$. Let $x_0^0 = -N$ and $x_0^{n+1} = N$. For $\lambda \in \mathbb{C}$ and $i = 0, \ldots, n$, define

$$\tilde{A}^i(\lambda, x) = \begin{pmatrix} 0 & I \\ \lambda I & Df(\bar{u}_0^i) - xI \end{pmatrix}, \quad x \in [x_0^i, x_0^{i+1}],$$

where $\bar{u}_0^i$ was defined in section 5.

LEMMA 6.1. *For each $\eta > 0$, there exist $\beta(\eta) < 0 < \alpha(\eta)$ such that, for all $\lambda \in \mathcal{R}(\eta)$, for all $i = 0, \dots, n$, and for all x in $[x_0^i, x_0^{i+1}]$, $\tilde{A}^i(\lambda, x)$ has n eigenvalues less than $\beta(\eta)$ and n eigenvalues greater than $\alpha(\eta)$. As $\eta \to 0$, α and β are $O(\eta)$.*

Proof. Fix an index i between 0 and n. Let $\nu_1^i < \cdots < \nu_n^i$ denote the eigenvalues of $Df(\bar{u}_0^i)$. Let μ be an eigenvalue of $\tilde{A}^i(\lambda, x)$. Then one of the following equations must hold:

$$\mu^2 + (x - \nu_j^i)\mu - \lambda = 0, \quad j = 1, \dots, n. \tag{6.12}$$

Let $p_j^i(x) := \frac{1}{2}(x - \nu_j^i)$, $x \in [x_0^i, x_0^{i+1}]$. The solutions of (6.12) are

$$\mu_j^{i\pm}(\lambda, x) := -p_j^i \pm \sqrt{p_j^{i\,2} + \lambda}.$$

The main branch of the square root is used.

Define

$$\mathcal{C}_j^i(\eta) := \left\{ (\sigma, \omega) : \sigma = \eta - \frac{\omega^2}{4(p_j^{i\,2} + \eta)} \right\},$$

$$\mathcal{R}_j^i(\eta) := \left\{ (\sigma, \omega) : \sigma \geq \eta - \frac{\omega^2}{4(p_j^{i\,2} + \eta)} \right\},$$

$$\alpha_j^i := -p_j^i + \sqrt{p_j^{i\,2} + \eta},$$

$$\beta_j^i := -p_j^i - \sqrt{p_j^{i\,2} + \eta}.$$

The vertex of the parabola $\mathcal{C}_j^i(\eta)$ is at $(\sigma, \omega) = (\eta, 0)$. The parabola opens to the left.

Using Lemma 5.5 with $p = p_j^i$, we have that if $\lambda \in \mathcal{R}_j^i(\eta)$, then

$$\mathrm{Re}\mu_j^{i-} \leq \beta_j^i < 0 < \alpha_j^i \leq \mathrm{Re}\mu_j^{i+}, \quad j = 1, \dots, n.$$

Define

$$p := \max |p_j^i(x)|, \quad \alpha := \min \alpha_j^i, \quad \beta := \max \beta_j^i,$$

$$\mathcal{C}(\eta) := \{(\sigma, \omega) | \sigma = \eta - \frac{\omega^2}{4(p^2 + \eta)}, \tag{6.13}$$

$$\mathcal{R}(\eta) := \cap_{i,j} \mathcal{R}_j^i(\eta) = \{(\sigma, \omega) | \sigma \geq \eta - \frac{\omega^2}{4(p^2 + \eta)}. \tag{6.14}$$

If $\lambda \in \mathcal{R}(\eta)$, then $\mu_j^{i-} < \beta < 0 < \alpha < \mu_j^{i+}$ for all i and j and for all $x \in [x_0^i, x_0^{i+1}]$. From their definitions, $\alpha_j^i = O(\eta)$ if $p_j^i > 0$ and $\beta_j^i = O(\eta)$ if $p_j^i < 0$. Notice that p_j^i can be both positive and negative. It follows that α and β are $O(\eta)$. $\square$

Let $x = \epsilon\xi$, and let $A^i(\lambda, \epsilon, \xi) := \tilde{A}^i(\lambda, \epsilon\xi)$. From the roughness theory of exponential dichotomies [7] and Lemma 6.1, we derive the following proposition.

PROPOSITION 6.2. *For each $i = 0, \dots, n$ and for each $\lambda \in \mathcal{R}(\eta)$, the slowly varying system*

$$W_\xi = A^i(\lambda, \epsilon, \xi)W, \quad \xi \in \left[-\frac{x_0^i}{\epsilon}, \frac{x_0^{i+1}}{\epsilon} \right],$$

has an exponential dichotomy with exponents $\beta(\eta) < 0 < \alpha(\eta)$. The unstable subspace of the exponential dichotomy in each subinterval is n-dimensional. As $\eta \to 0$, α and β are $O(\eta)$.

Using the information from Lemma 6.1, for each internal layer S^i and for each $\eta > 0$, we can define an Evans function $E^i(\lambda)$ for $\lambda \in \mathcal{R}(\eta)$. More precisely, rewrite (6.7) as

(6.15)
$$\begin{pmatrix} U_\xi \\ V_\xi \end{pmatrix} = B(\lambda,\xi)\begin{pmatrix} U \\ V \end{pmatrix}, \text{ where } B(\lambda,\xi) := \begin{pmatrix} 0 & I \\ \lambda I + D^2 f(q^i(\xi)) q^i_\xi & Df(q^i(\xi)) - x^i_0 I \end{pmatrix}.$$

The coefficient matrix approaches $\tilde{A}(\lambda, x^i_0 \pm)$ as $\xi \to \pm\infty$ exponentially. By Lemma 6.1, the limiting matrices $\tilde{A}(\lambda, x^i_0\pm)$ have n eigenvalues with real parts less than $\beta(\eta) < 0$ and the other n eigenvalues with real parts greater than $\alpha(\eta) > 0$. We conclude that for the system (6.15), there exist n linearly independent solutions $\{\phi^+_j(\lambda,\xi)\}^n_{j=1}$ such that each approaches zero as $\xi \to \infty$ and n linearly independent solutions $\{\phi^-_j(\lambda,\xi)\}^n_{j=1}$ such that each approaches zero as $\xi \to -\infty$.

The Evans function for the internal layer S^i is defined as

$$E^i(\lambda) := e^{-\int_0^\xi \mathrm{tr} B(\lambda,\zeta) d\zeta} a(\lambda,\xi) \wedge b(\lambda,\xi) = a(\lambda,0) \wedge b(\lambda,0). \tag{6.16}$$

Here $a(\lambda,\xi)$ and $b(\lambda,\xi)$ are n-forms associated to $\{\phi^-_j : i = 1,\dots,n\}$ and $\{\phi^+_j : i = 1,\dots,n\}$, respectively [11], [1], [14].

Since formula (6.16) is independent of η, the Evans function is actually defined on $\mathcal{R}(0)$. A zero of the Evans function corresponds to a complex number λ^i_{-1} for which (6.7) admits a nontrivial solution U^i_0 that approaches zero as $\xi \to \pm\infty$. The same Evans function arises in the study of the stability of the traveling wave solution $u(X,T) = q^i(X - x^i_0 T)$ of the system of viscous conservation laws (1.1).

According to [14], the Evans function extends analytically to a neighborhood of the origin. We always have $E^i(0) = 0$; an eigenfunction is q^i_ξ. By analyticity, there are no other zeros of $E^i(\lambda)$ near $\lambda = 0$. Therefore for any sufficiently small $\eta > 0$ and $\delta > 0$, all zeros of $E^i(\lambda)$ in $\{\lambda : \mathrm{Re}\lambda \geq -\delta\}$ are contained in $\mathcal{R}(\eta) \cap \{\lambda : \mathrm{Re}\lambda \geq -\delta\}$.

THEOREM 6.3. *Let $\eta > 0$ be given and let λ^i_{-1} be a zero of E^i in the region $\mathcal{R}(\eta) \cap \{\lambda : \mathrm{Re}\lambda \geq -\delta\}$. Assume that conditions* (H1)–(H3) *are satisfied. Then there exists $\epsilon_0(\eta) > 0$ such that if $0 < \epsilon < \epsilon_0(\eta)$, then the root λ^i_{-1} of E^i is associated to a finite number of curves of fast eigenvalues* (6.2).

To all orders in ϵ, the corresponding eigenfunction is zero in the regular layer and in singular layers other than the ith. The pair (λ^i_{-1}, U^i_0) satisfies (6.7) *and the boundary condition $U^i_0 \to 0$ as $\xi \to \pm\infty$. If the eigenspace of λ^i_{-1} for* (6.7) *is m_i-dimensional, then $U^i_0 = \sum_{j=1}^{m_i} c^i_j \phi^i_j$, where $\{\phi^i_j\}^{m_i}_{j=1}$ is a basis for the eigenspace. The m_i possible values of λ^i_0 and the corresponding vectors $\mathbf{c}^i$ are determined by the eigenvalue-eigenvector problem* (6.11).

Proof. Sketch of the proof: The procedure for finding the correction terms $\Delta\lambda$ and ΔU^i is similar to that for finding λ^i_0 and $\mathbf{c}^i$, followed by a contraction mapping argument. The necessary dichotomies in regular sublayers and singular layers come from Lemma 6.1 and Proposition 6.2. □

Remark 6.1. (1) We emphasize that Theorem 6.3 does not apply to $\lambda_{-1} = 0$. Indeed, by Proposition 6.2, as η decreases, the exponential dichotomy weakens, so the ϵ-interval on which the contraction mapping argument is valid shrinks. Thus, as $\eta \to 0$, $\epsilon_0(\eta) \to 0$. Moreover, as we shall see in the next section, there can be an infinite number of curves of eigenvalues (6.2) whose asymptotic expansion begins with $\lambda_{-1} = 0$; in the case $n = 2$, at least, this is typical.

(2) We also emphasize that we have not shown that for a fixed small $\epsilon > 0$, all eigenvalues near $\lambda_{-1} = 0$ are given by expansions of the form (6.2) with $\lambda_{-1} = 0$. We note, however, that $E'(0)$ is the product of two terms, one of which is nonzero if and only if Majda's inviscid stability condition holds [14], [3]. We expect that in the case $E'(0) \neq 0$, all eigenvalues near $\lambda_{-1} = 0$ are in fact given by such expansions.

7. $O(1)$ Eigenvalues. We look for eigenvalues of (6.1) of the form

$$\tilde{\lambda} = \sum_{j=0}^{\infty} \epsilon^j \lambda_j \tag{7.1}$$

and the corresponding eigenfunctions $U(x)$. We continue to work in the space $C^2(\gamma, \mathbb{R}_x)$, $\gamma > 0$. We rewrite (6.1) as

$$(\tilde{\lambda} + 1)U^R + ((Df(u_\epsilon) - xI)U^R)_x = \epsilon U^R_{xx} \qquad \text{in the regular layer,} \tag{7.2}$$

$$\epsilon(\tilde{\lambda} + 1)U^i + ((Df(u_\epsilon) - x^i(\epsilon) - \epsilon\xi I)U^i)_\xi = U^i_{\xi\xi} \qquad \text{in the singular layer } S^i. \tag{7.3}$$

Proposition 7.1. *To any order of ϵ, $\tilde{\lambda} = -1$ is an eigenvalue of* (7.2) *and* (7.3). *The corresponding eigenfunctions form an n-dimensional eigenspace. The ith basis vector is a homoclinic solution to* 0 *that, to lowest order in ϵ, equals q^i_ξ in the ith singular layer and is zero in other singular layers and in the regular layer.*

Proof. We need to find expansions of $U^R_\epsilon(x)$ and $U^i_\epsilon(\xi)$ to the following system:

$$((Df(u_\epsilon) - xI)U^R)_x = \epsilon U^R_{xx} \qquad \text{in the regular layer,} \tag{7.4}$$

$$((Df(u_\epsilon) - x^i(\epsilon) - \epsilon\xi I)U^i)_\xi = U^i_{\xi\xi} \qquad \text{in the singular layer } S^i. \tag{7.5}$$

By Lemma 7.2, proved below, for any $j \geq 0$, $U^R_j(x) = 0$ in the regular sublayer R^0.

Let $1 \leq i \leq n$. Assume that for all $j \geq 0$, $U^R_j(x) = 0$ in the regular sublayer R^{i-1}. We shall show that $U^i_0(\xi)$ is a constant multiple of q^i_ξ and that for every $j \geq 0$, $U^R_j(x) = 0$ in the regular sublayer R^i. Then, by induction on i, the proposition is proved.

In the singular layer S^i, in order to match the solution in R^{i-1}, we look for a bounded solution of (7.5) that approaches 0 as $\xi \to -\infty$. Integrating (7.5) from $-\infty$ to ξ, we have

$$U_\xi - (Df(u_\epsilon) - x^i(\epsilon) - \epsilon\xi I)U = 0. \tag{7.6}$$

By the definition of a Lax i-shock, at order ϵ^0, this system has exponential dichotomies for $\xi \in \mathbb{R}^\pm$. By the definition of a structurally stable Riemann solution, the unstable space of the dichotomy on $\mathbb{R}^-$ intersects the stable space of the dichotomy on $\mathbb{R}^+$ transversely at $\xi = 0$. The intersection is a one-dimensional space spanned by q^i_ξ. To have a bounded solution, we must set $U^i_0(\xi)$ equal to a constant multiple of q^i_ξ. Then $U^i_0(\xi)$ approaches zero exponentially as $\xi \to \pm\infty$.

At order ϵ^1, (7.6) becomes

$$U^i_{1\xi} - (Df(u^i_0(\xi)) - x^i_0)U^i_1 = (D^2 f(u^i_0(\xi))u^i_1 - (x^i_1 + \xi)I)U^i_0. \tag{7.7}$$

Since $U_0^i(\xi) = O(e^{-\alpha|\xi|})$, the nonhomogeneous term of (7.7) is $O((|\xi|+1)e^{-\alpha|\xi|})$, which approaches zero as $\xi \to \pm\infty$. Observe that the homogeneous part of (7.7) has exponential dichotomies in $\mathbb{R}^{\pm}$, and the unstable space of the dichotomy on $\mathbb{R}^-$ intersects the stable space of the dichotomy on $\mathbb{R}^+$ transversely at $\xi = 0$. Thus, if we assume that $U_1^i(0) \perp U_0^i(0)$, a unique solution $U_1^i = O((|\xi|+1)e^{-\alpha|\xi|})$ can be constructed using integral equations on $\mathbb{R}^{\pm}$ and the matching at $\xi = 0$. See [33], [25].

Proceeding inductively, at order ϵ^j, $j > 1$, we solve a nonhomogeneous system for U_j^i, with a nonhomogeneous term that is $O((1+|\xi|)^j e^{-\alpha|\xi|})$. The solution $U_j^i = O((1+|\xi|)^j e^{-\alpha|\xi|})$ approaches zero as $\xi \to \pm\infty$.

We now consider the solution in the regular sublayer R^i. By matching, for all $j \geq 0$, $U_j^R(x_0^i+) = U_j^i(\infty) = 0$.

We show inductively that for all $j \geq 0$, $U_j^R(x) = 0$ in R^i. At order ϵ^0, from (7.4), $(Df(u_0) - xI)U_0^R(x)$ is constant in R^i. Since it is zero at x_0^i+, $(Df(u_0) - xI)U_0^R(x) = 0$ in R^i. Since $Df(u_0) - xI$ is nonsingular in each regular sublayer, we see that $U_0^R = 0$ in R^i.

Next, at order ϵ^1, because $U_0^R = 0$, we can show similarly $(Df(u_0) - xI)U_1^R(x)$ is constant in R^i, and hence that $U_1^R = 0$ in this sublayer. Proceeding inductively, we see that for all $j \geq 0$, $U_j^R = 0$ in R^i. ∎

We remark that in the viscous regularization (1.1) of a system of conservation laws (1.3), traveling wave solutions always have a zero eigenvalue with eigenfunctions $U_0^i = c_0^i q_\xi^i$. Such an eigenfunction corresponds to a shift of the shock position from X_0^i to $X_0^i + c_0^i$. Using the self-similar variable $x = X/T$, the shock position is at $(X_0^i + c_0^i)/T$, which differs from X_0^i/T by a decay term c_0^i/T. Changing to the new time $t = \ln T$, the deviation of the shock position is $c_0^i e^{-t}$. This explains why (6.1) always has an eigenvalue (7.1) with $\lambda_0 = -1$, and why the eigenspace is as stated in Proposition 7.1.

To look for other slow eigenvalues, in the regular layer we substitute (3.4), (7.1), and (6.3) into (6.1) and expand in powers of ϵ. In singular layers, we substitute (3.6), (3.5), (7.1), and (6.4) into (6.6) and expand in powers of ϵ. For a fixed $\gamma > 0$, we shall look for solutions such that

$$|U(x)| \leq Ce^{-\gamma|x|} \text{ in the sublayers } R^0 = (-\infty, x_0^1) \text{ and } R^n = (x_0^n, \infty) \tag{7.8}$$

for some constant C.

At order ϵ^0 (the lowest order) we obtain

$$(\lambda_0 + 1)U_0^R + ((Df(\bar{u}_0^i) - xI)U_0^R)_x = 0 \qquad \text{in the sublayer } R^i, \tag{7.9}$$

$$((Df(q^i) - x_0^i I)U_0^i)_\xi = U_{0\xi\xi}^i \qquad \text{in the singular layer } S^i. \tag{7.10}$$

LEMMA 7.2. *To all orders of ϵ, eigenfunctions $U(x)$ that satisfy* (7.8) *are zero in the regular sublayers $R^0 = (-\infty, x_0^1)$ and $R^n = (x_0^n, \infty)$.*

Proof. First consider the lowest order ϵ^0. U_0^R satisfies (7.9). We consider only R^0. Let ν_j, $j = 1, \dots, n$, be the eigenvalues of $Df(\bar{u}_0^0)$. Notice that for each $j = 1, \dots, n$, $\nu_j - x > 0$ in R^0. Let $\mathbf{l}_j$, $j = 1, \dots, n$, be corresponding left eigenvectors. Let $v_j(x) = \langle \mathbf{l}_j, U_0(x) \rangle$, $x \in R^0$. Equation (7.9) becomes

$$\lambda_0 v_j + (\nu_j - x)v_{jx} = 0, \quad j = 1, \dots, n.$$

The general solution is $v_j = C_j(\nu_j - x)^{\lambda_0}$. Since $v_j = O(e^{-\gamma|x|})$, $\gamma > 0$, we must have $C_j = 0$ for all j. Therefore $U_0^R(x) = 0$ for all $x \in R^0$.

By an easy induction argument, we can show that $U_j^R = 0$ for all j on $R^0 \cup R^n$. $\square$

In the ith singular layer, we look for a solution U_0^i of (7.10) connecting the adjacent sublayers. Integration from $\xi = -\infty$ to $\xi = \infty$, together with matching, yields jump conditions that must be satisfied by U_0^R:

$$(7.11) \quad (Df(\bar{u}_0^i) - x_0^i I)U_0^R(x_0^i+) - (Df(\bar{u}_0^{i-1}) - x_0^i I)U_0^R(x_0^i-) = 0, \quad i = 1, \dots, n.$$

By Lemma 7.2, $U_0^R(x_0^1-) = 0$. Then setting $i = 1$ in (7.11) yields

$$(7.12) \qquad U_0^R(x_0^1+) = 0.$$

Solving the ODE (7.9) on the sublayer R^1 with the initial condition (7.12) yields $U_0^R(x) = 0$ for all $x \in R^1$. By induction, we have the following.

PROPOSITION 7.3. *Any solution of* (7.9)–(7.10) *that satisfies* (7.8) *has* $U_0^R(x) = 0$ *for all* x *in the regular layer.*

Proposition 7.3 implies that $U_0^i(\xi)$ approaches 0 as $\xi \to \pm\infty$ for all $i = 1, \dots, n$. Then assumption (S2′) implies the following proposition.

PROPOSITION 7.4. *Any solution of* (7.9)–(7.10) *that satisfies* (7.8) *has, for* $i = 1, \dots, n$, $U_0^i(\xi) = c_0^i q_\xi^i(\xi)$, $i = 1, \dots, n$, *for some constants* c_0^i.

The possible values of λ_0, along with the corresponding values of c_0^i, are determined at the ϵ^1-order expansion.

At order ϵ^1, we have

$$(7.13) \qquad \lambda_0 U_1 + (Df(\bar{u}_0^i) - xI)U_{1x} = 0 \quad \text{in the regular layer,}$$

$$(7.14) \qquad U_1^R(x) = 0 \quad \text{for } x \in R^0 \cup R^n,$$

$$(7.15) \qquad \begin{aligned} &(\lambda_0 + 1)U_0^i + ((D^2 f(q^i)u_1^i - (x_1^i + \xi)I)U_0^i)_\xi \\ &\qquad + ((Df(q^i) - x_0^i I)U_1^i)_\xi = U_{1\xi\xi}^i \quad \text{in the singular layer } S^i. \end{aligned}$$

In (7.15), $U_0^i(\xi) = c_0^i q_\xi^i(\xi)$, $i = 1, \dots, n$, for some constants c_0^i by Proposition 7.4.

In order to match with $U_1^R(x)$ in the regular layer, $U_1^i(\xi)$ must satisfy the following boundary conditions: $U_1^i(\xi) \to U_1^R(x_0^i-)$ exponentially as $\xi \to -\infty$ and $U_1^i(\xi) \to U_1^R(x_0^i+)$ exponentially as $\xi \to \infty$. Then, integrating (7.15) from $\xi = -\infty$ to $\xi = \infty$ and using $U_0^i = c_0^i q_\xi^i$, we have the jump condition

$$(7.16) \quad \begin{aligned} &(\lambda_0 + 1)c_0^i(\bar{u}_0^i - \bar{u}_0^{i-1}) + (Df(\bar{u}_0^i) - x_0^i I)U_1^R(x_0^i+) \\ &\qquad - (Df(\bar{u}_0^{i-1}) - x_0^i I)U_1^R(x_0^i-) = 0, \quad i = 1, \dots, n. \end{aligned}$$

By Lemma 3.1, condition (7.16) is sufficient for the existence of a solution $U_1^i(\xi)$ of (7.15) that approaches the desired limits exponentially as $\xi \to \pm\infty$. Thus if

$$(7.17) \qquad (\lambda_0, c_0^1, \dots, c_0^n, U_1^R(x))$$

satisfies (7.13) with auxiliary conditions (7.14) and (7.16), then there exist $U_1^i(\xi)$, $1 \le i \le n$, that satisfy (7.15). More precisely, if we write

$$U_1^i(\xi) = U_1^{i\perp}(\xi) + c_1^i q_\xi^i(\xi),$$

where $U_1^{i\perp}(0)$ is orthogonal to $q_\xi^i(0)$, then (7.15) uniquely determines $U_1^{i\perp}(\xi)$, but the values of c_1^i are determined at the ϵ^2-order expansion. In general, for each $j \ge 1$, we write $U_j^i(\xi) = U_j^{i\perp}(\xi) + c_j^i q_\xi^i(\xi)$ with $U_j^{i\perp}(0)$ orthogonal to $q_\xi^i(0)$. Then the ϵ^j-order expansion determines

$$(\lambda_{j-1}, c_{j-1}^1, \dots, c_{j-1}^n, U_j^{1\perp}(\xi), \dots, U_j^{n\perp}(\xi)),$$

leaving $(\lambda_j, c_j^1, \dots, c_j^n)$ to be determined at the ϵ^{j+1}-order expansion. In order to continue the expansion past the determination of (7.17), it is necessary to assume that $\lambda_0 + 1$ is a semisimple eigenvalue of a certain operator. This will be described in a later paper. See [25], [16] for related work on reaction-diffusion systems.

PROPOSITION 7.5. *For* $\lambda_0 = 0$ *there is no nontrivial solution of* (7.13) *with auxiliary conditions* (7.14) *and* (7.16).

Proof. If $\lambda_0 = 0$, then from (7.13), U_1^R is constant in each regular sublayer R^i, $i = 1, \dots, n-1$. Then (7.14) and assumption (S1) imply that the only solution of the system (7.16) is $U_1^R(x) \equiv 0$ for $x \in R^i$, $i = 1, \dots, n-1$, and $c_0^i = 0$ for all i. $\square$

Let $V^i(x) = (Df(\bar{u}_0^i) - xI)U_1^R(x)$, $x \in R^i$ for $i = 0, \dots, n$. Let $s^i := (\lambda_0 + 1)c_0^i$ and $\Delta^i = \bar{u}_0^i - \bar{u}_0^{i-1}$ for $i = 1, \dots, n$. Each Δ^i is nonzero. Equations (7.13), (7.16), and (7.14) become

$$V_x^i + (\lambda_0 + 1)(Df(\bar{u}_0^i) - xI)^{-1} V^i = 0, \quad i = 1, \dots, n-1, \tag{7.18}$$

$$V^i(x_0^i) - V^{i-1}(x_0^i) = -s^i \Delta^i, \quad i = 1, \dots, n, \tag{7.19}$$

$$V^0(x) \equiv 0 \quad \text{and} \quad V^n(x) \equiv 0. \tag{7.20}$$

PROPOSITION 7.6. *For* $\lambda_0 \neq -1$*, there is a nontrivial solution* (7.17) *of* (7.13), (7.14), (7.16) *if and only if there is a nontrivial solution*

$$(s^1, \dots, s^n, V^1, \dots, V^{n-1})$$

of the system (7.18)–(7.20).

In contrast to the $O(\frac{1}{\epsilon})$ eigenvalues, which reflect the dynamics in a single internal layer, the $O(1)$ eigenvalues reflect the dynamics of the first-order linear ODE (7.18) in the regular layer. Equations (7.19) and (7.20) provide boundary and interface conditions for (7.18).

We remark that the system (7.18)–(7.20) is similar to the SLEP system introduced by Nishiura and Fujii [35] to study the stability of internal layer solutions of reaction-diffusion systems. We now derive the analogue of the SLEP matrix of Nishiura and Fujii.

Let $X(x, y, \lambda_0)$ be the principal matrix solution of (7.18). Although the differential equation (7.18) has jumps at x_0^i, $i = 1, \dots, n$, the principal matrix solution $X(x, y, \lambda_0)$ does not have jumps. If, for example, $y < x_0^j < x_0^{j+1} < \cdots < x_0^i < x$, then

$$X(x, y, \lambda_0) = X(x, x_0^i, \lambda_0) \cdot X(x_0^i, x_0^{i-1}, \lambda_0) \cdot \ \cdots \ \cdot X(x_0^j, y, \lambda_0).$$

If we integrate (7.18) from x_0^1- to x_0^n+ and use the jump conditions (7.19) and the initial and terminal conditions (7.20), we obtain

$$\sum_{j=1}^{n} X(x_0^n, x_0^j, \lambda_0) s^j \Delta^j = 0. \tag{7.21}$$

Let $\mathcal{M}(\lambda_0)$ be the $n \times n$ matrix whose jth column is the n-vector $X(x_0^n, x_0^j, \lambda_0)\Delta^j$, and let $\mathbf{s} = (s^1, \dots, s^n)$. The matrix $\mathcal{M}(\lambda_0)$ is the analogue of the SLEP matrix. Finding the lowest order expansion of slow eigenvalues is equivalent to finding solutions of

$$\mathcal{M}(\lambda_0)\mathbf{s} = 0. \tag{7.22}$$

Note that Proposition 7.5 implies that $\mathcal{M}(0)$ is nonsingular.

We shall consider the existence of slow eigenvalues λ_0 other than -1 and 0 in more detail only for the case $n = 2$. In this case system (7.18)–(7.20) becomes

$$V_x + (\lambda_0 + 1)(Df(\bar{u}_0^1) - xI)^{-1}V = 0, \quad x_0^1 \le x \le x_0^2, \tag{7.23}$$

$$V(x_0^1) = -s^1\Delta^1, \tag{7.24}$$

$$V(x_0^2) = -s^2\Delta^2. \tag{7.25}$$

Since (7.23) is linear and Δ^1 and Δ^2 are nonzero, the system (7.23)–(7.25) has a nontrivial solution if and only if the following boundary value problem has a solution:

$$V_x + (\lambda_0 + 1)(Df(\bar{u}_0^1) - xI)^{-1}V = 0, \quad x_0^1 \le x \le x_0^2, \tag{7.26}$$

$$V(x_0^1) = \Delta^1, \tag{7.27}$$

$$V(x_0^2) = \text{ a nonzero multiple of } \Delta^2. \tag{7.28}$$

Let the eigenvalues of $Df(\bar{u}_0^1)$ be $\nu_1 < \nu_2$, with corresponding eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Let

$$V(x) = \sum_{j=1}^{2} a_j(x)\mathbf{r}_j,$$

where $a_j(x)$ is a scalar function. The function $a_j(x)$ satisfies

$$a_j'(x) + \frac{\lambda_0 + 1}{\nu_j - x}a_j(x) = 0. \tag{7.29}$$

Therefore the subspaces of $\mathbb{R}^2$ spanned by $\mathbf{r}_1$ and $\mathbf{r}_2$ are invariant under (7.26).

PROPOSITION 7.7. *For $n = 2$, if Δ^1 or Δ^2 is a multiple of one of the $\mathbf{r}_j$, then there is no λ_0 such that the system* (7.26)*–*(7.28) *has a solution.*

Proof. Without loss of generality, suppose Δ^1 is a multiple of one of the $\mathbf{r}_j$. Then Δ^2 cannot be a multiple of the same $\mathbf{r}_j$, since it is easy to check that in the case $n = 2$, the Riemann solution $u_0(x)$ satisfies condition (S1) for structural stability if and only if Δ^1 and Δ^2 are linearly independent. Therefore, since the subspaces of $\mathbb{R}^2$ spanned by $\mathbf{r}_1$ and $\mathbf{r}_2$ are invariant under (7.26), the system (7.26)–(7.28) cannot have a solution. ∎

The case in which neither Δ^1 nor Δ^2 is a multiple of one of the $\mathbf{r}_j$ is covered by the following result.

PROPOSITION 7.8. *For $n = 2$, let*

$$\Delta^i = \sum_{j=1}^{2} d_j^i \mathbf{r}_j, \quad i = 1, 2,$$

with all d_j^i nonzero. Then there is a countably infinite set of λ_0 for which (7.26)*–*(7.28) *has a solution. All such λ_0 have the same real part and have nontrivial U_1^R (hence they are nonlocal). Explicit formulas for λ_0 are given in* (7.36).

Proof. The solution of the initial value problem (7.23), (7.24) is

$$a_j(x) = \left(\frac{x-\nu_j}{x_0^1-\nu_j}\right)^{\lambda_0+1} d_j^1, \quad j=1,2. \tag{7.30}$$

Notice that $x-\nu_j$ and $x_0^1-\nu_j$ have the same sign in the interval $x_0^1 \le x \le x_0^2$, so the number being raised to a power is positive. The function t^{λ_0+1} used in (7.30) is in general multivalued. Since we must have $a_j(x_0^1) = d_j^1$, $j=1,2$, the branch used must be the one for which $1^{\lambda_0+1}=1$.

The boundary condition (7.28) implies that

$$\det\begin{pmatrix} a_1(x) & d_1^2 \\ a_2(x) & d_2^2 \end{pmatrix} = 0 \quad \text{when } x = x_0^2, \tag{7.31}$$

which reduces to

$$\left(\frac{(x-\nu_1)(x_0^1-\nu_2)}{(x_0^1-\nu_1)(x-\nu_2)}\right)^{\lambda_0+1} = \frac{d_2^1 d_1^2}{d_1^1 d_2^2} \quad \text{when } x = x_0^2. \tag{7.32}$$

Again, the branch of t^{λ_0+1} used in (7.32) is the one for which $1^{\lambda_0+1}=1$. In fact, let us define a change of variables by

$$t = \frac{(x-\nu_1)(x_0^1-\nu_2)}{(x_0^1-\nu_1)(x-\nu_2)}, \quad x_0^1 \le x \le x_0^2. \tag{7.33}$$

Then t is an increasing function of x on the interval $x_0^1 \le x \le x_0^2$, and $t(x_0^1)=1$. Let

$$b = t(x_0^2) = \frac{(x_0^2-\nu_1)(x_0^1-\nu_2)}{(x_0^1-\nu_1)(x_0^2-\nu_2)} > 1, \quad d = \frac{d_2^1 d_1^2}{d_1^1 d_2^2} \neq 0. \tag{7.34}$$

Then (7.32) reduces to $b^{\lambda_0+1} = d$ or

$$(\lambda_0+1)\log b = \log d. \tag{7.35}$$

Let the main branch of logarithm for which $\log 1 = 0$ be denoted $\ln x$. We must use the main branch $\log b = \ln b$ in order to have $1^{\lambda_0+1}=1$ for all complex λ_0 . However, in calculating $\log d$, we may use any branch of the natural logarithm.

Since $b>1$ is real and d is real and nonzero, there are two cases.

1. $d>0$. Then $\log d = \ln d + 2n\pi i, \quad n \in \mathbb{Z}$.
2. $d<0$. Then $\log d = \ln|c| + (2n+1)\pi i, \quad n\in\mathbb{Z}$.

Substituting $\log d$ into (7.35), we find

$$\begin{aligned} \mathrm{Re}\lambda_0 &= -1 + \frac{\ln|d|}{\ln b} \quad \text{for } d \neq 0, \\ \mathrm{Im}\lambda_0 &= \begin{cases} \frac{2n\pi}{\ln b} & \text{if } d>0, \\ \frac{(2n+1)\pi}{\ln b} & \text{if } d<0. \end{cases} \end{aligned} \tag{7.36}$$

$\square$

Remark 7.1. With $n=2$, consider a Riemann solution that consists of two weak Lax shocks connecting the states $\bar{u}_0^1$, $\bar{u}_0^2$, and $\bar{u}_0^3$. For the corresponding Riemann–Dafermos solution, Proposition 7.8 implies that the nonlocal slow eigenvalues are stable. In fact, for $i=1,2$, $\bar{u}_0^i - \bar{u}_0^{i-1}$ is approximately parallel to $\mathbf{r}_i$. Therefore $|d_2^1| << |d_1^1|$ and $|d_1^2| << |d_2^2|$, so $|d| << 1$. Hence $\mathrm{Re}\lambda_0 < -1$.

8. Slow eigenvalues and inviscid stability conditions. Let us consider the inviscid system (1.3) and its Riemann solution (2.4). In studying the linearized stability of (2.4) as a solution of (1.3), one considers the following system [22]:

$$U_T + \begin{cases} Df(\bar{u}^0)U_X = 0 & \text{for } X < \bar{s}^1 T, \\ Df(\bar{u}^i)U_X = 0 & \text{for } \bar{s}^i T < X < \bar{s}^{i+1} T, \quad i = 1, \dots, n-1, \\ Df(\bar{u}^n)U_X = 0 & \text{for } \bar{s}^n T < X, \end{cases} \tag{8.1}$$

$$\begin{aligned}(Df(\bar{u}^i) - \bar{s}^i I)U(\bar{s}^i T+, T) - (Df(\bar{u}^{i-1}) - \bar{s}^i I)U(\bar{s}^i T-, T) \\ - S^i(T)(\bar{u}^i - \bar{u}^{i-1}) = 0, \quad i = 1, \dots, n,\end{aligned} \tag{8.2}$$

where

$$U(\bar{s}^i T+, T) = \lim_{X \to \bar{s}^i T+} U(X, T), \quad U(\bar{s}^i T-, T) = \lim_{X \to \bar{s}^i T-} U(X, T). \tag{8.3}$$

In each sector, the matrix $Df(\bar{u}^i)$ is constant, so solutions (which may include discontinuities) propagate along straight-line characteristics. Along the lines $X = \bar{s}^i T$, data arrive from both sides along incoming characteristics, and one uses (8.2) to solve for S^i and for the continuation of the solution along outgoing characteristics. Majda's stability condition—which is that for each $i = 1, \dots, n$, the eigenvectors for the largest $i-1$ eigenvalues at $\bar{u}^{i-1}$, the eigenvectors for the smallest $n-i$ eigenvalues at $\bar{u}^i$, and the vector $\bar{u}^i - \bar{u}^{i-1}$ should constitute a basis for $\mathbb{R}^n$—is just the condition upon which one can do this.

In (8.1) and (8.2), let us make the change of variables $x = \frac{X}{T}$, $t = \ln T$. We obtain

$$U_t + \begin{cases} (Df(\bar{u}^0) - xI)U_x = 0 & \text{for } x < \bar{s}^1, \\ (Df(\bar{u}^i) - xI)U_x = 0 & \text{for } \bar{s}^i < x < \bar{s}^{i+1}, \quad i = 1, \dots, n-1, \\ (Df(\bar{u}^n) - xI)U_x = 0 & \text{for } \bar{s}^n < x, \end{cases} \tag{8.4}$$

$$\begin{aligned}(Df(\bar{u}^i) - \bar{s}^i I)U(\bar{s}^i+, t) - (Df(\bar{u}^{i-1}) - \bar{s}^i I)U(\bar{s}^i-, t) \\ - S^i(t)(\bar{u}^i - \bar{u}^{i-1}) = 0, \quad i = 1, \dots, n,\end{aligned} \tag{8.5}$$

where

$$U(\bar{s}^i+, t) = \lim_{x \to \bar{s}^i+} U(x, t), \quad U(\bar{s}^i-, t) = \lim_{x \to \bar{s}^i-} U(x, t). \tag{8.6}$$

The characteristics are no longer straight lines, but the lines $X = \bar{s}^i T$ become $x = \bar{s}^i$, so it is reasonable to look for eigenvalues and eigenfunctions. A solution of (8.4), (8.5) of the form $U(x,t) = e^{\lambda t}U(x)$, $S^i(t) = e^{\lambda t}S^i$ satisfies

$$\lambda U + \begin{cases} (Df(\bar{u}^0) - xI)U_x = 0 & \text{for } x < \bar{s}^1, \\ (Df(\bar{u}^i) - xI)U_x = 0 & \text{for } \bar{s}^i < x < \bar{s}^{i+1}, \quad i = 1, \dots, n-1, \\ (Df(\bar{u}^n) - xI)U_x = 0 & \text{for } \bar{s}^n < x, \end{cases} \tag{8.7}$$

$$\begin{aligned}(Df(\bar{u}^i) - \bar{s}^i I)U(\bar{s}^i+) - (Df(\bar{u}^{i-1}) - \bar{s}^i I)U(\bar{s}^i-) \\ - S^i(\bar{u}^i - \bar{u}^{i-1}) = 0, \quad i = 1, \dots, n,\end{aligned} \tag{8.8}$$

where

$$U(\bar{s}^i+) = \lim_{x\to\bar{s}^i+} U(x), \quad U(\bar{s}^i-) = \lim_{x\to\bar{s}^i-} U(x). \tag{8.9}$$

If we add the conditions $U(x) = 0$ for $x < \bar{s}^1$ and $\bar{s}^n < x$, then (8.7)–(8.8) is equivalent to the system (7.13)–(7.14), (7.16) that was studied in section 7.

Assuming Majda's stability condition, one can interpret (8.1)–(8.2) or (8.4)–(8.5) as describing the scattering of incoming small shock waves by the large shock waves that comprise the original Riemann solution. Several authors have found sufficient conditions that guarantee that, in some norm, the total weight of the scattered shocks is smaller than the total weight of the incoming shocks [42], [4], [5], [49], [22], [21]. For the case $n = 2$, the BV stability condition reads as follows in the notation of section 7 [49], [21]. Recall that $x_0^i = \bar{s}^i$ and $\bar{u}_0^i = \bar{u}^i$. Let

$$(\nu_1 I - Df(\bar{u}^1))^{-1}(\bar{u}^1 - \bar{u}^0) = a_1^1\mathbf{r}_1 + a_2^1\mathbf{r}_2, \tag{8.10}$$

$$(Df(\bar{u}^1) - \nu_2 I)^{-1}(\bar{u}^2 - \bar{u}^1) = a_1^2\mathbf{r}_1 + a_2^2\mathbf{r}_2. \tag{8.11}$$

Then

$$\left|\frac{a_1^2 a_2^1}{a_1^1 a_2^2}\right| < 1. \tag{8.12}$$

As in section 7, for $i = 1, 2$ let $\Delta^i = \bar{u}^i - \bar{u}^{i-1} = d_1^i\mathbf{r}_1 + d_2^i\mathbf{r}_2$, and define b and d by (7.34). Elementary computations show that

$$\frac{a_1^2 a_2^1}{a_1^1 a_2^2} = \frac{d_1^2 d_2^1(\bar{s}^1 - \nu_1)(\nu_2 - \bar{s}^2)}{d_1^1 d_2^2(\bar{s}^1 - \nu_2)(\nu_1 - \bar{s}^2)} = \frac{d}{b}, \tag{8.13}$$

and, since $b > 1$,

$$\frac{|d|}{b} < 1 \text{ if and only if } -1 + \frac{\ln|d|}{\ln b} < 0. \tag{8.14}$$

Thus the $n = 2$ BV inviscid stability condition holds if and only if all slow eigenvalues have negative real part.

9. Two Lax shocks in the *p*-system: An example. We consider the p-system

$$u_t - v_x = 0,$$

$$v_t + p(u)_x = 0,$$

with p a smooth function, $p'(u) < 0$ for all u, and $p''(u) \neq 0$ for all u.

The p-system has been used as a model for isentropic gas dynamics with $p(u) = ku^{-\gamma}$, $k > 0$, $\gamma \geq 1$ [37], [43]. The p-system is strictly hyperbolic with eigenvalues and eigenvectors

$$\nu_1(u,v) = -\sqrt{-p'(u)} < 0, \qquad \mathbf{r}_1(u,v) = (1, \sqrt{-p'(u)}),$$

$$\nu_2(u,v) = \sqrt{-p'(u)} > 0, \qquad \mathbf{r}_2(u,v) = (1, -\sqrt{-p'(u)}).$$

Consider a Riemann solution $(u_0, v_0)(x)$ that consists of two Lax shocks:

$$(u_0, v_0)(x) = \begin{cases} (\bar{u}^0, \bar{v}^0) & \text{for } x < \bar{s}^1, \\ (\bar{u}^1, \bar{v}^1) & \text{for } \bar{s}^1 < x < \bar{s}^2, \\ (\bar{u}^2, \bar{v}^2) & \text{for } \bar{s}^2 < x. \end{cases}$$

THEOREM 9.1. *To lowest order in ϵ, the corresponding Riemann–Dafermos solution has exactly the following slow eigenvalues:* (1) *a local eigenvalue with $\lambda_0 = -1$;* (2) *a family of nonlocal eigenvalues with $\lambda_0 = -2 + n\omega_0 i$, $n \in \mathbb{Z}$, $\omega_0 > 0$.*

Proof. We fix the middle state $(\bar{u}^1, \bar{v}^1)$ and look for (u, v) and s such that the Rankine–Hugoniot condition

$$-(\bar{v}^1 - v) - s(\bar{u}^1 - u) = 0, \quad p(\bar{u}^1) - p(u) - s(\bar{v}^1 - v) = 0$$

is satisfied. The solution set is two curves: Γ_1 given by

$$v = \phi(u) = \bar{v}^1 - \operatorname{sgn}(u - \bar{u}^1)\sqrt{(u - \bar{u}^1)(p(\bar{u}^1) - p(u))},$$

$$s = s^1(u) = -\sqrt{\frac{p(\bar{u}^1) - p(u)}{u - \bar{u}^1}},$$

and Γ_2 given by

$$v = \psi(u) = \bar{v}^1 + \operatorname{sgn}(u - \bar{u}^1)\sqrt{(u - \bar{u}^1)(p(\bar{u}^1) - p(u))},$$

$$s = s^2(u) = \sqrt{\frac{p(\bar{u}^1) - p(u)}{u - \bar{u}^1}}.$$

Γ_1 is a curve of 1-shocks, Γ_2 a curve of 2-shocks. Using Lax's condition for an i-shock, we easily check the following:

(1) If $(u, v, s^1) \in \Gamma_1$, then there is a 1-shock *from* (u, v) *to* $(\bar{u}^1, \bar{v}^1)$ with speed s^1 if and only if $u - \bar{u}^1 > 0$.

(2) If $(u, v, s^2) \in \Gamma_2$, then there is a 2-shock *from* $(\bar{u}^1, \bar{v}^1)$ *to* (u, v) with speed s^2 if and only if $u - \bar{u}^1 > 0$.

Therefore we have, in the notation of section 7,

$$\begin{aligned}
\Delta^1 &= (\bar{u}^1 - \bar{u}^0, \bar{v}^1 - \bar{v}^0) = (\bar{u}^1 - \bar{u}^0, \bar{v}^1 - \phi(\bar{u}^0)), \quad \bar{u}^0 - \bar{u}^1 > 0,\\
\Delta^2 &= (\bar{u}^2 - \bar{u}^1, \bar{v}^2 - \bar{v}^1) = (\bar{u}^2 - \bar{u}^1, \psi(\bar{u}^2) - \bar{u}^1), \quad \bar{u}^2 - \bar{u}^1 > 0.
\end{aligned}$$

Let

$$q(u) = \sqrt{\frac{(u - \bar{u}^1)(p(\bar{u}^1) - p(u))}{-p'(\bar{u}^1)}}.$$

Then

$$\Delta^i = \sum_{j=1}^{2} d_j^i \mathbf{r}_j, \quad i = 1, 2,$$

with

$$d_1^1 = \frac{1}{2}(-(\bar{u}^0 - \bar{u}^1) - q(\bar{u}^0)), \quad d_2^1 = \frac{1}{2}(-(\bar{u}^0 - \bar{u}^1) + q(\bar{u}^0)),$$

$$d_1^2 = \frac{1}{2}(\bar{u}^2 - \bar{u}^1 - q(\bar{u}^2)), \quad d_2^2 = \frac{1}{2}(\bar{u}^2 - \bar{u}^1 + q(\bar{u}^2)).$$

Therefore

$$d = \frac{d_2^1 d_1^2}{d_1^1 d_2^2} = \frac{(\bar{u}^0 - \bar{u}^1 - q(\bar{u}^0))(\bar{u}^2 - \bar{u}^1 - q(\bar{u}^2))}{(\bar{u}^0 - \bar{u}^1 + q(\bar{u}^0))(\bar{u}^2 - \bar{u}^1 + q(\bar{u}^2))}.$$

By Lemma 9.2 below, the numerator of this fraction is positive. Therefore $d > 0$.

Let $\nu_i = \nu_i(\bar{u}^1, \bar{v}^1)$, $i = 1, 2$. Then

$$b = \frac{(\bar{s}^2 - \nu_1)(\bar{s}^1 - \nu_2)}{(\bar{s}^1 - \nu_1)(\bar{s}^2 - \nu_2)} > 1.$$

An easy computation shows that $b = \frac{1}{d}$. The result now follows from Proposition 7.8. ∎

LEMMA 9.2. *For $u > \bar{u}^1$, the sign of $u - \bar{u}^1 - q(u)$ is independent of u.*

Proof. We shall assume $p''(u) > 0$ for all u. The case $p''(u) < 0$ for all u is similar. Let $u > \bar{u}^1$. Since $p'' > 0$ everywhere,

$$p'(\bar{u}^1) < \frac{p(\bar{u}^1) - p(u)}{\bar{u}^1 - u}.$$

Therefore

$$(u - \bar{u}^1)^2 > \frac{(u - \bar{u}^1)(p(\bar{u}^1) - p(u))}{-p'(\bar{u}^1)},$$

so $u - \bar{u}^1 > q(u)$. ∎

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