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Classification of Codimension-One Riemann Solutions

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We investigate solutions of Riemann problems for systems of two conservation laws in one spatial dimension. Our approach is to organize Riemann solutions into strata of successively higher codimension. The codimension-zero stratum consists of Riemann solutions that are structurally stable: the number and types of waves in a solution are preserved under small perturbations of the flux function and initial data. Codimension-one Riemann solutions, which constitute most of the boundary of the codimension-zero stratum, violate structural stability in a minimal way. At the codimension-one stratum, either the qualitative structure of Riemann solutions changes or solutions fail to be parameterized smoothly by the flux function and the initial data. In this paper, we give an overview of the phenomena associated with codimension-one Riemann solutions. We list the different kinds of codimension-one solutions, and we classify them according to their geometric properties, their roles in solving Riemann problems, and their relationships to wave curves.

KEY WORDS: Conservation law; Riemann problem; viscous profile.
1991 Mathematics Subject Classifications: 35L65, 35L67, 34D30.

1. INTRODUCTION

1.1. Riemann Solutions

We consider systems of two conservation laws in one space dimension, partial differential equations of the form

$$U_t + F(U)_x = 0 \tag{1.1}$$

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with $t>0$, $x \in \mathbb{R}$, $U(x, t) \in \mathbb{R}^2$, and $F: \mathbb{R}^2 \to \mathbb{R}^2$ a smooth map. The most basic initial-value problem for Eq. (1.1) is the *Riemann problem*, in which the initial data are piecewise constant with a single jump at $x=0$:

$$U(x, 0) = \begin{cases} U_L & \text{for} \quad x<0 \\ U_R & \text{for} \quad x>0 \end{cases} \tag{1.2}$$

This paper is the second in a series in which we study the structure of solutions of Riemann problems.

We seek piecewise continuous weak solutions of Riemann problems in the scale-invariant form $U(x, t) = \hat{U}(x/t)$ consisting of a finite number of constant parts, continuously changing parts (*rarefaction waves*), and jump discontinuities (*shock waves*). Shock waves occur when

$$\lim_{\xi \to s-} \hat{U}(\xi) = U_- \neq U_+ = \lim_{\xi \to s+} \hat{U}(\xi) \tag{1.3}$$

They are required to satisfy the following *viscous profile admissibility criterion*: a shock wave is admissible provided that the ordinary differential equation

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \tag{1.4}$$

has a heteroclinic solution, or a finite sequence of such solutions, leading from the equilibrium U_- to a second equilibrium U_+.

By the term *Riemann solution* for Eqs. (1.1) and (1.2) we mean a weak solution U of this kind [or, equivalently, the scale-invariant function $\hat{U}$, or the sequence of waves in U, or the quadruple $(\hat{U}, U_L, U_R, F)$]. There are various *types* of rarefaction and shock waves (e.g., 1-family rarefaction waves and transitional shock waves); the *type* of a Riemann solution is the sequence of types of its waves.

Riemann solutions have been studied by many authors. For systems that are strictly hyperbolic and genuinely nonlinear, local solutions were found by Lax [10], and global solutions were obtained, for a certain class of systems, by Smoller [20]. This work was extended to allow for loss of genuine nonlinearity by Wendroff [23] for gas dynamics and by Liu [11] for a broad class of systems. Many examples of systems that fail to be strictly hyperbolic have been analyzed over the last two decades; see, e.g., Refs. 5, 6, 15, and 18. A common feature of this analysis is the construction of wave curves, one-parameter families of Riemann solutions. Special assumptions about the system of conservation laws lead to wave curves with simple geometry and permit the formulation of simple, but effective, admissibility criteria for shock waves. For general systems, however, the wave curve geometry can be exceedingly complicated and a more fundamental admissibility criterion, such as the viscous profile admissibility criterion, is needed.

Our approach to understanding Riemann solutions is to investigate the local structure of the set of Riemann solutions: we consider a particular solution $(\hat{U}^*, U_L^*, U_R^*, F^*)$ and construct nearby ones. More precisely, we define an open neighborhood $\mathscr{X}$ of $\hat{U}^*$ in a Banach space of scale-invariant functions $\hat{U}$, open neighborhoods $\mathscr{U}_L$ and $\mathscr{U}_R$ of U_L^* and U_R^* in $\mathbb{R}^2$, respectively, and an open neighborhood $\mathscr{B}$ of F^* in a Banach space of smooth flux functions F. Then our goal is to construct a set $\mathscr{R}$ of Riemann solutions $(\hat{U}, U_L, U_R, F) \in \mathscr{X} \times \mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$ near $(\hat{U}^*, U_L^*, U_R^*, F^*)$. To guide this construction, we view $\mathscr{R}$ as organized into strata of successively higher codimension.

The largest stratum of $\mathscr{R}$, which has codimension zero within $\mathscr{R}$, consists of structurally stable Riemann solutions. For such solutions, $\hat{U}$ changes continuously, and its type remains unchanged, when (U_L, U_R, F) varies in certain open subsets of $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$. Moreover, the left and right states and speeds of each wave in $\hat{U}$ depend smoothly on (U_L, U_R, F). In contrast, at a stratum of higher codimension, either Riemann solutions are degenerate in some way or the parameterization of waves by U_L, U_R, and F loses smoothness. For example, solutions can change type upon crossing the stratum, or the parameterization can have a fold.

Structurally stable Riemann solutions were studied in the first paper [17] in the present series. In this second paper, we begin the study of codimension-one Riemann solutions, for which structural stability fails in a minimal fashion.

1.2. Structurally Stable Riemann Solutions

A quadruple $(\hat{U}^*, U_L^*, U_R^*, F^*)$ represents a *structurally stable Riemann solution* if for each (U_L, U_R, F) near (U_L^*, U_R^*, F^*), there is a Riemann solution $\hat{U}$ near $\hat{U}^*$ such that (1) $\hat{U}$ has the same type as $\hat{U}^*$ and (2) the left and right states and speeds of each wave in $\hat{U}$ depend smoothly on (U_L, U_R, F). In particular, we obtain a set $\mathscr{R}$ of Riemann solutions, represented as a graph of a function from an open subset of $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$ to $\mathscr{X}$.

In Ref. 17, we identified a set of sufficient conditions for structural stability of strictly hyperbolic Riemann solutions. Briefly, these conditions have the following character.

(H0) There is a restriction on the sequence of wave types in the solution.

(H1) Each wave satisfies certain nondegeneracy conditions.

(H2) The "wave group interaction condition" is satisfied. In the simplest case, the forward wave curve and the backward wave curve are transverse.

(H3) If a shock wave represented by a connection *to* a saddle is followed by another represented by a connection *from* a saddle, the shock speeds differ.

The methods by which these conditions were derived strongly suggest that they are also necessary for structural stability.

1.3. Codimension-One Riemann Solutions

In this paper, we begin an investigation of strictly hyperbolic Riemann solutions that lie in the boundary of the set of structurally stable Riemann solutions in that they violate precisely one of the structural stability conditions (H0)–(H3). Under appropriate nondegeneracy conditions, these Riemann solutions constitute a graph over a codimension-one submanifold of $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$.

A point $(\hat{U}^*, U_L^*, U_R^*, F^*)$ represents a *codimension-one Riemann solution* if there exists a codimension-one submanifold $\mathscr{S}$ of $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$ with the following properties. For each point $(U_L, U_R, F) \in \mathscr{S}$ near (U_L^*, U_R^*, F^*), there is a structurally unstable Riemann solution $\hat{U}$ near $\hat{U}^*$ such that (1) $\hat{U}$ has the same type as $\hat{U}^*$ and (2) the endpoints and speeds of each wave in $\hat{U}$ depend smoothly on (U_L, U_R, F). Furthermore, (3) $\mathscr{S}$ bounds a region in $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$ that corresponds to structurally stable solutions. In particular, we obtain a set $\mathscr{R}$ of Riemann solutions as a graph of a function from a manifold-with-boundary in $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$ to $\mathscr{X}$. Finally, (4) $\mathscr{S}$ is situated with a certain regularity in $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$: either $\mathscr{S}$ is in general position relative to planes of constant (U_L, F) and planes of constant (U_R, F), so that (U_L, F) and (U_R, F) both serve as good coordinates for $\mathscr{S}$, or $\mathscr{S}$ is a cylinder over a hypersurface in (U_L, F)-, (U_R, F)-, or F-space.

In most cases, a codimension-one submanifold $\mathscr{S}$ can be regarded as bounding structurally stable Riemann solutions not only on one side but also on the other side. Usually the number and types of waves in a structurally stable Riemann solution change at a codimension-one boundary; as a result, there is a change in the number and form of equations that define solutions. To accommodate such changes, we use several technical devices; for instance, we allow shock and rarefaction waves to have zero-strength, and we sometimes represent a Riemann solution in different, but equivalent, ways.

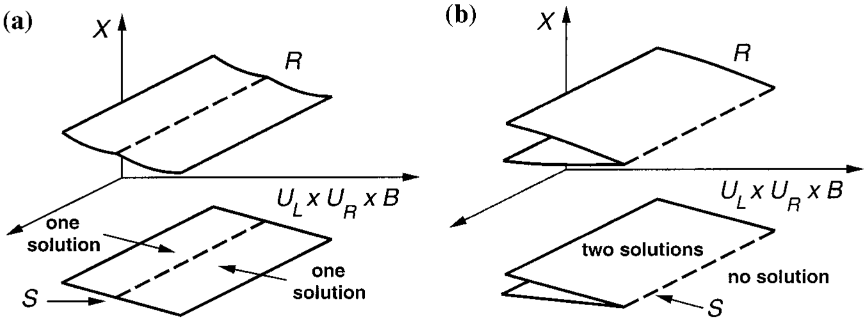

Fig. 1. Riemann solution joins.

1.4. Classification of Codimension-One Riemann Solutions

Codimension-one Riemann solutions can be classified in several ways.

A. They can be classified with respect to the structure of $\mathscr{R}$. The following possibilities arise.

1. *Joins:* $\mathscr{R}$ is formed from two manifolds-with-boundary joined along their common boundary. As the boundary is crossed, a structurally stable Riemann solution becomes degenerate and then turns into a structurally stable solution of a different type. See Figs. 1a and b for schematic illustrations.
2. *Folds:* $\mathscr{R}$ is a manifold homeomorphic to $\mathbb{R}^4 \times \mathscr{B}$, and there is no change in type of the Riemann solution upon crossing the codimension-one submanifold, but there is a fold in the projection to $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$. Thus $\mathscr{R}$ fails to be a graph over (U_L, U_R, F)-space, as illustrated in Fig. 2a.

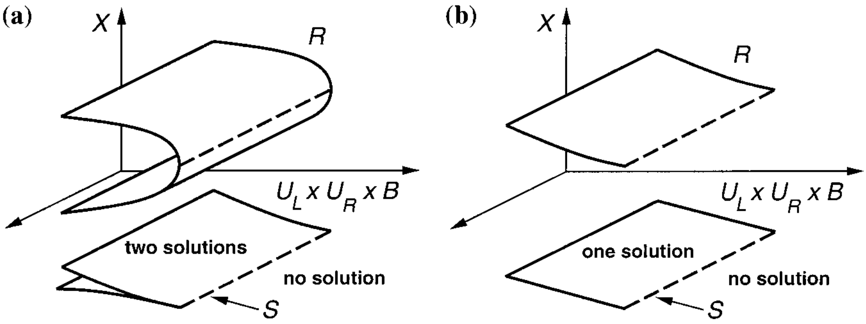

Fig. 2. Riemann solution folds and frontiers.

3. *Frontiers:* $\mathscr{R}$ is a manifold-with-boundary homeomorphic to $\mathbb{R}^3 \times \mathbb{R}^+ \times \mathscr{B}$. Riemann solutions exist only on one side of the codimension-one submanifold; see Fig. 2b.

The reader will notice that, in each case, the Riemann solution set $\mathscr{R}$ arising naturally in our construction is either a manifold or a manifold-with-boundary. This is in contrast to the situation when one uses the Lax admissibility criterion for shock waves [10], where branching can occur [2]. We emphasize, however, that it is possible for two such sets, $\mathscr{R}_1$ and $\mathscr{R}_2$, to intersect within $\mathscr{X} \times \mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$; this happens when the solution $\hat{U}^*$ contains a shock wave that has two distinct viscous profiles.

B. The codimension-one Riemann solutions can also be classified with respect to how $\mathscr{S}$ is situated in $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$. In our experience, either $\mathscr{S}$ has regular projections onto coordinate planes or $\mathscr{S}$ is a cylinder; when the projection of $\mathscr{S}$ onto a coordinate plane has a fold, for example, we believe that $\hat{U}^*$ should be regarded as having higher codimension. For this reason, our definition of codimension-one Riemann solution requires $\mathscr{S}$ to be one of the following types of boundaries.

1. *Intermediate boundary:* The submanifold $\mathscr{S}$ is transverse to both of the two-dimensional planes $\{(U_L, U_R, F) : (U_L, F) = (U_L^*, F^*)\}$ and $\{(U_L, U_R, F) : (U_R, F) = (U_R^*, F^*)\}$. Thus for each (U_L, F) near (U_L^*, F^*), $\mathscr{S}$ meets the corresponding copy of the U_R-plane in a curve; and for each (U_R, F) near (U_R^*, F^*), $\mathscr{S}$ meets the corresponding copy of the U_L-plane in a curve. In other words, if U_L and F are fixed, codimension-one Riemann solutions correspond to a curve in the U_R-plane; and if U_R and F are fixed, they correspond to a curve in the U_L-plane.
2. *U_L-boundary:* There is a codimension-one submanifold $\tilde{\mathscr{S}}$ in (U_L, F)-space, transverse to the two-dimensional plane $\{(U_L, F): F = F^*\}$, such that $(U_L, U_R, F) \in \mathscr{S}$ if and only if $(U_L, F) \in \tilde{\mathscr{S}}$. Thus for each F near F^* there is a curve $\mathscr{C}(F)$ in the U_L-plane such that $(U_L, U_R, F) \in \mathscr{S}$ if and only if $U_L \in \mathscr{C}(F)$. That is, for a specific system of conservation laws, codimension-one Riemann solutions occur when U_L lies on a fixed curve. Another type of boundary is obtained through duality by reversing the roles of U_L and U_R in this definition.
3. *F-boundary:* There is a codimension-one submanifold $\tilde{\mathscr{S}}$ in $\mathscr{B}$ such that $(U_L, U_R, F) \in \mathscr{S}$ if and only if $F \in \tilde{\mathscr{S}}$.

Remark. What we call an "intermediate boundary" is called a "U_R-boundary" in Ref. 15. We use a different terminology to avoid confusion between intermediate boundaries and duals of U_L-boundaries.

These boundaries are useful in describing the solutions of Riemann problems [2, 4, 5, 14, 19]. In solving Riemann problems for a flux function that is not on an F-boundary, the first step is to locate the U_L-boundaries, which divide the U_L-plane into several open regions. In the second step, for a representative choice of U_L in each region, the intermediate boundaries are located; these curves divide the U_R-plane into open regions. Finally, Riemann problems are solved for a representative choice of U_R in each U_R-region. The qualitative structure of Riemann solutions is the same for all U_L in a U_L-region and U_R in a U_R-region. Examples of U_L-boundaries include the inflection, double sonic, and secondary bifurcation loci (see Ref. 2 for a comprehensive list in the context of the Lax admissibility criterion); examples of intermediate boundaries include the rarefaction and shock curves drawn through U_L.

Codimension-one F-boundaries seem to be new. To our knowledge, they do not occur in the systems of conservation laws that have been investigated so far.

C. The codimension-one Riemann solutions can be classified with respect to the number of solutions of nearby Riemann problems. In the case of a fold, for data (U_L, U_R, F) on one side of $\mathscr{S}$, there are two nearby structurally stable Riemann solutions; for data in $\mathscr{S}$, there is a locally unique codimension-one solution; and for data on the other side of $\mathscr{S}$, there is no nearby Riemann solution. This case is depicted schematically in Fig. 2a. The same situation can occur along some of the Riemann solution joins. In classical examples, the two manifolds-with-boundary, which meet along their common boundary, project to different sides of $\mathscr{S}$, so that there is local existence and uniqueness of Riemann solutions, as in Fig. 1a. It is possible, however, for the two manifolds-with-boundary to project to the same side of $\mathscr{S}$, so that for nearby data there are two, one, or zero nearby Riemann solutions, as in the case of a fold; see Fig. 1b. For a frontier, there is a locally unique solution on $\mathscr{S}$ and on one side of $\mathscr{S}$, but no solution on the other side, as illustrated in Fig. 2b.

1.5. Overview of the Paper

Several simplifying assumptions are made in the current paper. First, the differential equation (1.4) used to determine the admissibility of shock waves has a special form. [More generally, the left-hand side could be replaced by $D(U)\,\dot{U}$, where $D(U)$ is called the viscosity matrix.] Second, we consider only Riemann solutions that are strictly hyperbolic, in that all states $\hat{U}(\xi)$ in the weak solution lie inside the region where characteristic speeds are distinct; however, we do not require viscous profiles for shock waves to lie entirely within this region. Third, we exclude shock waves with viscous profiles that form homoclinic orbits. These simplifying assumptions are adopted to reduce the number of cases to be studied. Notice that certain waves and wave configurations are excluded by these assumptions; examples include transitional rarefaction waves and shock waves with saddle-to-spiral or homoclinic connections.

Given these assumptions, our goal is to list and classify codimension-one Riemann solutions. Specifically, we do the following.

1. We give a precise definition of codimension-one Riemann solutions.

2. We list the ways in which hypotheses (H0)–(H3) can be violated on the boundary of the set of structurally stable, strictly hyperbolic Riemann solutions. Violations of hypothesis (H0) can be identified with violations of hypothesis (H1). Also, we can amalgamate all violations of hypothesis (H2) into a single case. In order to reduce the number of violations of hypotheses (H1) and (H3) that must be considered, we amalgamate those cases that are analogous under a duality between slow and fast waves.

3. We note that certain violations of hypothesis (H1) lead to failure of strict hyperbolicity, and we argue that others have codimension higher than one. We discard these from further consideration.

4. There are 63 remaining violations of hypotheses (H0)–(H3). It is expected that each of them gives rise, under appropriate nondegeneracy conditions, to codimension-one Riemann solutions. Indeed, many occur in the literature. For some of these degeneracies, we mention one or two of the more obvious nondegeneracy conditions that are required. However, precise statements of the required nondegeneracy conditions in each case are left to later papers.

To prove rigorously that each of the 63 violations of hypotheses (H0)–(H3) gives rise to codimension-one Riemann solutions, one must check, in each case, that under the appropriate nondegeneracy conditions, certain matrices of partial derivatives of an appropriate mapping are invertible. In this paper we give the mappings, but we do not make any of the checks. The knowledgeable reader will realize that in may cases the necessary computations are well-known, or are at least present in the literature. We plan to provide the necessary computations for at least some of the cases in later papers.

5. We classify the 63 types of codimension-one Riemann solutions according to how they are situated in $\mathscr{R}$. (Again, proofs are omitted.) Four are folds; 5 are frontiers; and 54 form 27 pairs of related degeneracies that give rise to 27 joins.

The five frontier cases involve overcompressive waves [15]. Except for these cases, whenever one arrives at a strictly hyperbolic codimension-one boundary of a set of structurally stable Riemann solutions of a type identified in Ref. 17, another set of structurally stable Riemann solutions of a type identified in Ref. 17 lies on the other side. We regard this as further indication that the list of types of structurally stable, strictly hyperbolic Riemann solutions in Ref. 17 is complete.

6. We classify the codimension-one manifolds of unstable solutions with respect to how $\mathscr{S}$ is situated in $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$. In most cases, the type of violation of (H0)–(H3) does not determine whether it is an intermediate boundary, a U_L-boundary or dual, or an F-boundary; one must also know where in the wave sequence the violation occurs. Again, we do not give proofs.

7. We explain how the results of this paper are related to wave curves. Many of the Riemann solution degeneracies we describe can be understood in terms of "junction points" in wave curves. The results of this paper explain how the complexity of wave curves increases and decreases as they are extended.

The structure of solutions of Riemann problems is very rich, and the list of codimension-one bifurcations of this structure is long and complicated. The present paper only formulates this list. Supplying proofs for each case constitutes a rather large program, to which we plan to contribute in later papers. We hope to have compensated for the lack of proofs in the current paper by presenting a clear overview of the range of bifurcation phenomena.

The remainder of the paper is organized as follows. In Section 2 we review terminology and results about structurally stable Riemann solutions from Ref. 17. In Section 3 we introduce some new terminology needed for our treatment of codimension-one Riemann solutions, and provide a definition of such solutions. In Section 4 we consider the possible violations of hypotheses (H0)–(H3), discard those that give rise to failure of strict hyperbolicity or to phenomena of codimension greater than one, and give appropriate mappings for the remaining codimension-one phenomena. In Sections 5–7 we carry out steps 5–7 described above. The proof of one lemma is given in Appendix A, and details of several cases are relegated to Appendix B.

2. STRUCTURALLY STABLE, STRICTLY HYPERBOLIC RIEMANN SOLUTIONS

In this section we review standard terminology concerning conservation laws [21] along with results about structurally stable Riemann solutions from Ref. 17.

We consider the system (1.1) with $t > 0$, $x \in \mathbb{R}$, $U(x, t) \in \mathbb{R}^2$, and $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ a C^2 map. Let

$$\mathscr{U}_F = \{ U \in \mathbb{R}^2 : DF(U) \text{ has distinct real eigenvalues} \} \tag{2.1}$$

be the *strictly hyperbolic region* in state-space. We call a Riemann solution $\hat{U}$ *strictly hyperbolic* if $\hat{U}(\xi) \in \mathscr{U}_F$ for all $\xi \in \mathbb{R}$. In this paper, all Riemann solutions are assumed to be strictly hyperbolic. (Notice, however, that we do not require that viscous profiles for shock waves lie entirely within the strictly hyperbolic region.)

In the following we fix an open set $\mathscr{U} \subseteq \mathbb{R}^2$ and an open set $\mathscr{B}$ in a Banach space of smooth (i.e., C^2) flux functions. These sets depend on the Riemann solution whose stability we are investigating; they are specified in Section 2.2 below. For the discussion in the next subsection, it suffices to assume two properties of $\mathscr{U}$ and $\mathscr{B}$. First, $\mathscr{U} \subseteq \mathscr{U}_F$ for all $F \in \mathscr{B}$. Second, there exists a closed, bounded interval $I \subseteq \mathbb{R}$ such that the eigenvalues of $DF(U)$ belong to I for all $U \in \mathscr{U}$ and $F \in \mathscr{B}$. In this case, if $\hat{U}$ is a scale-invariant solution of the Riemann problem (1.1)–(1.2), then $\hat{U}(\xi) = U_L$ for $\xi \leqslant \min I$ and $\hat{U}(\xi) = U_R$ for $\xi \geqslant \max I$. Therefore we can regard scale-invariant solutions as belonging to the open set

$$\mathscr{X} := \{ \hat{U}\colon I \to \mathscr{U} \mid \hat{U} \in L^1(I, \mathbb{R}^2) \} \tag{2.2}$$

in the Banach space $L^1(I, \mathbb{R}^2)$.

2.1. Elementary Waves

Riemann solutions are composed of elementary waves. The definitions of elementary waves given in this section are not the most general, but they suffice in the context of strictly hyperbolic structurally stable Riemann solutions. In Section 3 we adopt more general definitions.

For $F \in \mathscr{B}$ and $U \in \mathscr{U}$, let $\lambda_1(U) < \lambda_2(U)$ denote the eigenvalues of $DF(U)$. Also let $\ell_i(U)$ and $r_i(U)$, $i = 1, 2$, denote corresponding left and right eigenvectors, normalized so that $\ell_i(U)\, r_j(U) = \delta_{ij}$. For suitable neighborhoods $\mathscr{N} \subseteq \mathscr{U}$ (described below), we can choose these eigenvectors to depend smoothly on $U \in \mathscr{N}$. More generally, with each eigenvalue family, $i = 1, 2$, is associated the smooth line field of null directions for $DF(U) - \lambda_i(U)\, I$.

A *rarefaction wave* of *type* R_i is a differentiable map $\hat{U}\colon [\alpha, \beta] \to \mathscr{U}$, where $\alpha < \beta$, such that $\hat{U}'(\xi)$ is an eigenvector of $DF(\hat{U}(\xi))$ with eigenvalue $\xi = \lambda_i(\hat{U}(\xi))$ for each $\xi \in [\alpha, \beta]$. The states $U = \hat{U}(\xi)$ with $\xi \in [\alpha, \beta]$

constitute the *rarefaction curve* $\bar{\Gamma}$. This definition implies that if $U = \hat{U}(\xi) \in \bar{\Gamma}$, then since $D\lambda_i(U)\,\hat{U}'(\xi) = 1$,

$$\ell_i(U)\,D^2F(U)(r_i(U), r_i(U)) = D\lambda_i(U)\,r_i(U) \neq 0 \tag{2.3}$$

Condition (2.3) is *genuine nonlinearity* of characteristic family i at U. The definition also implies that $\hat{U}$ is actually C^1 and that $\lambda_i(U_-) < \lambda_i(U_+)$, where $U_- = \hat{U}(\alpha)$ and $U_+ = \hat{U}(\beta)$ are the *left* and *right states* of the rarefaction wave, respectively. We will find it convenient to associate a specific *speed* s to a rarefaction wave: for a rarefaction wave of type R_1, $s = \lambda_1(U_+)$; for a rarefaction wave of type R_2, $s = \lambda_2(U_-)$.

A *shock wave* consists of a *left state* $U_- \in \mathscr{U}$, a *right state* $U_+ \in \mathscr{U}$ (with $U_+ \neq U_-$), a *speed* s, and a *connecting orbit* $\Gamma \subset \mathbb{R}^2$, i.e., an orbit of the ordinary differential equation (1.4) leading from the equilibrium U_- to the equilibrium U_+. In particular, the speed and the left and right states of a shock wave are related by the Rankine–Hugoniot condition

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0 \tag{2.4}$$

which states that U_+ is an equilibrium for Eq. (1.4). The orbit Γ is the range of some solution $\tilde{U}$ of Eq. (1.4) such that $\lim_{\eta \to \pm\infty} \tilde{U}(\eta) = U_\pm$. Corresponding to $\tilde{U}$ are traveling wave solutions $U(x, t) = \tilde{U}((x - x_0 - st)/\varepsilon)$ of the parabolic equation

$$U_t + F(U)_x = \varepsilon U_{xx} \tag{2.5}$$

each traveling wave tends to the shock wave as $\varepsilon \to 0$.

For any equilibrium $U \in \mathscr{U}$ of Eq. (1.4), the eigenvalues of the linearization of Eq. (1.4) at U are $\lambda_i(U) - s$, $i = 1, 2$. We use the terminology defined in Table I for such an equilibrium.

Our name for an equilibrium accounts only for the signs of the eigenvalues; it does not necessarily reflect the topological type of the phase

Table I. Types of Equilibria

Name	Symbol	Eigenvalues	
Repeller	R	+	+
Repeller-saddle	RS	0	+
Saddle	S	−	+
Saddle-attractor	SA	−	0
Attractor	A	−	−

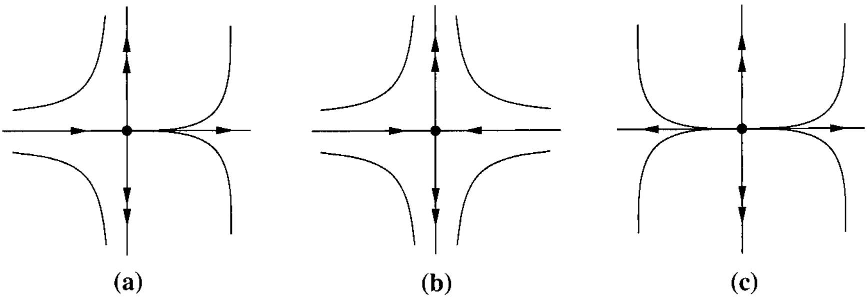

Fig. 3. Possible phase portraits for a repeller-saddle. The differential equation on the center manifold is $\dot{\sigma} = b\sigma^k + \ldots$, where $k \geqslant 2$. In case (a), k is even and $b \neq 0$; in case (b), k is odd and $b < 0$; in case (c), k is odd and $b > 0$.

portrait if there is a zero eigenvalue. Figures 3 and 4 show the possible phase portraits for repeller-saddles and saddle-attractors.

If w is a shock wave, its *type* is determined by the equilibrium types of its left and right states. (For example, w is of type $R \cdot S$ if its left state is a repeller and its right state is a saddle.) These types are listed in Figs. 5–8. (In these figures, the phase portraits are drawn for the case that repeller-saddles and saddle-attractors are nondegenerate, i.e., $k = 2$ in Figs. 3 and 4.) Shock types are grouped into four sets of four: *slow*, *fast*, *overcompressive*, and *transitional* shock waves. Slow and fast shock waves are called *classical shock waves.*

Remark. It is helpful, when thinking about shock waves involving saddle-nodes, to regard an RS equilibrium as a saddle S on the left and a repeller R on the right, as in Fig. 3a. Similarly, an SA equilibrium is an

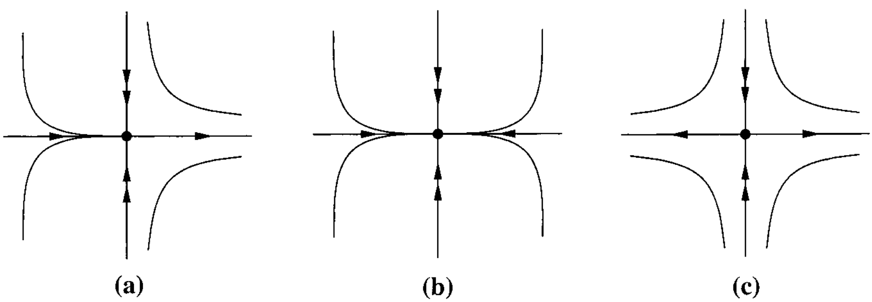

Fig. 4. Possible phase portraits for a saddle-attractor. The differential equation on the center manifold is $\dot{\sigma} = b\sigma^k + \ldots$, where $k \geqslant 2$. In case (a), k is even and $b \neq 0$; in case (b), k is odd and $b < 0$; in case (c), k is odd and $b > 0$.

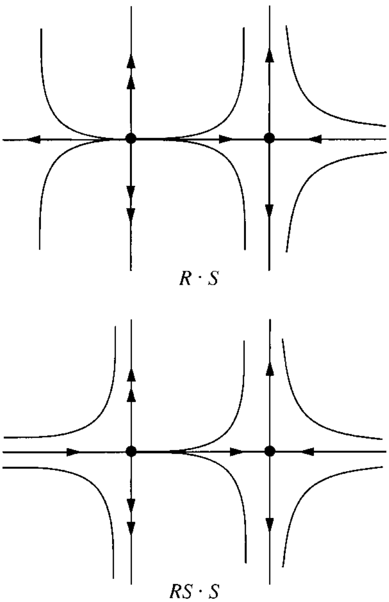

Fig. 5. Slow shock waves.

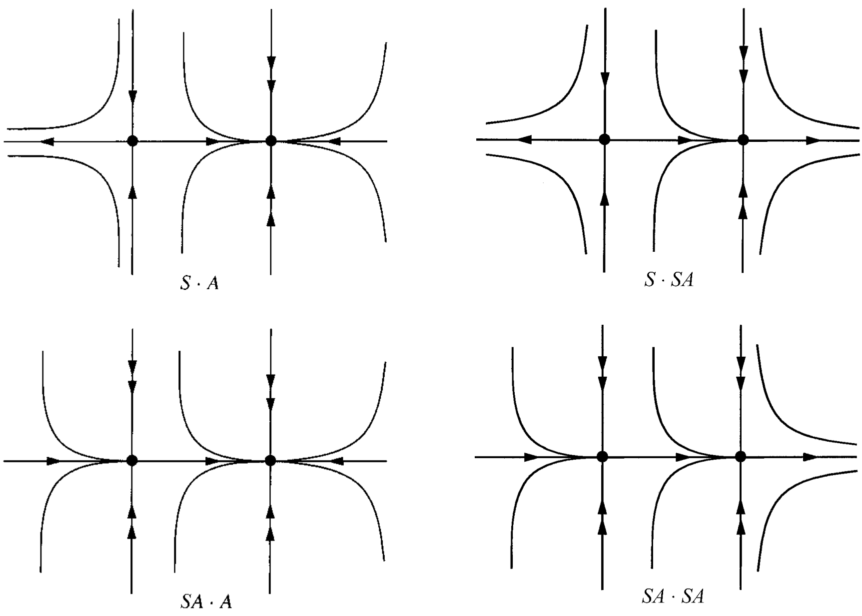

Fig. 6. Fast shock waves.

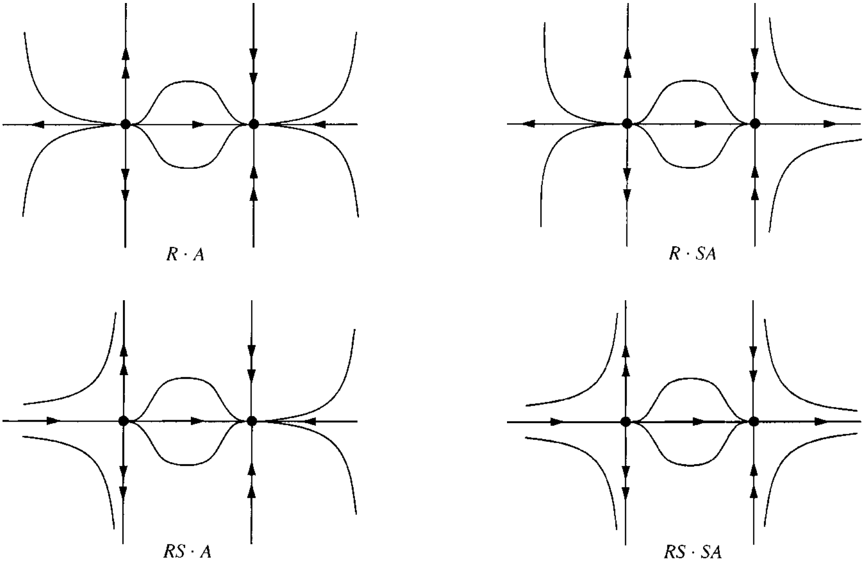

Fig. 7. Overcompressive shock waves.

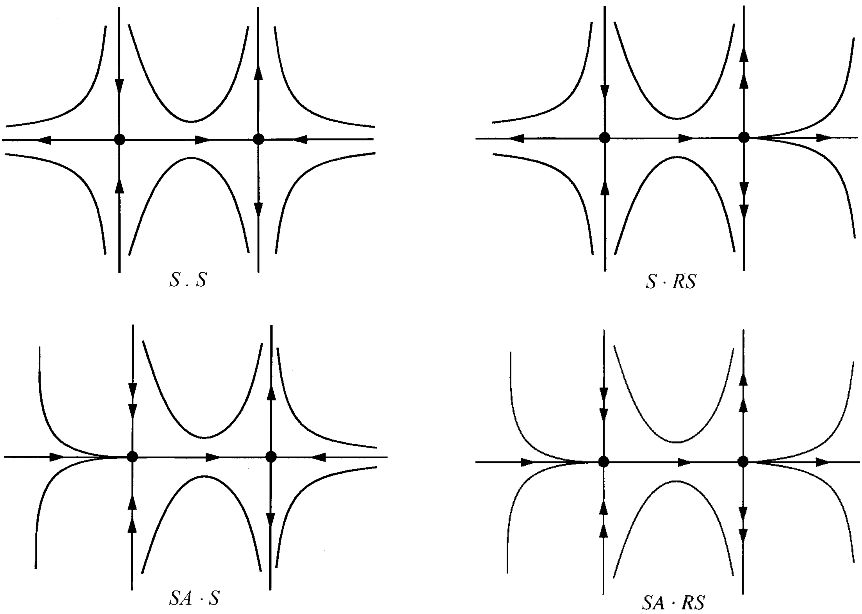

Fig. 8. Transitional shock waves.

attractor A on the left and a saddle S on the right, as in Fig. 3a. For instance, for an $RS \cdot S$ shock wave, the orbit leaves the repeller half of the RS and connects to the saddle; thus the connection is generally stable, and this is a classical wave. In contrast, for an $S \cdot RS$ wave, the connection leads from the saddle to the saddle half of the RS, so that the connection is generally unstable, and this is a transitional wave.

An *elementary wave* w is either a rarefaction wave or a shock wave. We write

$$w\colon U_- \xrightarrow{s} U_+ \tag{2.6}$$

if w has left state U_-, right state U_+, and speed s. Notice that an elementary wave also has a *type* T, as defined above.

Associated with each elementary wave is a *speed interval* σ: for a rarefaction wave of type R_i, $\sigma = [\lambda_i(U_-), \lambda_i(U_+)]$, whereas for a shock wave of speed s, $\sigma = [s, s]$. If σ_1 and σ_2 are speed intervals, we write $\sigma_1 \leqslant \sigma_2$ if $s_1 \leqslant s_2$ for every $s_1 \in \sigma_1$ and $s_2 \in \sigma_2$.

Also associated with each elementary wave is a compact set $\bar{\Gamma}$: if w is a rarefaction wave, $\bar{\Gamma}$ denotes its rarefaction curve; if w is a shock wave, then $\bar{\Gamma}$ denotes the closure of its connecting orbit. We shall say that an open set $\mathcal{N} \subseteq \mathbb{R}^2$ is a *neighborhood* of the elementary wave $w\colon U_- \xrightarrow{s} U_+$ if $\bar{\Gamma} \subset \mathcal{N}$.

2.2. Structurally Stable Riemann Solutions

A wave sequence $(w_1, w_2,..., w_n)$ is said to be *allowed* if

(W1) for each $i = 1,..., n-1$, the right state of w_i coincides with the left state of w_{i+1};

(W2) the speed intervals σ_i for w_i satisfy

$$\sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_n \tag{2.7}$$

(W3) no two successive waves are rarefaction waves of the same type.

For such a wave sequence we write

$$(w_1, w_2,..., w_n) : U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n \tag{2.8}$$

If $U_0 = U_L$ and $U_n = U_R$, then associated with an allowed wave sequence $(w_1, w_2,..., w_n)$ is a Riemann solution U of Eqs. (1.1) and (1.2). Therefore we refer to an allowed wave sequence as a *Riemann solution*. The *type* of a Riemann solution is simply the sequence of types of its waves.

In the following, we begin with an unperturbed Riemann solution,

$$(w_1^*, w_2^*, \ldots, w_n^*)\colon U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{2.9}$$

for the system of conservation laws $U_t + F^*(U)_x = 0$ and initial data $U_L^* = U_0^*$ and $U_R^* = U_n^*$; then we seek Riemann solutions when the data and the flux are perturbed from U_L^*, U_R^*, and F^*.

To this end, we first fix a compact set $K \subset \mathbb{R}^2$ such that Int K is a neighborhood of each w_i^* for $i = 1, \ldots, n$. Second, we choose an open neighborhood $\mathscr{U} \subseteq \operatorname{Int} K$ of the states $U_0^*, \ldots, U_n^*$ and of all rarefaction waves appearing in the unperturbed solution. Third, we choose an open neighborhood $\mathscr{B}$ of F^* in the Banach space of C^2 functions $F\colon K \to \mathbb{R}^2$, equipped with the C^2 norm. The sets $\mathscr{U}$ and $\mathscr{B}$ are chosen to have two properties: (i) $\mathscr{U} \subseteq \mathscr{U}_F$ for all $F \in \mathscr{B}$; and (ii) there exists a closed, bounded interval $I \subseteq \mathbb{R}$ such that the eigenvalues of $DF(U)$ belong to I for all $U \in \mathscr{U}$ and $F \in \mathscr{B}$. (Notice that we do not require $\mathscr{U}$ to be a neighborhood of the shock waves appearing in the unperturbed solution; thus we allow shock orbits to leave the region of strict hyperbolicity.)

Let $\mathscr{H}(\operatorname{Int} K)$ denote the set of nonempty, closed subsets of Int K, which we equip with the Hausdorff metric.

Definition 2.1. We shall say that the Riemann solution (2.9) is *structurally stable* if there are neighborhoods $\mathscr{U}_i \subseteq \mathscr{U}$ of U_i^*, $\mathscr{I}_i \subseteq I$ of s_i^*, and $\mathscr{F} \subseteq B$ of F^* and a C^1 map

$$G\colon \mathscr{U}_0 \times \mathscr{I}_1 \times \mathscr{U}_1 \times \mathscr{I}_2 \times \cdots \times \mathscr{I}_n \times \mathscr{U}_n \times \mathscr{F} \to \mathbb{R}^{3n-2} \tag{2.10}$$

with $G(U_0^*, s_1^*, U_1^*, s_2^*, \ldots, s_n^*, U_n^*, F^*) = 0$ such that

(P1) $G(U_0, s_1, U_1, s_2, \ldots, s_n, U_n, F) = 0$ implies that there exists a Riemann solution

$$(w_1, w_2, \ldots, w_n)\colon U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n \tag{2.11}$$

for $U_t + F(U)_x = 0$ with successive waves of the same types as those of the wave sequence (2.9), with each w_i contained in Int K, and with each rarefaction wave contained in $\mathscr{U}$;

(P2) $DG(U_0^*, s_1^*, U_1^*, s_2^*, \ldots, s_n^*, U_n^*, F^*)$, restricted to the $(3n-2)$-dimensional space of vectors $\{(\dot U_0, \dot s_1, \dot U_1, \dot s_2, \ldots, \dot s_n, \dot U_n, \dot F)\colon \dot U_0 = 0 = \dot U_n, \dot F = 0\}$, is an isomorphism onto $\mathbb{R}^{3n-2}$.

Condition (P2) implies, by the implicit function theorem, that $G^{-1}(0)$ is a graph of a function defined on a neighborhood of (U_0^*, U_n^*, F^*), which we may as well take to be $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$. Therefore for each wave w_i we can define a map $\bar{\Gamma}_i: \mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F} \to \mathscr{H}(\text{Int } K)$; namely, $\bar{\Gamma}_i(U_0, U_n, F)$ is the rarefaction curve or the closure of the connecting orbit of the wave w_i. We further require that

(P3) $(w_1, w_2, ..., w_n)$ can be chosen so that each map $\bar{\Gamma}_i$ is continuous.

The map G will be said to *exhibit* the structural stability of the Riemann solution (2.9).

Associated with $(w_1^*, w_2^*, ..., w_n^*)$ is a solution $U^*(x, t) = \hat{U}^*(x/t)$ of the Riemann problem (1.1) and (1.2) with $U_L = U_L^* := U_0$, $U_R = U_R^* := U_n$, and $F = F^*$. Similarly, for each (U_L, U_R, F) near (U_L^*, U_R^*, F^*), there is a Riemann solution $\hat{U}$ near $\hat{U}^*$ associated with the point in $G^{-1}(0)$ that has left state U_L, right state U_R, and flux F.

2.3. Local Defining Maps

To construct maps G that exhibit structural stability, we use *local defining maps* for each type of elementary wave. Let $w^*: U_-^* \xrightarrow{s^*} U_+^*$ be an elementary wave of type T for $U_t + F^*(U)_x = 0$. The local defining map G_T has as its domain a set of the form $\mathscr{U}_- \times \mathscr{I} \times \mathscr{U}_+ \times \mathscr{F}$ (with $\mathscr{U}_\pm$ being neighborhoods of $U_\pm^*$, $\mathscr{I}$ being a neighborhood of s^*, and $\mathscr{F}$ being a neighborhood of F^*). The range is some $\mathbb{R}^e$; the number e depends only on the wave type T. The local defining map is such that $G_T(U_-^*, s^*, U_+^*, F^*) = 0$. Moreover, if certain *wave nondegeneracy conditions* are satisfied at (U_-^*, s^*, U_+^*, F^*), then there is a neighborhood $\mathscr{N}$ of w^* such that

(D1) $G_T(U_-, s, U_+, F) = 0$ if and only if there exists an elementary wave $w: U_- \xrightarrow{s} U_+$ of type T for $U_t + F(U)_x = 0$ contained in $\mathscr{N}$;

(D2) $DG_T(U_-^*, s^*, U_+^*, F^*)$, restricted to the space $\{(\dot{U}_-, \dot{s}, \dot{U}_+, \dot{F}) : \dot{F} = 0\}$, is surjective.

Condition (D2) implies, by the implicit function theorem, that $G_T^{-1}(0)$ is a manifold of codimension e. Therefore we can define a map $\bar{\Gamma}$ from this manifold to $\mathscr{H}(\text{Int } K)$ (just as above). We have that

(D3) w can be chosen so that $\bar{\Gamma}$ is continuous and reduces to $\bar{\Gamma}^*$ at the point in $G_T^{-1}(0)$ corresponding to (U_-^*, s^*, U_+^*, F^*).

The system of equations $G_T(U_-, s, U_+, F) = 0$ is called a system of *local defining equations*. We now discuss local defining equations and nondegeneracy conditions for each type of elementary wave.

2.4. Rarefaction Waves

Suppose that $w^*: U_-^* \xrightarrow{s^*} U_+^*$ is a rarefaction of family 1 for the equation $U_t + F^*(U)_x = 0$. Then there exist neighborhoods $\mathscr{F}$ of F^* and $\mathscr{N}$ of w^* such that for all $F \in \mathscr{F}$: (a) the eigenvector $r_1(U)$ of $DF(U)$ corresponding to the eigenvalue $\lambda_1(U)$ can be chosen to depend smoothly on U throughout $\mathscr{N}$; and (b) $D\lambda_1(U)\, r_1(U) \neq 0$ for all $U \in \mathscr{N}$. We can therefore normalize $r_1(U)$ so that $D\lambda_1(U)\, r_1(U) \equiv 1$.

For each $U_- \in \mathscr{N}$, define ψ_1 to be the maximal solution of the initial-value problem

$$\frac{\partial \psi_1}{\partial s}(U_-, s) = r_1(\psi_1(U_-, s)), \tag{2.12}$$

$$\psi_1(U_-, \lambda_1(U_-)) = U_- \tag{2.13}$$

Then, for $F \in \mathscr{F}$ and $U_-, U_+ \in \mathscr{N}$, there exists a rarefaction wave of type R_1 for the equation $U_t + F(U)_x = 0$ that leads from U_- to U_+, has speed s, and lies within $\mathscr{N}$ if and only if

$$U_+ - \psi_1(U_-, s) = 0 \tag{2.14}$$

$$s = \lambda_1(U_+) > \lambda_1(U_-) \tag{2.15}$$

Similarly, we can define ψ_2 to be the solution of

$$\frac{\partial \psi_2}{\partial s}(s, U_+) = r_2(\psi_2(s, U_+)) \tag{2.16}$$

$$\psi_2(\lambda_2(U_+), U_+) = U_+ \tag{2.17}$$

Then there is a rarefaction wave of type R_2 for $U_t + F(U)_x = 0$ from U_- to U_+ with speed s if and only if

$$U_- - \psi_2(s, U_+) = 0 \tag{2.18}$$

$$s = \lambda_2(U_-) < \lambda_2(U_+) \tag{2.19}$$

Equations (2.14) and (2.18) are defining equations for rarefaction waves of types R_1 and R_2, respectively. The nondegeneracy conditions for rarefaction waves of type R_i, which are implicit in our definition of rarefaction wave, are the speed inequality (2.15) or (2.19), and the genuine nonlinearity condition (2.3). To define codimension-one Riemann solutions, we must allow these nondegeneracy conditions to be violated. Therefore the definition of rarefaction wave is generalized in Section 3.

2.5. Shock Waves

If there is to be a shock wave solution of $U_t + F(U)_x = 0$ from U_- to U_+ with speed s, we must have that:

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0 \tag{E0}$$

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \quad \text{has an orbit from} \quad U_- \text{ to } U_+ \tag{C0}$$

The two-component equation (E0) is a defining equation. In the context of structurally stable Riemann solutions, condition (C0) can be regarded as a nondegeneracy condition except for transitional shock waves, for which it is a defining equation. Codimension-one Riemann solutions can violate condition (C0), necessitating the generalized definition of shock waves given in Section 3; in this context, the role of condition (C0) is more subtle, as we discuss in Section 4.

In Tables II–IV we list additional defining equations and nondegeneracy conditions for shock waves of various types. The additional defining equations are either equality of the shock speed with a characteristic speed or, for transitional shock waves, the vanishing of a separation function. The wave nondegeneracy conditions are open conditions. The tables omit several types of nondegeneracy conditions, which we assume implicitly: (a) $U_- \neq U_+$; (b) inequality conditions on the eigenvalues that are implied by the shock type (e.g., for an $R \cdot S$ shock wave, $\lambda_1(U_-) < \lambda_2(U_-) < s$ and $\lambda_1(U_+) < s < \lambda_2(U_+)$); and (c) condition (C0) when it is an open condition (given the defining equations and the listed nondegeneracy conditions).

Table II. Additional Defining Equations and Nondegeneracy Conditions for Slow Shock Waves

Type of shock	Additional defining equation(s)		Nondegeneracy conditions	
$R \cdot S$	None		None	
$R \cdot RS$	$\lambda_1(U_+) - s = 0$	(E1)	$D\lambda_1(U_+)\, r_1(U_+) \neq 0$	(G1)
			$\ell_1(U_+)(U_+ - U_-) \neq 0$	(B1)
$RS \cdot S$	$\lambda_1(U_-) - s = 0$	(E2)	$D\lambda_1(U_-)\, r_1(U_-) \neq 0$	(G2)
			Not distinguished connection	(C1)
$RS \cdot RS$	$\lambda_1(U_-) - s = 0$	(E3)	$D\lambda_1(U_-)\, r_1(U_-) \neq 0$	(G3)
	$\lambda_1(U_+) - s = 0$	(E4)	$D\lambda_1(U_+)\, r_1(U_+) \neq 0$	(G4)
			$\ell_1(U_+)(U_+ - U_-) \neq 0$	(B2)
			Not distinguished connection	(C2)

Table III. Additional Defining Equations and Nondegeneracy Conditions for Fast Shock Waves

Type of shock	Additional defining equation(s)		Nondegeneracy conditions	
$S \cdot A$	Nnone		None	
$SA \cdot A$	$\lambda_2(U_-) - s = 0$	(E5)	$D\lambda_2(U_-)\, r_2(U_-) \neq 0$	(G5)
			$\ell_2(U_-)(U_+ - U_-) \neq 0$	(B3)
$S \cdot SA$	$\lambda_2(U_+) - s = 0$	(E6)	$D\lambda_2(U_+)\, r_2(U_+) \neq 0$	(G6)
			Not distinguished connection	(C3)
$SA \cdot SA$	$\lambda_2(U_-) - s = 0$	(E7)	$D\lambda_2(U_-)\, r_2(U_-) \neq 0$	(G7)
	$\lambda_2(U_+) - s = 0$	(E8)	$D\lambda_2(U_+)\, r_2(U_+) \neq 0$	(G8)
			$\ell_2(U_-)(U_+ - U_-) \neq 0$	(B4)
			Not distinguished connection	(C4)

The additional defining equations and nondegeneracy conditions for classical shock waves are given in Tables II–III; the reader should also refer to Figs. 5–7. In these tables, conditions (C1)–(C4) are that the connection Γ is not *distinguished*; this means the following. For $RS \cdot *$ shock waves, the connection Γ should not lie in the unstable manifold of U_- (i.e., the unique invariant curve tangent to an eigenvector with positive eigenvalue). For $* \cdot SA$ shock waves, the connection Γ should not lie in the stable manifold of U_+.

Table IV. Additional Defining Equations and Nondegeneracy Conditions for Transitional Shock Waves

Type of shock	Additional defining equation(s)		Nondegeneracy conditions	
$S \cdot S$	$S(U_-, s) = 0$	(S1)	$DS(U_-, s) \neq 0$	(T1)
$S \cdot RS$	$\lambda_1(U_+) - s = 0$	(E13)	$D\lambda_1(U_+)\, r_1(U_+) \neq 0$	(G13)
	$S(U_-, s) = 0$	(S2)	Transversality	(T2)
$SA \cdot S$	$\lambda_2(U_-) - s = 0$	(E14)	$D\lambda_2(U_-)\, r_2(U_-) \neq 0$	(G14)
	$\tilde{S}(s, U_+) = 0$	(S3)	Transversality	(T3)
$SA \cdot RS$	$\lambda_2(U_-) - s = 0$	(E15)	$D\lambda_2(U_-)\, r_2(U_-) \neq 0$	(G15)
	$\lambda_1(U_+) - s = 0$	(E16)	$D\lambda_1(U_+)\, r_1(U_+) \neq 0$	(G16)
	$S(U_-, s) = 0$	(S4)	Transversality	(T4)

2.6. Transitional Shock Waves

Referring to Fig. 8, suppose that $w^*: U_-^* \xrightarrow{s^*} U_+^*$ is a shock wave for $U_t + F^*(U)_x = 0$ of type $S \cdot S$, $S \cdot RS$, or $SA \cdot RS$. Thus we suppose that, for the differential equation

$$\dot{U} = F^*(U) - F^*(U_-^*) - s^*(U - U_-^*) \tag{2.20}$$

U_-^* is an equilibrium of saddle or saddle-attractor type, U_+^* is an equilibrium of saddle or repeller-saddle type, and there is a solution $\tilde{U}: \mathbb{R} \to \mathbb{R}^2$ such that $\lim_{\eta \to \pm\infty} \tilde{U}(\eta) = U_\pm^*$ and $\tilde{U}(\eta) \in \Gamma^*$ for all $\eta \in \mathbb{R}$.

If U_-^* is a saddle of Eq. (2.20), let $W_-(U_-^*, s^*)$ denote its unstable manifold; if U_+^* is a saddle of Eq. (2.20), let $W_+(U_-^*, s^*)$ denote its stable manifold. Similarly, if $U_\pm^*$ is a repeller-saddle or saddle-attractor, let $W_\pm(U_-^*, s^*)$ denote one of its center manifolds. The manifolds $W_\pm(U_-^*, s^*)$ both perturb smoothly to invariant manifolds of Eq. (1.4), denoted $W_\pm(U_-, s)$. When U_-^* is a saddle, $W_-(U_-, s)$ is just the unstable manifold of the saddle U_- of Eq. (1.4); when U_+^* is a saddle, $W_+(U_-, s)$ is the stable manifold of the saddle of Eq. (1.4) near U_+^*.

Let Σ be a line segment through $\tilde{U}(0)$ transverse to $\tilde{U}(0)$ in the direction V. Then $W_\pm(U_-, s)$ meet Σ in points $\bar{U}_\pm(U_-, s)$, and

$$\bar{U}_-(U_-, s) - \bar{U}_+(U_-, s) = S(U_-, s)\, V \tag{2.21}$$

The function S is called the *separation function*; it is defined on a neighborhood of (U_-^*, s^*), and, of course, $S(U_-^*, s^*) = 0$. The partial derivatives of S are given as follows [16]. The linear differential equation

$$\dot{\phi} + \phi[DF(\tilde{U}(\eta)) - s^* I] = 0 \tag{2.22}$$

has, up to constant multiple, a unique bounded solution. For the correct choice of this constant,

$$\frac{\partial S}{\partial s}(U_-^*, s^*) = -\int_{-\infty}^{\infty} \phi(\eta)(\tilde{U}(\eta) - U_-^*)\, d\eta \tag{2.23}$$

$$D_{U_-} S(U_-^*, s^*) = -\left(\int_{-\infty}^{\infty} \phi(\eta)\, d\eta\right) \{DF(U_-^*) - s^* I\}. \tag{2.24}$$

One can treat $SA \cdot S$ shock waves analogously to $S \cdot RS$ waves; one obtains a separation function $\tilde{S}(s, U_+)$ [17].

The additional local defining equations and nondegeneracy conditions (T1)–(T4) for transitional shock waves are given in Table IV. Condition (T2) is that there is a vector W such that

$$\begin{pmatrix} \ell_1(U_+) \\ \int_{-\infty}^{\infty} \phi(\eta)\, d\eta \end{pmatrix} W \quad \text{and} \quad \begin{pmatrix} \ell_1(U_+)(U_+ - U_-) \\ \int_{-\infty}^{\infty} \phi(\eta)(U(\eta) - U_-)\, d\eta \end{pmatrix}$$
$$\text{are linearly independent} \tag{T2}$$

Condition (T3) is that there is a vector W such that

$$\begin{pmatrix} \ell_2(U_-) \\ \int_{-\infty}^{\infty} \phi(\eta)\, d\eta \end{pmatrix} W \quad \text{and} \quad \begin{pmatrix} \ell_2(U_-)(U_- - U_+) \\ \int_{-\infty}^{\infty} \phi(\eta)(U(\eta) - U_+)\, d\eta \end{pmatrix}$$
$$\text{are linearly independent} \tag{T3}$$

Condition (T4) is that

$$\begin{pmatrix} \ell_1(U_+) \\ \int_{-\infty}^{\infty} \phi(\eta)\, d\eta \end{pmatrix} r_1(U_-) \quad \text{and} \quad \begin{pmatrix} \ell_1(U_+)(U_+ - U_-) \\ \int_{-\infty}^{\infty} \phi(\eta)(U(\eta) - U_-)\, d\eta \end{pmatrix}$$
$$\text{are linearly independent} \tag{T4}$$

2.7. Riemann Numbers

For the Riemann solution (2.9), let w_i^* have type T_i and local defining map G_{T_i}, with range $\mathbb{R}^{e_i}$. For appropriate neighborhoods $\mathcal{U}_i$ of U_i^*, $\mathcal{I}_i$ of s_i^*, $\mathcal{F}$ of F^*, and $\mathcal{N}_i$ of w_i^*, we can define a map $G: \mathcal{U}_0 \times \mathcal{I}_1 \times \cdots \times \mathcal{I}_n \times \mathcal{U}_n \times \mathcal{F} \to \mathbb{R}^{e_1 + \cdots + e_n}$ by $G = (G_1,..., G_n)$, where

$$G_i(U_0, s_1,..., s_n, U_n, F) = G_{T_i}(U_{i-1}, s_i, U_i, F) \tag{2.25}$$

The map G is called the *local defining map* of the wave sequence (2.9). Assuming the wave nondegeneracy conditions, if $G(U_0, s_1,..., s_n, U_n, F) = 0$, then for each $i = 1,..., n$, there is an elementary wave $w_i: U_{i-1} \xrightarrow{s_i} U_i$ of type T_i for $U_t + F(U)_x = 0$ contained in $\mathcal{N}_i$, for which $\bar{\Gamma}_i$ is continuous.

In view of the requirement in Definition 2.1 that the local defining map have range $\mathbb{R}^{3n-2}$, a necessary condition for $G = (G_1,..., G_n)$ to exhibit the structural stability of the wave sequence (2.9) is that

$$\sum_{i=1}^{n} e_i = 3n - 2, \qquad \text{i.e.,} \quad \sum_{i=1}^{n} (3 - e_i) = 2 \tag{2.26}$$

Table V. Riemann Numbers of Shock Waves

$U_-\backslash U_+$	RS	S	SA	A
R	0	1	0	1
RS	−1	0	−1	0
S	−1	0	0	1
SA	−2	−1	−1	0

We are therefore led to define the *Riemann number* of an elementary wave type T to be

$$\rho(T) = 3 - e(T) \tag{2.27}$$

where $e(T)$ is the number of defining equations for a wave of type T. For convenience, if w is an elementary wave of type T, we shall sometimes write $\rho(w)$ instead of $\rho(T)$. Because of Eq. (2.26), a necessary condition for an allowed sequence of elementary waves $(w_1, \ldots, w_n)$ to be structurally stable is that $\sum_{i=1}^{n} \rho(w_i) = 2$.

The Riemann number of a rarefaction wave is 1. The Riemann numbers of shock waves are given in Table V.

2.8. Wave Groups

A *1-wave group* is either a single $R \cdot S$ wave or an allowed sequence of elementary waves of the form

$$(R \cdot RS)(R_1 \quad RS \cdot RS) \cdots (R_1 \quad RS \cdot RS)\, R_1\, (RS \cdot S) \tag{2.28}$$

where the terms in parentheses are optional. If any of the terms in parentheses are present, the group is termed *composite*.

A *transitional wave group* is either a single $S \cdot S$ wave or an allowed sequence of elementary waves of the form

$$S \cdot RS(R_1 \quad RS \cdot RS) \cdots (R_1 \quad RS \cdot RS)\, R_1\, (RS \cdot S) \tag{2.29}$$

or

$$(S \cdot SA)\, R_2\, (SA \cdot SA\ R_2) \cdots (SA \cdot SA\ R_2)\, SA \cdot S \tag{2.30}$$

the terms in parentheses being optional. In cases (2.29) and (2.30), the group is termed *composite*.

A *2-wave group* is either a single $S \cdot A$ wave or an allowed sequence of elementary waves of the form

$$(S \cdot SA)\, R_2\, (SA \cdot SA\ R_2) \cdots (SA \cdot SA\ R_2)\, (SA \cdot A) \tag{2.31}$$

where again the terms in parentheses are optional. If any of the terms in parentheses are present, the group is termed *composite*.

An $SA \cdot RS$ wave is called a *doubly sonic transitional wave*.

The reader should note a symmetry between the wave groups (2.28) and (2.31), as well as between the groups (2.29) and (2.30). The wave groups $R \cdot S$, (2.28), and (2.29) are termed *slow*; the wave groups $S \cdot A$, (2.31), and (2.30) are termed *fast*. A solution U for the equation $U_t + F(U)_x = 0$ that consists of a fast wave group corresponds to a solution $\check{U}$ for the equation $\check{U}_t - F(\check{U})_x = 0$ that consists of a slow wave group; the correspondence is

$$\check{U}(x, t) = U(-x, t) \tag{2.32}$$

Under this duality, rarefaction waves of type R_1 correspond to those of type R_2. Shock waves of type $R \cdot RS$, for example, correspond to those of type $SA \cdot A$; in general, to find the dual of a shock type, one reverses its name and interchanges the letters R and A. This symmetry will be exploited throughout this paper to shorten the treatment.

2.9. Wave Group Interaction Condition

The wave group interaction condition is a transversality condition appearing as a hypothesis in the structural stability theorem. In order to state this condition precisely, we recall some results from Ref. 17 concerning wave curves and transitional wave groups.

First, consider a 1-wave group

$$U_0^* \xrightarrow{s_1^*} \cdots \xrightarrow{s_k^*} U_k^* \tag{2.33}$$

for $U_t + F^*(U)_x = 0$ with local defining map $G^{(1)}$. Assume that each wave in the 1-wave group satisfies its nondegeneracy conditions. Then the solutions of $G^{(1)}(U_0, s_1, \ldots, s_k, U_k, F) = 0$ can be parameterized locally by U_0, F, and one additional parameter σ. [This is to be expected because the sum of the Riemann numbers for the wave sequence (2.33) is 1.] More precisely, there exist neighborhoods $\mathcal{U}_0 \subseteq \mathcal{U}$ of U_0^*, $\mathcal{F} \subseteq \mathcal{B}$ of F^*, and $\mathcal{J} \subseteq \mathbb{R}$ of a parameter value σ^*, and smooth mappings $s_i^f : \mathscr{U}_0 \mathscr{S} F \times \mathscr{J} \to \mathbb{R}$ and U_i^f: $\mathscr{U}_0 \mathscr{S} F \times \mathscr{J} \to \mathscr{U}$ for $i = 1,..., k$, with $s_i^f(U_0^*, F^*, \sigma^*) = s_i^*$ and $U_i^f(U_0^*, F^*, \sigma^*) = U_i^*$, such that $G^{(1)}(U_0, s_1^f(U_0, F, \sigma),..., s_k^f(U_0, F, \sigma), U_k^f(U_0, F, \sigma), F) = 0$ for each $(U_0, F, \sigma) \in \mathscr{U}_0 \mathscr{S} F \times \mathscr{J}$. In particular, there exists a family

$$U_0 \xrightarrow{s_1^f(U_0, F, \sigma)} \cdots \xrightarrow{s_k^f(U_0, F, \sigma)} U_k^f(U_0, F, \sigma) \tag{2.34}$$

of wave sequences for $U_t + F(U)_x = 0$ with successive waves of the same type as those of the 1-wave group (2.33). The curve $U_k^f(U_0^*, F^*, \sigma)$ parameterized by σ is called the *forward wave curve* through U_k^* associated with the wave sequence (2.33).

Similarly, if a 2-wave group

$$U_k^* \xrightarrow{s_{k+1}^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{2.35}$$

for $U_t + F^*(U)_x = 0$ has local defining map $G^{(2)}$, and each wave satisfies its nondegeneracy conditions, then solutions of $G^{(2)}(U_k, s_{k+1},..., s_n, U_n, F) = 0$ can be parameterized locally by U_n, F, and a parameter τ, giving a family

$$U_k^b(U_n, F, \tau) \xrightarrow{s_{k+1}^b(U_n, F, \tau)} \cdots \xrightarrow{s_n^b(U_n, F, \tau)} U_n \tag{2.36}$$

of wave sequences. The curve $U_k^b(U_n^*, F^*, \tau)$ parameterized by τ is called the *backward wave curve* through U_k^* associated with the wave sequence (2.35).

For the simplest case of a structurally stable Riemann solution

$$U_0^* \xrightarrow{s_1^*} \cdots \xrightarrow{s_k^*} U_k^* \xrightarrow{s_{k+1}^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{2.37}$$

comprising only a 1-wave group and a 2-wave group (joined at the state U_k^*), the *wave group interaction condition* is that the forward wave curve $U_k^f(U_0^*, F^*, \sigma)$ and the backward wave curve $U_k^b(U_n^*, F^*, \tau)$ should be transverse, i.e., the tangent vectors $\partial U_k^f / \partial \sigma$ at (U_0^*, F^*, σ^*) and $\partial U_k^b / \partial \tau$ at (U_n^*, F^*, τ^*) should be linearly independent.

Next consider a transitional wave group

$$U_k^* \xrightarrow{s_{k+1}^*} \cdots \xrightarrow{s_\ell^*} U_\ell^* \tag{2.38}$$

for $U_t + F^*(U)_x = 0$ with local defining map $G^{(t)}$. Assume that each wave satisfies its nondegeneracy conditions. Then there exists a subspace Δ of

U_k-space, of dimension 0 or 1, such that the following statements are equivalent for any $V \neq 0$ in U_k-space:

(a) $V \notin \Delta$;

(b) the linear map $DG^{(t)}(U_k^*, s_{k+1}^*, \ldots, s_\ell^*, U_\ell^*, F^*)$, restricted to the subspace

$$\{(\dot{U}_k, \dot{s}_{k+1}, \ldots, \dot{s}_\ell, \dot{U}_\ell, \dot{F}) : \dot{U}_k \text{is a multiple of } V, \dot{F} = 0\} \tag{2.39}$$

is surjective onto $\mathbb{R}^{3(\ell - k)}$ and the projection of its one-dimensional kernel onto U_ℓ-space is one-dimensional.

The significance of this result is that, if V is tangent to the forward wave curve through U_k^* for the 1-wave group (2.33), and if $V \notin \Delta$, then solutions of the pair of equations

$$G^{(1)}(U_0, s_1, \ldots, s_k, U_k, F) = 0 \tag{2.40}$$

$$G^{(t)}(U_k, s_{k+1}, \ldots, s_\ell, U_\ell, F) = 0 \tag{2.41}$$

can be parameterized by U_0, F, and an additional parameter σ, giving a family

$$U_0 \xrightarrow{s_1^f(U_0, F, \sigma)} \cdots \xrightarrow{s_k^f(U_0, F, \sigma)} U_k^f(U_0, F, \sigma) \xrightarrow{s_{k+1}^f(U_0, F, \sigma)} \cdots \xrightarrow{s_\ell^f(U_0, F, \sigma)} U_\ell^f(U_0, F, \sigma) \tag{2.42}$$

of wave sequences. In this manner, a forward wave curve can be extended by attaching a transitional wave group, provided either that $\Delta = \{0\}$ or that Δ is transverse to the forward wave curve. The curve $U_\ell^f(U_0^*, F^*, \sigma)$ parameterized by σ is likewise called the *forward wave curve* through U_ℓ^*. Similarly, using a dual version of the result just quoted, a backward wave curve can be extended by attaching a transitional wave group.

Suppose that a structurally stable Riemann solution consists of a 1-wave group g_0, $r \geq 1$ transitional wave groups $g_1, \ldots, g_r$, and a 2-wave group, g_{r+1}. Then the *wave group interaction condition* is that (a) inductively for $j = 1, \ldots, r$, the transitional wave group g_j should satisfy the transversality condition allowing it to be attached to the forward wave curve associated with groups $g_0, \ldots, g_{j-1}$; and (b) the forward wave curve associated with groups $g_0, \ldots, g_r$ should be transverse to the backward wave curve associated with group g_{r+1}.

Remark. The wave group interaction condition could be formulated in alternative ways. Under a certain transversality condition, similar to the one for attaching a transitional wave group to a forward wave curve, a transitional wave group can be attached to a backward wave curve. For each $s \leqslant r$, an equivalent formulation of the wave group interaction condition consists of (a) the transversality conditions allowing wave groups $g_1,..., g_s$ to be attached to g_0 to form a forward wave curve; (b) the transversality conditions allowing wave groups $g_r,..., g_{s+1}$ to be attached to g_{r+1} to form a backward wave curve; and (c) the transversality of these forward and backward wave curves at the state between g_s and g_{s+1}.

The most general structurally stable Riemann solution consists of waves of type $SA \cdot RS$ separating wave sequences of the form already discussed (see Theorem 2.2 below). In this case, the *wave group interaction condition* is that each of these wave sequences should satisfy its wave group interaction condition.

2.10. Wave Structure and Structural Stability

In Ref. 17 the following results are proved.

Theorem 2.4 (Wave Structure). *Consider the allowed wave sequence (2.9). Then*

(A) $\sum_{i=1}^{n} \rho(w_i^*) \leqslant 2;$

(B) $\sum_{i=1}^{n} \rho(w_i^*) = 2$ *if and only if the following conditions are satisfied.*

(1) Suppose that the wave sequence (2.9) includes no $SA \cdot RS$ waves. Then it consists of one 1-wave group, followed by an arbitrary number of transitional wave groups (in any order), followed by one 2-wave group.

(2) Suppose that the wave sequence (2.9) includes $m \geqslant 1$ waves of type $SA \cdot RS$. Then these waves separate $m+1$ wave sequences $g_0,..., g_m$. Each g_i is exactly as in (1) with the restrictions that:

(a) if $i < m$, the last wave in the group has type R_2;

(b) if $i > 0$, the first wave in the group has type R_1.

Theorem 2.5 (Structural Stability). *Suppose that the allowed wave sequence (2.9) has total Riemann number $\sum_{i=1}^{n} \rho(w_i^*) = 2$. Assume that:*

(H1) each wave satisfies the appropriate wave nondegeneracy conditions;

(H2) the wave group interaction condition is satisfied;

(H3) if w_i^* has type $* \cdot S$ and w_{i+1}^* has type $S \cdot *$, then $s_i^* < s_{i+1}^*$.

Then the wave sequence (2.9) is structurally stable.

In fact, more can be concluded: not only can the connecting orbit Γ_i of the perturbed shock wave w_i be chosen to vary continuously, but also there is a neighborhood $\mathcal{N}_i$ such that if $\Gamma_i \subset \mathcal{N}_i$, then Γ_i is unique.

3. CODIMENSION-ONE RIEMANN SOLUTIONS

In this section we define codimension-one Riemann solutions. To describe these solutions conveniently and to relate them to structurally stable Riemann solutions, we must generalize the definitions of rarefaction and shock waves in Section 2, for the following reason. When a structurally stable Riemann solution becomes a codimension-one Riemann solution, the number and types of waves can change. However, the number and types of waves determine the dimensions of the domain and range for the defining map. To keep these dimensions fixed, we broaden the definitions of rarefaction and shock waves so that a codimension-one solution can be regarded as having the same number and types of waves as the structurally stable Riemann solutions bordering it.

3.1. Generalized Elementary Waves

The definition in Section 2 of a rarefaction wave requires the solution $\hat{U}$ to be differentiable as a function of $\xi = x/t$, which entails that genuine nonlinearity holds all along the rarefaction curve $\bar{\Gamma}$. For codimension-one Riemann solutions, however, we must allow genuine nonlinearity to fail within rarefaction waves. In this situation, $\bar{\Gamma}$ is not parameterized smoothly by the eigenvalue λ_i.

To construct a rarefaction wave of family *i*, we consider a closed segment $\bar{\Gamma}$ of an integral curve of the line field of null directions for $DF(U) - \lambda_i(U)\, I$, and we assume that $\lambda_i(U)$ is strictly monotone along $\bar{\Gamma}$. Then $\bar{\Gamma}$ can be parameterized by a continuous function $\hat{U}$ of $\xi = \lambda_i(U)$, but this parameterization fails to be smooth at the critical values of λ_i, *viz.*, speeds $\xi = \lambda_i(U)$ corresponding to states $U \in \bar{\Gamma}$ at which $D\lambda_i(U)\, r_i(U) = 0$. Nevertheless, $\hat{U}$ is a scale-invariant weak solution of Eq. (1.1), i.e., it satisfies the equation

$$\frac{d}{d\xi} F(\hat{U}(\xi)) - \xi \frac{d}{d\xi} \hat{U}(\xi) = 0 \tag{3.1}$$

in the sense of distributions. [To see this, let $U = \check{U}(\tau)$ parameterize $\bar{\Gamma}$ smoothly, so that $\hat{U}(\xi) = \check{U}(\tau)$ when $\xi = \lambda_i(\check{U}(\tau)$.] Then, in the weak formulation of Eq. (3.1), change the variable of integration from ξ to τ.)

Therefore we define a *generalized rarefaction wave* of type R_i to be a continuous map $\hat{U}: [\alpha, \beta] \to \mathscr{U}$, where $\alpha \leqslant \beta$, such that (i) the rarefaction curve $\bar{\Gamma} = \{\hat{U}(\xi) : \xi \in [\alpha, \beta]\}$ is an integral curve for the line field associated to family i and (ii) $\xi = \lambda_i(\hat{U}(\xi))$ for all $\xi \in [\alpha, \beta]$. Thus we generalize the definition in Section 2 in two ways: (1) α may equal β, giving a *rarefaction wave of zero strength*; and (2) $\hat{U}$ may fail to be differentiable. In other words, we allow the strength of a rarefaction wave to vanish, and we permit genuine nonlinearity to fail, as a codimension-one boundary is approached.

A *generalized shock wave* consists of a *left state* U_-, a *right state* U_+ (possibly equal to U_-), a *speed* s, and a sequence of connecting orbits $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2,..., \tilde{\Gamma}_k$ of Eq. (1.4), $k \geqslant 1$, from $U_- = \tilde{U}_0$ to $\tilde{U}_1$, $\tilde{U}_1$ to $\tilde{U}_2,..., \tilde{U}_{k-1}$ to $\tilde{U}_k = U_+$. Notice that $\tilde{U}_0, \tilde{U}_1,..., \tilde{U}_k$ must be equilibria of Eq. (1.4). We allow for the possibility that $\tilde{U}_{j-1} = \tilde{U}_j$, in which case we assume that $\tilde{\Gamma}_j$ is the trivial orbit $\{\tilde{U}_j\}$. (As explained below, we exclude homoclinic orbits.) In particular, we generalize the definition in Section 2 to allow (1) *zero-strength shock waves*, which we define to have $U_- = U_+$ and a single trivial orbit $\tilde{\Gamma}_1 = \{U_-\}$; and (2) multiple orbits ($k > 1$). As a codimension-one boundary is approached, the strength of a shock wave can vanish, just as for rarefaction waves. Moreover, the orbit Γ connecting U_- to U_+ can, in this limit, break into two orbits: $\tilde{\Gamma}_1$ from U_- to an equilibrium $\tilde{U}_1$ and $\tilde{\Gamma}_2$ from $\tilde{U}_1$ to U_+. (More precisely, the closure $\bar{\Gamma}$ tends, in the Hausdorff metric, to $\overline{\tilde{\Gamma}_1} \cup \overline{\tilde{\Gamma}_2}$.)

Remark. The definition of generalized shock wave allows nontrivial cycles of orbits, when $\tilde{U}_m$ coincides with $\tilde{U}_n$ for some $n \geqslant m+2$ and the union of the orbits $\tilde{\Gamma}_{m+1},..., \tilde{\Gamma}_n$ is not simply $\{\tilde{U}_m\}$. For example, a 2-cycle of shock waves can occur in a codimension-one Riemann solution [1]. However, for simplicity, and in keeping with Ref. 17, we do not consider shock waves with homoclinic orbits (e.g., joining a saddle point to itself or a repeller-saddle to itself). Of course, there is no distinction between trivial orbits and homoclinic orbits or cycles on the level of the weak solution $\hat{U}(\xi)$.

Associated with each generalized rarefaction or shock wave is a speed s, defined as before, and a curve $\bar{\Gamma}$: the rarefaction curve or the shock profile $\overline{\tilde{\Gamma}_1} \cup ... \cup \overline{\tilde{\Gamma}_k}$. The type of a generalized wave is also defined as before. In particular, the type of a generalized shock wave is defined by the equilibrium types of U_- and U_+ and is unrelated to the equilibrium types of any of the intermediate equilibria $\tilde{U}_1, \tilde{U}_2, ..., \tilde{U}_{k-1}$.

A *generalized allowed wave sequence* is a sequence of generalized rarefaction and shock waves that satisfies conditions (W1)–(W3). If $U_0 = U_L$ and $U_n = U_R$, then associated with a generalized allowed wave sequence $(w_1, w_2, ..., w_n)$ is a Riemann solution U of Eqs. (1.1) and (1.2). Therefore we shall refer to a generalized allowed wave sequence as a *Riemann solution.*

In a generalized allowed wave sequence, rarefaction and shock waves may have zero strength, and adjacent shock waves may have the same speed. This permits us to represent a Riemann solution by different wave sequences. Wave sequences representing the same Riemann solution are viewed as equivalent. More precisely, we make the following definitions.

A generalized allowed wave sequence is *minimal* if

1. there are no rarefaction or shock waves of zero strength;
2. no two successive shock waves have the same speed.

Among the minimal generalized allowed wave sequences we include sequences with no waves; such a sequence is given by a single $U_0 \in \mathbb{R}^2$ and represents a constant solution of Eq. (1.1). We *shorten* a generalized allowed wave sequence by omitting a rarefaction or shock wave of zero strength and by amalgamating adjacent shock waves of nonzero strength with the same speed. Every generalized allowed wave sequence can be shortened to a unique minimal generalized allowed wave sequence. Two generalized allowed wave sequences are *equivalent* if their minimal shortenings are the same. Equivalent generalized allowed wave sequences represent the same weak solution $U(x, t) = \hat{U}(x/t)$ of Eq. (1.1).

Remark. Equivalence of two generalized allowed wave sequences does not necessarily follow from equality of the corresponding weak solutions in L^1_{loc}. Indeed, when two shock waves are equivalent, the corresponding orbit sets $\bar{\Gamma} = \overline{\tilde{\Gamma}_1} \cup \cdots \cup \overline{\tilde{\Gamma}_k}$ coincide. For example, a shock wave with $U_- = U_+$ and a sequence of orbits forming a homoclinic cycle is not regarded as equivalent to a zero strength shock wave, which by definition has a trivial orbit.

3.2. Codimension-One Riemann Solutions

A generalized allowed wave sequence (2.9) is a *codimension-one Riemann solution* provided that there is a sequence of wave types

$(T_1^*, ..., T_n^*)$ with $\sum_{i=1}^n \rho(T_i^*) = 2$, neighborhoods $\mathcal{U}_i \subseteq \mathcal{U}$ of U_i^*, $\mathcal{I}_i \subseteq I$ of s_i^*, and $\mathcal{F} \subseteq \mathcal{B}$ of F^*, and a C^1 map

$$(G, H): \mathcal{U}_0 \times \mathcal{I}_1 \times \cdots \times \mathcal{I}_n \times \mathcal{U}_n \times \mathcal{F} \to \mathbb{R}^{3n-2} \times \mathbb{R} \tag{3.2}$$

with $G(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*) = 0$ and $H(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*) = 0$ such that the following conditions, (Q1)–(Q7), are satisfied.

(Q1) If $G(U_0, s_1, ..., s_n, U_n, F) = 0$ and $H(U_0, s_1, ..., s_n, U_n, F) \geqslant 0$, then there is a generalized allowed wave sequence

$$(w_1, w_2, ..., w_n) : U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n \tag{3.3}$$

for $U_t + F(U)_x = 0$ with each w_i contained in Int K and each rarefaction wave contained in $\mathcal{U}$;

(Q2) if $G(U_0, s_1, ..., s_n, U_n, F) = 0$ and $H(U_0, s_1, ..., s_n, U_n, F) > 0$, then $(w_1, w_2, ..., w_n)$ is a structurally stable Riemann solution of type $(T_1^*, ..., T_n^*)$ and G exhibits its structural stability;

(Q3) if $G(U_0, s_1, ..., s_n, U_n, F) = 0$ and $H(U_0, s_1, ..., s_n, U_n, F) = 0$, then $(w_1, w_2, ..., w_n)$ is not equivalent to a structurally stable Riemann solution;

(Q4) $D(G, H)(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*)$, restricted to some $(3n-1)$-dimensional space of vectors that contains $\{(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, ..., \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = 0 = \dot{U}_n, \dot{F} = 0\}$, is an isomorphism.

Condition (Q4) implies, by the implicit function theorem, that $\mathcal{M} := (G, H)^{-1}(\{0\} \times \mathbb{R}^+)$ is a manifold-with-boundary of codimension $3n-2$ within $\mathcal{U}_0 \times \mathcal{I}_1 \times \cdots \times \mathcal{I}_n \times \mathcal{U}_n \times \mathcal{F}$, and that $\partial\mathcal{M} = (G, H)^{-1}(0, 0)$ is a graph over a codimension-one manifold $\mathcal{S}$ in $\mathcal{U}_0 \times \mathcal{U}_n \times \mathcal{F}$. As before, we can define maps $\bar{\Gamma}_i: \mathcal{M} \to \mathcal{H}(\text{Int } K)$ giving the rarefaction curve or closure of the union of orbits for the corresponding wave. We require further that

(Q5) $(w_1, w_2, ..., w_n)$ can be chosen so that each map $\bar{\Gamma}_i$ is continuous.

As an additional condition, the surface $\mathcal{S}$ is required to be regularly situated with respect to the foliation of (U_0, U_n, F)-space by planes of constant (U_0, F) and planes of constant (U_n, F). More precisely, let

$$\Sigma_0 = \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = 0 \quad \text{and} \quad \dot{F} = 0\},$$

$$\Sigma_n = \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_n = 0 \text{ and } \dot{F} = 0\}.$$

Then we require that one of the following hold:

(Q6$_1$) $D(G, H)(U_0^*, s_1^*,..., s_n^*, U_n^*, F^*)$ restricted to Σ_0 and to Σ_n, respectively, are surjective;

(Q6$_2$) $D(G, H)(U_0^*, s_1^*,..., s_n^*, U_n^*, F^*)$ restricted to Σ_n is surjective, and there is a codimension-one manifold $\tilde{\mathscr{S}}$ through (U_0^*, F^*) in (U_0, F)-space such that $\mathscr{S} = U_n \times \tilde{\mathscr{S}}$;

(Q6$_3$) $D(G, H)(U_0^*, s_1^*,..., s_n^*, U_n^*, F^*)$ restricted to Σ_0 is surjective, and there is a codimension-one manifold $\tilde{\mathscr{S}}$ through (U_n^*, F^*) in (U_n, F)-space such that $\mathscr{S} = \mathscr{U}_0 \times \tilde{\mathscr{S}}$;

(Q6$_4$) there is a codimension-one manifold $\tilde{\mathscr{S}}$ through F^* in F-space such that $\mathscr{S} = U_0 \times \mathscr{U}_n \times \tilde{\mathscr{S}}$.

When (Q6$_1$), (Q6$_2$) or (Q6$_3$), or (Q6$_4$) holds, then the codimension-one Riemann solution is termed an *intermediate boundary*, a *U_L-boundary* or dual, or an *F-boundary*, respectively.

Finally, we require one of the following conditions to hold:

(Q7$_1$) the linear map

$$DG(U_0^*, s_1^*,..., s_n^*, U_n^*, F^*) \qquad \text{restricted to} \quad \Sigma_0 \cap \Sigma_n \tag{3.6}$$

is an isomorphism. (In this case, $\mathscr{M}$ is a smooth graph over a manifold-with-boundary in $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$ with boundary $\mathscr{S}$, consisting of $\mathscr{S}$ and an open set on one side of $\mathscr{S}$.)

(Q7$_2$) the linear map (3.6) is not surjective, but the projection of $G^{-1}(0)$ to $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$ has a fold along $(G, H)^{-1}(0, 0)$. [In this case, $\mathscr{M}$ is again a graph over a manifold-with-boundary in $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$ with boundary $\mathscr{S}$, but the mapping $(U_0, U_n, F) \mapsto (s_1, U_1,..., U_{n-1}, s_n)$ loses smoothness along $\mathscr{S}$.]

More precisely, condition (Q7$_2$) means that: (i) the kernel of the linear map (3.6) is one-dimensional, spanned by a vector V; and (ii) if ℓ is a nonzero left null vector of the linear map (3.6), then for some $a \neq 0$, $\ell G((U_0^*, s_1^*,..., s_n^*, U_n^*, F^*) + tV) = at^2 + o(t^2)$ as $t \to 0$.

3.3. Folds, Frontiers, and Joins

Let $(U_0^*, s_1^*, U_1^*, s_2^*,..., s_n^*, U_n^*, F^*)$ be a generalized allowed wave sequence that is a codimension-one Riemann solution of type $(T_1^*,..., T_n^*)$. Let $\mathscr{M}$ denote the associated manifold-with-boundary, $\partial\mathscr{M}$ being a graph over the manifold $\mathscr{S}$.

1. Suppose that $(U_0^*, s_1^\sharp, U_1^\sharp, s_2^\sharp, ..., s_m^\sharp, U_m^*, F^*)$ is an equivalent generalized allowed wave sequence that is a codimension-one Riemann solution of type $(T_1^\sharp, ..., T_m^\sharp)$, with associated manifold-with-boundary $\mathcal{N}$. Suppose further that $\partial\mathcal{N}$ is a graph over the same manifold $\mathcal{S}$ as is $\partial\mathcal{M}$, and that the points in $\partial\mathcal{M}$ and $\partial\mathcal{N}$ above the same point in $\mathcal{S}$ are equivalent. Then $\mathcal{S}$ is called a *Riemann solution join*, and the structurally stable Riemann solutions of types $(T_1^*, ..., T_n^*)$ and $(T_1^\sharp, ..., T_m^\sharp)$ are said to be *adjacent.*

2. Suppose that there is no other generalized allowed wave sequence that is equivalent to $(U_0^*, s_1^*, U_1^*, s_2^*, ..., s_n^*, U_n^*, F^*)$. Then $\mathcal{S}$ is called a *Riemann solution frontier.*

3. Suppose that (a) the linear map (3.6) is not surjective; (b) the generalized wave sequence of condition (Q1) exists whenever $G(U_0, s_1, ..., s_n, U_n, F) = 0$ (no matter what the value of $H(U_0, s_1, ..., s_n, U_n, F)$ is); (c) this sequence is a structurally stable Riemann solution of type $(T_1^*, ..., T_n^*)$ whenever $H(U_0, s_1, ..., s_n, U_n, F) \neq 0$, and G exhibits its structural stability; and (d) each map $\bar{\Gamma}_i\colon G^{-1}(0) \to \mathcal{H}(\text{Int } K)$, giving the rarefaction curve or closure of the union of orbits, can be chosen to be continuous. Then $\mathcal{S}$ is called a *Riemann solution fold.*

4. VIOLATION OF THE NONDEGENERACY CONDITIONS OF THE STRUCTURAL STABILITY THEOREM

We now discuss violations of the hypotheses of the Structural Stability Theorem: (1) wave nondegeneracy conditions; (2) the wave group interaction condition; and (3) the requirement that, when a $* \cdot S$ shock wave is followed by an $S \cdot *$ shock wave, the shock speeds differ. We identify the violations of these conditions that lead to loss of strict hyperbolicity or phenomena of codimension greater that one, which we do not consider further, and we label the remaining cases, which we expect to give rise to codimension-one Riemann solutions.

Except in three cases, which involve failure of genuine nonlinearity at a point of a rarefaction wave, the map G in definition (3.2) is the defining map that would be used if the nondegeneracy condition held and the Riemann solution were structurally stable. [This would be true as well of the three exceptions if Eqs. (4.2) and (4.3) below were used instead of Eq. (2.12) as the defining equation for a rarefaction wave.] We refer to the equation $H = 0$ as the "extra equation." We give the equation $H = 0$ for each situation, but we do not attempt in this paper to verify conditions (Q4)–(Q7) for each situation. In seven cases, the description of the extra equation is a little more involved than in the others; these cases are treated in Appendix B.

Throughout this section we consider the generalized allowed wave sequence

$$(w_1, w_2, \ldots, w_n) : U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n \tag{4.1}$$

4.1. Wave Nondegeneracy Conditions

We omit fast wave groups from the discussion, since they correspond to slow wave groups under the duality (2.32). Therefore we discuss the nondegeneracy conditions only for the elementary waves appearing in 1-wave groups and slow transitional wave groups, and for doubly sonic transitional waves: R_1, $R \cdot S$, $R \cdot RS$, $RS \cdot S$, $RS \cdot RS$, $S \cdot S$, $S \cdot RS$, and $SA \cdot RS$.

4.1.1. R_1 Rarefaction Waves

The nondegeneracy conditions are that (1) the interval of definition of the rarefaction wave has positive length and (2) genuine nonlinearity of the first characteristic field holds at each point of the rarefaction curve. The violations of these conditions therefore divide into two groups.

4.1.1.1. Rarefaction Waves of Zero Strength. The nature of a rarefaction wave of zero strength depends on the types of the waves that precede and follow it. A rarefaction wave of type R_1 in a structurally stable Riemann solution either is the initial wave in its wave group or has a predecessor of type $R \cdot RS$, $RS \cdot RS$, $S \cdot RS$, or $SA \cdot RS$. Similarly, it either is the last wave in its wave group or has a successor of type $RS \cdot RS$ or $RS \cdot S$. There are thus $5 \times 3 = 15$ cases in which an R_1 rarefaction wave in a structurally stable Riemann solution can shrink to zero strength. Each is a codimension-one phenomenon. If the rarefaction wave that shrinks to zero strength is w_i: $U_{i-1} \xrightarrow{s_i} U_i$, then for the extra equation $H = 0$ we may use $s_i - \lambda_1(U_{i-1}) = 0$.

In Table VI we indicate how we label four of the cases in which there is a rarefaction wave of zero strength. The other 11 cases are labeled analogously.

4.1.1.2. Failure of Genuine Nonlinearity. Near a Riemann solution that contains a rarefaction wave for which genuine nonlinearity fails, the defining Eq. (2.14) cannot be used. Instead we proceed as follows. If there is a rarefaction wave of type R_1 from U_-^* to U_+^*, these points are joined by an integral curve for the line field associated with family 1, i.e., the line

Table VI. Labeling of Four Cases in Which There is a Rarefaction Wave of Zero Strength

Predecessor	Successor	Label
None	None	0
None	$RS \cdot S$	$0 \ RS \cdot S$
$R \cdot RS$	None	$R \cdot RS \ 0$
$R \cdot RS$	$RS \cdot S$	$R \cdot RS \ 0 \ RS \cdot S$

field of null directions for $DF(U) - \lambda_1(U)\,I$. For a suitable neighborhood $\mathcal{N}$ of this integral curve, we may choose a smooth map $\eta_1 : \mathcal{N} \to \mathbb{R}$ such that $D\eta_1 \neq 0$ and the level sets of η_1 are the integral curves of the 1-family line field. (The map η_1 is known as a Riemann invariant.) Then there is a rarefaction wave of type R_1 with speed s from a point U_- near U_-^* to a point U_+ near U_+^* if and only if

$$\eta_1(U_+) - \eta_1(U_-) = 0 \tag{4.2}$$

$$\lambda_1(U_+) - s = 0 \tag{4.3}$$

$\lambda_1(U)$ is strictly increasing along the level curve

$$\eta_1(U) = \eta_1(U_-) \text{ from } U_- \text{ to } U_+ \tag{4.4}$$

Equations (4.2) and (4.3) serve in place of Eq. (2.14) as defining equations for rarefaction waves of type R_1.

We may assume that the left and right eigenvectors $\ell_1(U)$ and $r_1(U)$ of $DF(U)$ depend smoothly on U for $U \in \mathcal{N}$. An integral curve of the 1-family line field may then be parameterized by $U = \check{U}(\tau)$, where $\check{U}'(\tau) = r_1(\check{U}(\tau))$. Let $\check{\lambda}_1(\tau) = \lambda_1(\check{U}(\tau))$, so that $\check{\lambda}_1'(\tau) = D\lambda_1(\check{U}(\tau))\,\check{U}'(\tau) = D\lambda_1(U)\,r_1(U)$.

Genuine nonlinearity of the 1-family characteristic line field fails at points U where

$$D\lambda_1(U)\,r_1(U) = 0 \tag{4.5}$$

This equation typically defines a curve $\mathcal{I}$ in $\mathcal{U}$ (called the *inflection locus*). When an integral curve of the 1-family line field meets $\mathcal{I}$, say at $U^* = \check{U}(\tau^*)$, we have

$$\check{\lambda}_1'(\tau^*) = 0 \tag{4.6}$$

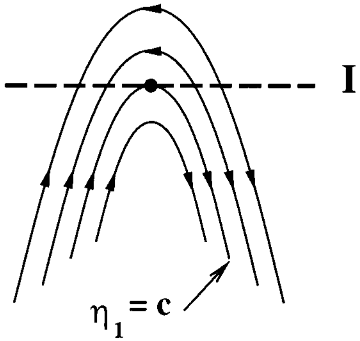

Fig. 9. Rarefaction waves of type R_1 near the inflection locus. The inflection locus $\mathscr{I}$ is dashed; curves on which η_1 is constant are solid. Arrows indicate the directions of increasing λ_1. There is one curve, $\eta_1 = c$, that has a quadratic tangency with $\mathscr{I}$.

The integral curve is transverse to $\mathscr{I}$ provided that

$$\check{\lambda}_1''(\tau^*) = D^2\lambda_1(U^*)(r_1(U^*), r_1(U^*)) + D\lambda_1(U^*)\, Dr_1(U^*)\, r_1(U^*) \neq 0 \tag{4.7}$$

Typically, integral curves are transverse to $\mathscr{I}$ except at isolated points of $\mathscr{I}$, where $\mathscr{I}$ has a quadratic tangency with an integral curve, as in Fig. 9. At such a point, we have Eq. (4.6) and the conditions

$$\check{\lambda}_1''(\tau^*) = 0 \tag{4.8}$$

$$\check{\lambda}_1'''(\tau^*) \neq 0 \tag{4.9}$$

Notice that if a rarefaction wave begins at a point on $\mathscr{I}$, then $\check{\lambda}_1''(\tau^*) \geqslant 0$ at this point; and if a rarefaction wave ends at a point on $\mathscr{I}$, then $\check{\lambda}_1''(\tau^*) \leqslant 0$ at this point.

Genuine nonlinearity can fail at either the initial point, the terminal point, or an interior point of a rarefaction wave w_i: $U_{i-1} \xrightarrow{s_i} U_i$. Each possibility gives rise to a codimension-one phenomenon for R_1 rarefaction waves.

1. *Initial point*: As the extra equation $H = 0$, we may use the equation expressing that U_{i-1} lies in $\mathscr{I}$: $D\lambda_1(U_{i-1})\, r_1(U_{i-1}) = 0$. A necessary nondegeneracy condition is the inequality $D^2\lambda_1(U_{i-1})(r_1(U_{i-1}), r_1(U_{i-1})) + D\lambda_1(U_{i-1})\, Dr_1(U_{i-1})\, r_1(U_{i-1}) > 0$.
2. *Terminal point*: As the extra equation $H = 0$, we may use $D\lambda_1(U_i)\, r_1(U_i) = 0$. The inequality $D^2\lambda_1(U_i)(r_1(U_i), r_1(U_i)) + D\lambda_1(U_i)\, Dr_1(U_i)\, r_1(U_i) < 0$ is needed as a nondegeneracy condition.

3. *Interior point*: To have a codimension-one Riemann solution, the rarefaction curve $\eta_1(U)=c$ must have a quadratic tangency with $\mathscr{I}$, as in Fig. 9. For the extra equation $H=0$ we can use $\eta_1(U_{i-1})-c=0$. At the point of quadratic tangency, we have Eqs. (4.6) and (4.8) together with the necessary nondegeneracy condition (4.9).

Remark. Rarefaction waves of zero strength and of types $R_1(1)$, $R_1(2)$, and $R_1(3)$ above satisfy the definition of generalized rarefaction wave in Section 3, but not the definition of rarefaction wave in Section 2.

4.1.2. Shock Waves: Generalities

The nondegeneracy conditions for a shock wave w: $U_- \xrightarrow{s} U_+$ are that (1) $U_- \neq U_+$; (2) the inequalities on the eigenvalues of $DF(U_\pm)$ and s implicit in the definition of the shock type; (3) the existence of a connection for Eq. (1.4) from U_- to U_+; and (4) the nondegeneracy conditions listed in Tables II–IV.

The sense in which condition (3) is a nondegeneracy condition is as follows. We regard the existence of an orbit from U_- to U_+ as comprising two parts:

$$U_- \text{ is joined to } U_+ \text{ by a closed invariant curve } \bar{\Gamma} \text{ of Eq. (1.4)} \tag{C0$_a$}$$

$$\text{the vector field (1.4) is nonzero on } \bar{\Gamma} \text{ except at } U_- \text{ and } U_+ \tag{C0$_b$}$$

For classical shock waves, condition (C0$_a$) is an open condition. Nevertheless, we do not consider it to be a nondegeneracy condition because, if it were violated, then w would not be a generalized shock wave. For transitional shock waves, condition (C0$_a$) is a defining equation; again, it is not a nondegeneracy condition. Condition (C0$_b$), in contrast, is a nondegeneracy condition for both classical and transitional shock waves. Therefore, by the nondegeneracy condition (3), we mean condition (C0$_b$).

Violation of condition (1) gives rise, by definition, to a zero-strength shock wave. Violation of condition (2) or (4) does not require the introduction of a generalized shock wave, but usually leads to a change of shock type. Violation of condition (3) [i.e., condition (C0$_b$)] entails that the shock wave becomes a generalized shock wave represented by a sequence of connecting orbits. To have a codimension-one Riemann solution, there must be two orbits, one joining the left state to an intermediate state and the other joining the intermediate state to the right state. We describe this situation by saying that the connection is broken by the intermediate equilibrium. If a shock wave of type $E_- \cdot E_+$ is broken by an equilibrium of type E, then the resulting generalized shock wave is represented by a $E_- \cdot E$ connection followed by a $E \cdot E_+$ connection. Notice that the intermediate equilibrium must be of type *RS*, *SA*, or *S*, since types *R* and *A* are obviously excluded.

Remark. A shock profile from U_- to U_+ can, of course, be broken by other mechanisms: the orbit can escape to infinity (giving rise to a delta shock wave [9, 12, 22] or a singular shock wave [7, 13]), or it can be broken by a nonhyperbolic limit cycle. In these cases, the states U_- and U_+ are not joined by a sequence of orbits and, hence, do not define a generalized shock wave. Therefore, these mechanisms do not lead to codimension-one Riemann solutions as we have defined them.

Certain violations of nondegeneracy conditions are expected to cause phenomena of codimension higher than one, which we shall exclude from consideration. In deciding which violations to exclude, we make use of the following observations.

Lemma 4.1. *If the connection for a slow shock wave is broken by a saddle-attractor, then the total Riemann number for the wave sequence is reduced from 2 to 0.*

Proof. If the connection for an $R \cdot S$ shock wave, which has Riemann number 1, is broken by a saddle-attractor, then it is replaced in the wave sequence by $R \cdot SA$ and $SA \cdot S$ shock waves, which have Riemann numbers 0 and -1, respectively. Therefore the total Riemann number is reduced by 2. Similarly reductions occur when the connections for $R \cdot RS$, $RS \cdot S$, and $RS \cdot RS$ shock waves are broken by saddle-attractors. □

Wave sequences with total Riemann number zero are expected to give rise to phenomena of codimension at least two.

Next we relate the differential equation on the center manifold for a repeller-saddle to the behavior of the 1-family eigenvalue along the rarefaction curve. For this purpose, we introduce the following notation.

(a) Let U^* be an equilibrium of the differential equation

$$\dot{U} = F(U) - F(U_-) - s^*(U - U_-) \tag{4.10}$$

with

$$s^* = \lambda_1(U^*) \tag{4.11}$$

The eigenvalues of the linearization of Eq. (4.10) at U^* are 0 and $\lambda_2(U^*) - \lambda_1(U^*) > 0$. Each local center manifold of Eq. (4.10) at U^* can be parameterized by a curve $U_c(\sigma)$, where we may assume that

$$U_c(0) = U^* \tag{4.12}$$

$$\ell_1(U^*)[U_c(\sigma) - U^*] = \sigma \tag{4.13}$$

Equation (4.13) fixes the parameterization locally and implies that

$$U_c'(0) = r_1(U^*) \tag{4.14}$$

since $r_1(U^*)$ is tangent to the center manifold at U^*. Let the differential equation (4.10) on the local center manifold at U^* be denoted

$$\dot{\sigma} = g(\sigma) \tag{4.15}$$

(b) Let $\check{U}(\tau)$ be an integral curve of the 1-family line field with $\check{U}(0) = U^*$. We may assume that

$$\ell_1(U^*)[\check{U}(\tau) - U^*] = \tau \tag{4.16}$$

near $\tau = 0$. Equation (4.16) fixes the parameterization locally and implies that

$$\check{U}'(0) = r_1(U^*) \tag{4.17}$$

Define $\check{\lambda}_i(\tau) = \lambda_i(\check{U}(\tau))$, $i = 1, 2$.

Lemma 4.2. *Suppose that $k \geqslant 1$ is such that $D^i\check{\lambda}_1(0) = 0$ for $i = 1, \ldots, k-1$ and $D^k\check{\lambda}_1(0) \neq 0$. Then $D^i g(0) = 0$ for $i = 0, \ldots, k$ and $D^{k+1}g(0) \neq 0$; in fact, $D^{k+1}g(0) = D^k\check{\lambda}_1(0)$.*

The proof of this lemma is given in Appendix A.

Lemma 4.3. *Let $w_i \colon U_{i-1} \xrightarrow{s_i} U_i$ be a shock wave of type $* \cdot RS$ followed by a 1-family rarefaction wave w_{i+1}. Suppose that $D\lambda_1(U_i)\, r_1(U_i) = 0$ but $D^2\lambda_1(U_i)\,(r_1(U_i), r_1(U_i)) + D\lambda_1(U_i)\, Dr_1(U_i)\, r_1(U_i) \neq 0$. Then either w_i or w_{i+1} has zero strength.*

The analogous statement holds for a shock wave of type $RS \cdot *$ preceded by a 1-family rarefaction wave.

Proof. Consider the differential equation

$$\dot{U} = F(U) - F(U_{i-1}) - s_i(U - U_{i-1}) \tag{4.18}$$

near the equilibrium $U = U_i$. According to Eqs. (4.6) and (4.7), the hypotheses of Lemma 4.2 hold for this differential equation with $k = 2$. Therefore, the corresponding differential equation on the center manifold of U_i is $\dot{\sigma} = b\sigma^3 + \cdots$, where $b \neq 0$ has the same sign as does the quantity $D^2\lambda_1(U_i)(r_1(U_i), r_1(U_i)) + D\lambda_1(U_i)\, Dr_1(U_i)\, r_1(U_i)$. If $U_{i-1} \neq U_i$, then because the shock wave has a connection, we must have $b < 0$ (see Figs. 3b and c). Therefore no 1-family rarefaction wave can begin at U_i. □

For example, if condition (G1) is violated for an $R \cdot RS$ wave, then another nondegeneracy condition must be violated simultaneously: either this shock wave has zero strength, the rarefaction wave following it has zero strength, or U_i is a degenerate point on the inflection locus. Therefore we expect this phenomenon to have codimension at least two.

We now discuss the nondegeneracy conditions for the various shock types.

4.1.3. $R \cdot S$ Shock Waves

The $R \cdot S$ shock wave must be the first wave, w_1: $U_0 \xrightarrow{s_1} U_1$. The nondegeneracy conditions are $U_0 \neq U_1$, $\lambda_1(U_0) > s_1$, $\lambda_2(U_1) > s_1 > \lambda_1(U_1)$, and ($C0_b$). Breaking of the connection from U_0 to U_1 by a saddle-attractor is expected to have codimension at least two, according to Lemma 4.1. Therefore the codimension-one phenomena are as follows.

(1) $U_0 = U_1$: the repeller and saddle point coalesce, forming a repeller-saddle.

(2) $\lambda_1(U_0) = s_1$: the shock type of w_1 becomes $RS \cdot S$.

(3) $\lambda_2(U_1) = s_1$: the shock type becomes $R \cdot SA$ with the connection being distinguished.

(4) $\lambda_1(U_1) = s_1$: the shock type becomes $R \cdot RS$.

(5) The connection from U_0 to U_1 is broken by a repeller-saddle.

(6) The connection is broken by a saddle.

In cases (1)–(4), the extra equations $H = 0$ are (1) $\lambda_1(U_0) - s_1 = 0$, (2) $\lambda_1(U_0) - s_1 = 0$, (3) $\lambda_2(U_1) - s_1 = 0$, and (4) $s_1 - \lambda_1(U_1) = 0$, respectively. In case (3), the connection is distinguished because it arises as the limit of $R \cdot S$ connections. In case (6), the extra equation is that a separation function for two saddles vanishes. To find the extra equation in case (5), let $(U_0, s_1, U_1, s_2, \ldots, s_n, U_n, F)$ lie in the boundary of the set of structurally stable Riemann solutions of type $(R \cdot S, T_2, ..., T_n)$, with the connection from U_0 to U_1 broken by an equilibrium $\bar{U}$ of type RS for the equation

$$\dot{U} = F(U) - F(U_0) - s_1(U - U_0) \tag{4.19}$$

Once such an equilibrium exists, the condition that there be a connection from U_0 to $\bar{U}$, and another from $\bar{U}$ to U_1, is an open condition. Such repeller-saddle equilibria exist along a codimension-one surface in (U_0, s_1)-space, across which Eq. (4.19) undergoes a saddle-node bifurcation. The extra function H, which depends only on (U_0, s_1), is defined to have this saddle-node bifurcation surface as its zero set.

4.1.4. $R \cdot RS$ Shock Waves

The $R \cdot RS$ shock wave must be the first wave, $w_1 \colon U_0 \xrightarrow{s_1} U_1$. The nondegeneracy conditions are $U_0 \neq U_1$, $\lambda_1(U_0) > s_1$, $\lambda_2(U_1) > s_1$, (C0$_b$), (G1), and (B1). Since $\lambda_1(U_1) = s_1$, violation of $\lambda_2(U_1) > s_1$ would entail that $\lambda_2(U_1) = \lambda_1(U_1)$, i.e., U_1 would not belong to the strictly hyperbolic region; we therefore exclude this possibility. Breaking of the connection from U_0 to U_1 by a saddle-attractor is expected to have codimension at least two, according to Lemma 4.1. Violation of condition (G1) when $U_0 \neq U_1$ is also expected to have codimension at least two, according to Lemma 4.3. Therefore the codimension-one phenomena are as follows.

(1) $U_0 = U_1$: the repeller and repeller-saddle coalesce, forming a repeller-saddle with equation $\dot{\sigma} = b\sigma^3 + \cdots$, where $b > 0$, on its center manifold (see Fig. 3c). Thus violation of $U_0 \neq U_1$ entails violation of condition (G1).

(2) $\lambda_1(U_0) = s_1$: the shock type of w_1 becomes $RS \cdot RS$.

(3) The connection from U_0 to U_1 is broken by a repeller-saddle.

(4) The connection is broken by a saddle.

(5) Condition (B1) is violated: the appropriate nondegeneracy condition here is that as s varies in the equation $\dot{U} = F(U) - F(U_0) - s(U - U_0)$, a transcritical bifurcation [3] (instead of a saddle-node bifurcation) occurs at $s = s_1$, $U = U_1$.

In cases (1)–(3), the extra equation $H = 0$ is similar to ones already discussed. In case (4), the extra equation says that the separation between the unstable manifold of the intermediate saddle $\bar{U}$ and the center manifold of U_1 vanishes. In case (5), it is $l_1(U_1)(U_1 - U_0) = 0$.

4.1.5. $RS \cdot S$ Shock Waves

Let the $RS \cdot S$ shock wave be $w_i \colon U_{i-1} \xrightarrow{s_i} U_i$. The nondegeneracy conditions are $U_{i-1} \neq U_i$, $\lambda_2(U_{i-1}) > s_i$, $\lambda_2(U_i) > s_i > \lambda_1(U_i)$, (C0$_b$), (G2), and (C1). Violation of $\lambda_2(U_{i-1}) > s_i$ leads to loss of strict hyperbolicity. Breaking of the connection from U_{i-1} to U_i by a saddle-attractor is expected to have codimension at least two, according to Lemma 4.1. Violation of condition (G2) when $U_{i-1} \neq U_i$ is also expected to have codimension at least two, according to Lemma 4.3. Therefore the codimension-one phenomena are as follows.

(1) $U_{i-1} = U_i$: the equilibria coalesce, forming an equilibrium with equation $\dot{\sigma} = b\sigma^3 + \cdots$, where $b < 0$, on its center manifold (see Fig. 3b). Thus violation of $U_{i-1} \neq U_i$ entails violation of condition (G2).

(2) $\lambda_1(U_i) = s_i$: the shock type of w_i becomes $RS \cdot RS$.

(3) $\lambda_2(U_i) = s_i$: the shock type becomes $RS \cdot SA$ with the connection being distinguished.

(4) The connection from U_{i-1} to U_i is distinguished.

(5) The connection is broken by a repeller-saddle.

(6) The connection is broken by a saddle.

In cases (1)–(3), (5), and (6), the extra equations are similar to ones already discussed. In case (4), the extra equation says that the separation between the unstable manifold of U_{i-1} and the stable manifold of U_i vanishes.

4.1.6. RS · RS Shock Waves

Let the $RS \cdot RS$ shock wave be w_i: $U_{i-1} \xrightarrow{s_i} U_i$. The nondegeneracy conditions are $U_{i-1} \neq U_i$, $\lambda_2(U_{i-1}) > s_i$, $\lambda_2(U_i) > s_i$, ($C0_b$), (G3), (G4), (B2), and (C2). Violation of $\lambda_2(U_{i-1}) > s_i$ or $\lambda_2(U_i) > s_i$ leads to loss of strict hyperbolicity. Breaking of the connection from U_{i-1} to U_i by a saddle-attractor is expected to have codimension at least two, according to Lemma 4.1. Violation of either condition (G3) or condition (G4) when $U_{i-1} \neq U_i$ is also expected to have codimension at least two, according to Lemma 4.3. Therefore the codimension-one phenomena are as follows.

(1) $U_{i-1} = U_i$: the equilibria coalesce, forming an equilibrium with equation $\dot{\sigma} = b\sigma^4 + \cdots$, where $b \neq 0$, on its center manifold (see Fig. 3a). Thus violation of $U_{i-1} \neq U_i$ entails violation of conditions (G3) and (G4).

(2) Condition (B2) is violated: as in Section 4.1.4, the appropriate nondegeneracy condition here is that as s varies in $\dot{U} = F(U) - F(U_{i-1}) - s(U - U_{i-1})$, a transcritical bifurcation (instead of a saddle-node bifurcation) occurs at $s = s_i$, $U = U_i$.

(3) The connection from U_{i-1} to U_i is distinguished.

(4) The connection is broken by a repeller-saddle.

(5) The connection from by a saddle.

In case (1), the extra equation is $D\lambda_1(U_{i-1})\, r_1(U_{i-1}) = 0$. In cases (2)–(5), the extra equations are similar to ones already discussed.

4.1.7. $S \cdot S$ Shock Waves

Let the $S \cdot S$ shock wave be w_i: $U_{i-1} \xrightarrow{s_i} U_i$. The nondegeneracy conditions are $U_{i-1} \neq U_i$, $\lambda_2(U_{i-1}) > s_i > \lambda_1(U_{i-1})$, $\lambda_2(U_i) > s_i > \lambda_1(U_i)$, ($C0_b$), and (T1). Violation of $U_{i-1} \neq U_i$ would entail violation of strict hyperbolicity. Since violation of condition (T1) requires that all entries of a 3-component row vector vanish, it is expected to have codimension at least three. Therefore the codimension-one phenomena are as follows.

(1) $\lambda_1(U_{i-1}) = s_i$: w_i becomes an $RS \cdot S$ shock wave, which necessarily has a distinguished connection. The degeneracy $\lambda_2(U_i) = s_i$, which gives rise to an $S \cdot SA$ shock with distinguished connection, is dual to this case.

(2) $\lambda_1(U_i) = s_i$: the shock type of w_i becomes $S \cdot RS$. The degeneracy $\lambda_2(U_{i-1}) = s_i$, which gives rise to an $SA \cdot S$ shock wave, is dual to this case.

(3) The connection from U_{i-1} to U_i is broken by a repeller-saddle. Breaking the connection by a saddle-attractor is dual to this case.

(4) The connection is broken by a saddle.

In cases (1) and (2), the extra equations are similar to ones already discussed. Cases (3) and (4) are discussed in Appendix B.

4.1.8. $S \cdot RS$ Shock Waves

Let the $S \cdot RS$ shock wave be w_i: $U_{i-1} \xrightarrow{s_i} U_i$. The nondegeneracy conditions are $U_{i-1} \neq U_i$, $\lambda_2(U_{i-1}) > s_i > \lambda_1(U_{i-1})$, $\lambda_2(U_i) > s_i$, ($C0_b$), (G13), and (T2). Violation of $U_{i-1} \neq U_i$ would entail violation of strict hyperbolicity. Violation of condition (G13) when $U_{i-1} \neq U_i$ is expected to have codimension at least two, according to Lemma 4.3. Violation of condition (T2) requires a 2×3 matrix to have rank 1, which should be a phenomenon of codimension at least two. Therefore the codimension-one phenomena are as follows.

(1) $\lambda_1(U_{i-1}) = s_i$: w_i becomes an $RS \cdot RS$ shock wave, which necessarily has a distinguished connection.

(2) $\lambda_2(U_{i-1}) = s_i$: the shock type of w_i becomes $SA \cdot RS$.

(3) The connection from U_{i-1} to U_i is broken by a repeller-saddle.
(4) The connection is broken by a saddle-attractor.
(5) The connection is broken by a saddle.

In cases (1) and (2), the extra equations are similar to ones already discussed. Cases (3)–(5) are discussed in Appendix B.

4.1.9. $SA \cdot RS$ Shock Waves

Let the $SA \cdot RS$ shock wave be w_i: $U_{i-1} \xrightarrow{s_i} U_i$. The nondegeneracy conditions are $U_{i-1} \neq U_i$, $s_i > \lambda_1(U_{i-1})$, $\lambda_2(U_i) > s_i$, (C0$_b$), (G15), (G16), and (T4). Violation of one of the first three conditions leads to loss of strict hyperbolicity. Violation of either condition (G15) or condition (G16) when $U_{i-1} \neq U_i$ is expected to have codimension at least two, according to Lemma 4.3. Therefore the codimension-one phenomena are as follows.

(1) Violation of condition (T4).
(2) The connection from U_{i-1} to U_i is broken by a repeller-saddle. Breaking the connection by a saddle-attractor is dual to this case.
(3) The connection is broken by a saddle.

In case (1), the extra equation is similar to one already discussed. Cases (2) and (3) are discussed in Appendix B.

4.2. Wave Group Interaction Condition

The wave group interaction condition (H2), as stated in Section 2.9, consists of several requirements, each being that a forward wave curve $U_\ell^f(U_0^*, F^*, \sigma)$ through U_ℓ^* should be transverse to a certain line L (either the line $\Delta \neq \{0\}$ associated with a transitional wave group, or the line tangent to the backward wave curve associated to a 2-wave group). For a codimension-one Riemann solution, it is possible for one of these transversality conditions to be violated. The extra equation specifies that the tangent $\partial U_\ell^f / \partial \sigma$ at (U_0^*, F^*, σ^*) lies in L. The nondegeneracy condition is that the wave curve $U_\ell^f(U_0^*, F^*, \sigma)$ has a quadratic tangency to L.

Remark. If L is the line $\Delta \neq \{0\}$ associated with a transitional wave group, then L is tangent to the backward wave curve $U_\ell^b(U_0^*, F^*, \tau)$ through U_ℓ^*. Indeed, if tangency of $\partial U_\ell^f / \partial \sigma$ and L is the only degeneracy, then the backward wave curve $U_\ell^b(U_0^*, F^*, \tau)$ that includes all transitional wave groups beyond U_ℓ^* is well-defined, and, moreover, its tangent line is Δ. (This result is established for a special case in the proof Prop. 6.3 in

Table VII. $* \cdot S\ S \cdot *$ Wave Sequences

$w_{i-1} \backslash w_i$	$S \cdot RS$	$S \cdot S$	$S \cdot SA$	$S \cdot A$
$R \cdot S$	1	2	3	4
$RS \cdot S$	5	6	7	3_d
$S \cdot S$	8	9	6_d	2_d
$SA \cdot S$	10	8_d	5_d	1_d

Ref. 17; the general case is proved the same way.) Therefore the failure of the wave group interaction condition can always be interpreted as tangency of a forward and a backward wave curve.

4.3. Different Shock Speeds

If a $* \cdot S$ wave w_{i-1} is followed by an $S \cdot *$ wave w_i in a structurally stable Riemann solution, 1 of the 16 situations listed and numbered in Table VII occurs. For $k = 1$, 2, 3, 5, 6, and 8, the wave sequence labeled k_d is dual to the wave sequence labeled k. There are thus 10 essentially distinct cases. In each case, violation of condition (H3) occurs when $s_i - s_{i-1} = 0$; this is the extra equation. We label these degeneracies using the notation, such as

$$R \cdot S \leftrightarrow S \cdot RS \tag{4.20}$$

4.4. Codimension-One Riemann Solutions

Sixty-three candidates for codimension-one Riemann solutions have been listed in this section: 52 in Section 4.1, 1 in Section 4.2, and 10 in Section 4.3. Many of these cases have already been analyzed (see, e.g., Refs. 23 and 11). We are led to the following conjecture.

Conjecture 4.4. *The 63 degeneracies of structurally stable Riemann solutions listed above give rise, under suitable nondegeneracy hypotheses, to codimension-one Riemann solutions.*

5. FOLDS, FRONTIERS, AND JOINS

In this section, we classify the 63 codimension-one cases listed in Section 4 as folds, frontiers, or joins.

5.1. Folds

Four cases—the transcritical bifurcation cases $R \cdot RS(5)$ and $RS \cdot RS(2)$, the doubly sonic transitional case $SA \cdot RS(1)$, and violation of the wave group interaction condition—are expected to be folds. The reasons are as follows.

(A) For the transcritical bifurcation case $R \cdot RS(5)$, the nondegeneracy condition (B1) is violated. Let $(U_0^*, s_1^*, U_1^*, s_2^*, ..., s_n^*, U_n^*, F^*)$ lie in the boundary of the set of structurally stable Riemann solutions of type $(R \cdot RS, R_1, T_3, ..., T_n)$; assume that the bifurcation diagram of the differential equation

$$\dot{U} = F^*(U) - F^*(U_0^*) - s_1(U - U_0^*) \tag{5.1}$$

has a transcritical bifurcation at $s_1 = s_1^*$, $U = U_1^*$, as in Fig. 10. We are interested in U_0 near U_0^* for which we can find a nondegenerate $R \cdot RS$ shock wave, with speed s_1 near s_1^*, from U_0 to a state near U_1^*. For such U_0 we will have a Riemann solution of type $(R \cdot RS, R_1, T_3, ..., T_n)$ to any right state near U_n^*.

In other words, the bifurcation diagram of the equation

$$\dot{U} = F^*(U) - F^*(U_0) - s_1(U - U_0) \tag{5.2}$$

should have a saddle-node bifurcation near $s_1 = s_1^*$, $U = U_1^*$. However, we expect that in U_0-space there is a curve for which we have the bifurcation diagram shown in Fig. 10b; on one side of this curve we have the bifurcation diagram shown in Fig. 10a, and on the other side we have the bifurcation

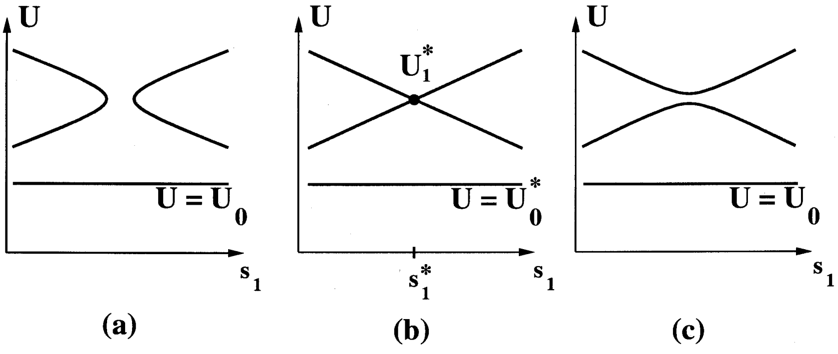

Fig. 10. Perturbation of a transcritical bifurcation.

diagram shown in Fig. 10c. Thus for U_0 on one side of the curve, and for U_n near U_n^*, there should be two nearby Riemann solutions; on the other side, there should be none. This indicates that there is a fold in the Riemann solution manifold $\mathscr{R}$.

Similarly, the transcritical bifurcation case $RS \cdot RS(2)$ also gives rise to a fold in the Riemann solution manifold $\mathscr{R}$. In this case, however, it may be necessary to change F (not just U_0) in order to change the bifurcation diagram of Fig. 10b. Indeed, this degeneracy can give rise to an F-boundary as well as a U_L-boundary; see Section 6.

(B) The case $SA \cdot RS(1)$, in which condition (T4) is violated for a doubly sonic transitional wave w^*: $U_-^* \xrightarrow{s^*} U_+^*$, is a fold for the following reason. Consider the family of differential equations

$$\dot{U} = F^*(U) - F^*(U_-) - s(U - U_-) \tag{5.3}$$

parameterized by (U_-, s). We assume that there is a curve $(U_-(\tau), s(\tau))$ through (U_-^*, s^*) such that Eq. (5.3) has an equilibrium of type SA at $U_-(\tau)$ and one of type RS at some corresponding equilibrium $U_+(\tau)$. The separation function $S_{F^*}(\tau)$ between the center manifolds of $U_-(\tau)$ and $U_+(\tau)$ vanishes at any τ^* for which there is a connection from $U_-(\tau)$ to $U_+(\tau)$. Condition (T4) means that $S'_{F^*}(\tau^*) \neq 0$; if it is violated, then we have that $S'_{F^*}(\tau^*) = 0$, and typically that $S''_{F^*}(\tau^*) \neq 0$. For F near F^* we have an analogous function $S_F(\tau)$. On one side of a codimension-one surface in F-space, there are two solutions of $S_F = 0$; on the other side there are none. Again, there is a fold in the Riemann solution manifold $\mathscr{R}$.

(C) Violation of the wave group interaction condition is similarly easy to understand. If the forward wave curve $U_\ell^f(U_0^*, F^*, \sigma)$ and the line L defined in Section 4.2 have a quadratic tangency, perturbation of U_0, U_n, or F should produce two, one, or zero intersections. Again, there is a fold in the Riemann solution manifold.

5.2. Frontiers

The cases $R \cdot S \leftrightarrow S \cdot A$, $R \cdot S \leftrightarrow S \cdot SA$, $R \cdot S(3)$, $RS \cdot S \leftrightarrow S \cdot SA$, and $RS \cdot S(3)$ are frontiers, as we now explain.

Consider a codimension-one Riemann solution of type $R \cdot S \leftrightarrow S \cdot A$, defined by the wave sequence $U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} U_2^*$ of type $(R \cdot S, S \cdot A)$ with $s_2^* = s_1^*$. The phase portrait of the corresponding equation

$$\dot{U} = F(U) - F(U_0^*) - s_1^*(U - U_0^*) \tag{5.4}$$

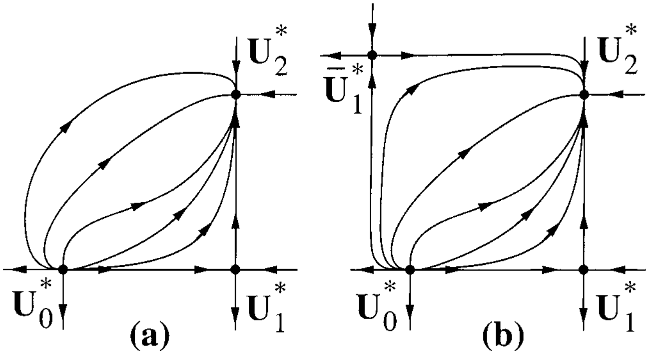

Fig. 11. Possible phase portraits for Riemann solutions of type $R \cdot S \leftrightarrow S \cdot A$.

is shown in Fig. 11a. Such a solution lies in a frontier of $\mathscr{R}$. Indeed, the connections for the shock waves $U_0^* \xrightarrow{s_1^*} U_1^*$ and $U_1^* \xrightarrow{s_2^*} U_2^*$ are each stable under perturbation; therefore, nearby shock waves $U_0 \xrightarrow{s_1} U_1$ and $U_1 \xrightarrow{s_2} U_2$ have types $R \cdot S$ and $S \cdot A$, respectively. To combine these shock waves into a Riemann solution $U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} U_2$, we need $s_1 \leqslant s_2$. Thus $\mathscr{R}$ is a manifold-with-boundary.

However, a frontier does not necessarily lead to nonexistence of solutions of Riemann problems. Suppose that the phase portrait of Eq. (5.4) contains a second saddle point $\bar{U}_1^*$ that is connected to both U_0^* and U_2^*, as in Fig. 11b. Then $U_0^* \xrightarrow{\bar{s}_1^*} \bar{U}_1^* \xrightarrow{\bar{s}_2^*} U_2^*$ is also a Riemann solution of type $(R \cdot S, S \cdot A)$ with $\bar{s}_2^* = \bar{s}_1^* (= s_2^* = s_1^*)$. This solution belongs to the codimension-one surface bounding a second region of structurally stable Riemann solutions $U_0 \xrightarrow{\bar{s}_1} \bar{U}_1 \xrightarrow{\bar{s}_2} U_2$ of type $(R \cdot S, S \cdot A)$. Moreover, the weak solutions $\hat{U}(\xi)$ corresponding to $U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} U_2^*$ and to $U_0^* \xrightarrow{\bar{s}_1^*} \bar{U}_1^* \xrightarrow{\bar{s}_2^*} U_2^*$ are identical, despite that these Riemann solutions are not equivalent (in the sense of Section 3.1). Thus solutions of type $(R \cdot S, S \cdot A)$ are continued by other solutions of the same type, although the shock waves involved have profiles that change abruptly at the codimension-one boundary. Depending on how the second region of structurally stable solutions projects onto $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$, it might be that solutions of Riemann problems exist throughout a neighborhood of (U_0^*, U_2^*, F^*).

Similarly, the case $R \cdot S \leftrightarrow S \cdot SA$ is a frontier. In this case, the phase portrait for Eq. (5.4) is shown in Fig. 12a. Just as before, $\mathscr{R}$ is a manifold-with-boundary. However, the phase portrait might contain a distinguished connection from the repeller U_0^* to the saddle-attractor U_2^*, as in Fig. 12b. Such a connection corresponds to a codimension-one Riemann solution of

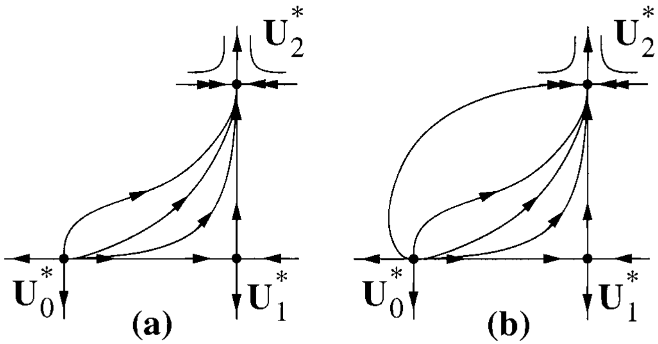

Fig. 12. Possible phase portraits for Riemann solutions of type $R \cdot S \leftrightarrow S \cdot SA$.

type $R \cdot S(3)$, which is also a frontier. Moreover, the weak solutions corresponding to the $R \cdot S \leftrightarrow S \cdot SA$ and $R \cdot S(3)$ solutions are the same. Thus structurally stable Riemann solutions containing the types $(R \cdot S,\ S \cdot SA,\ R_2)$ can sometimes be continued by ones containing the types $(R \cdot S,\ R_2)$, and vice versa, although the profiles for the shock waves change discontinuously at the codimension-one boundary.

Analogously, cases $RS \cdot S \leftrightarrow S \cdot SA$ and $RS \cdot S(3)$ are frontiers. Structurally stable Riemann solutions containing the types $(R_1,\ RS \cdot S,\ S \cdot SA,\ R_2)$ and $(R_1,\ RS \cdot S,\ R_2)$ can be continuations of each other, with a discontinuous change in the shock profiles at the boundary.

The reader should notice that in the five frontier cases, overcompressive shock waves, of types $R \cdot A$, $R \cdot SA$, and $RS \cdot A$, are present. No overcompressive waves appear in other codimension-one Riemann solutions.

5.3. Joins

Other than the 4 fold cases and the 5 frontier cases, there are 54 conjectured codimension-one cases. These cases are listed as 27 pairs in Tables VIII–XI. As can be verified easily, the members of each pair lead to equivalent generalized allowed wave sequences, in the sense of Section 3. (For the joins in Table IX, the verification uses Lemma 4.2.) Therefore these pairs form 27 Riemann solution joins.

6. F-, U_L-, AND INTERMEDIATE BOUNDARIES

Codimension-one Riemann solutions can be parameterized (in a suitable neighborhood) by the points of a codimension-one submanifold $\mathscr{S}$ of (U_L, U_R, F)-space, $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathscr{B}$. In this section, we classify such solutions,

Table VIII. Joins Caused by a Zero-Strength Rarefaction Wave

Zero-strength rarefaction	Matching degeneracy
0	$R \cdot S(1)$
$0\ RS \cdot RS$	$R \cdot RS(2)$
$0\ RS \cdot S$	$R \cdot S(2)$
$R \cdot RS\ 0$	$R \cdot S(4)$
$R \cdot RS\ 0\ RS \cdot RS$	$R \cdot RS(3)$
$R \cdot RS\ 0\ RS \cdot S$	$R \cdot S(5)$
$RS \cdot RS\ 0$	$RS \cdot S(2)$
$RS \cdot RS\ 0\ RS \cdot RS$	$RS \cdot RS(4)$
$RS \cdot RS\ 0\ RS \cdot S$	$RS \cdot S(5)$
$S \cdot RS\ 0$	$S \cdot S(2)$
$S \cdot RS\ 0\ RS \cdot RS$	$S \cdot RS(3)$
$S \cdot RS\ 0\ RS \cdot S$	$S \cdot S(3)$
$SA \cdot RS\ 0$	Dual of $S \cdot RS(2)$
$SA \cdot RS\ 0\ RS \cdot RS$	$SA \cdot RS(2)$
$SA \cdot RS\ 0\ RS \cdot S$	Dual of $S \cdot RS(4)$

Table IX. Joins in Which Genuine Nonlinearity Fails Within a Rarefaction Wave

Rarefaction wave degeneracy	Matching degeneracy
$R_1(1)$	$R \cdot RS(1)$
$R_1(2)$	$RS \cdot S(1)$
$R_1(3)$	$RS \cdot RS(1)$

Table X. Joins Occurring When Shock Speeds Coincide

Wave sequence with coinciding shock speeds	Matching degeneracy
$R \cdot S \leftrightarrow S \cdot RS$	$R \cdot RS(4)$
$R \cdot S \leftrightarrow S \cdot S$	$R \cdot S(6)$
$RS \cdot S \leftrightarrow S \cdot RS$	$RS \cdot RS(5)$
$RS \cdot S \leftrightarrow S \cdot S$	$RS \cdot S(6)$
$S \cdot S \leftrightarrow S \cdot RS$	$S \cdot RS(5)$
$S \cdot S \leftrightarrow S \cdot S$	$S \cdot S(4)$
$SA \cdot S \leftrightarrow S \cdot RS$	$SA \cdot RS(3)$

Table XI. Joins Caused by a Distinguished Connection

Distinguished connection degeneracy	Matching degeneracy
$RS \cdot RS(3)$	$S \cdot RS(1)$
$RS \cdot S(4)$	$S \cdot S(1)$

as identified in Section 4, in terms of how $\mathscr{S}$ is situated in $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathscr{B}$, i.e., whether they are F-boundaries, U_L-boundaries or duals, or intermediate boundaries.

Consider a codimension-one Riemann solution (2.9) of type $(T_1^*, ..., T_n^*)$ with defining map G and extra equation $H = 0$. According to condition (Q7), the solution set $\mathscr{M}$ of the equation $G(U_0, s_1, ..., s_n, U_n, F) = 0$, subject to $H(U_0, s_1, ..., s_n, U_n, F) \geqslant 0$, is a graph over a manifold-with-boundary in $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$ with boundary $\mathscr{S}$. Thus $(s_1, U_1, ..., U_{n-1}, s_n)$ can be expressed locally as a continuous function of (U_0, U_n, F). [If condition (Q7$_2$) holds, this function is not smooth, but smoothness is not needed for our purpose here.] Using this function we can write $H(U_0, s_1, ..., s_n, U_n, F)$ as $\bar{H}(U_0, U_n, F)$. The equation $\bar{H}(U_0, U_n, F) = 0$ defines the codimension-one manifold $\mathscr{S}$ within (U_0, U_n, F)-space. The classification of the Riemann solution (2.9) is determined by how $\bar{H}$ depends on U_0, U_n, and F: if $\bar{H}$ is independent of U_0 and U_n, the Riemann solution is an F-boundary; otherwise, if $\bar{H}$ is independent of U_n (respectively, of U_0), the Riemann solution is a U_L-boundary (resp., dual); and if $\bar{H}$ depends on both U_0 and U_n, the Riemann solution is an intermediate boundary. As we will see, the classification is determined by the position of the degeneracy with respect to rarefaction waves.

Each of the conjectured codimension-one Riemann solutions listed in Section 4 involves one of the following phenomena:

(a) degeneracy in a shock or rarefaction wave w_ℓ^* (Section 4.1);

(b) tangency of a forward wave curve $U_\ell^f(U_0^*, F^*, \sigma)$ with a certain line L (Section 4.2);

(c) equality of speeds for shock waves $w_{\ell-1}^*$ and w_ℓ^* (Section 4.3).

In case (a), the function H depends solely on $(U_{\ell-1}, s_\ell, U_\ell, F)$. Suppose that w_ℓ^* occurs in a sequence $(w_1^*, ..., w_p^*)$ such that $\sum_{i=1}^{p} \rho(T_i^*) = 0$. Then the defining equation can be split into two equations of the form

$$G^{(1)}(U_0, s_1, ..., s_p, U_p, F) = 0 \qquad (6.1)$$

$$G^{(2)}(U_p, s_{p+1}, ..., s_n, U_n, F) = 0 \qquad (6.2)$$

Since the sum of the Riemann numbers is 0, the first of these equations can be solved for $(s_1, U_1, ..., s_p, U_p)$ in terms of (U_0, F). In particular, $\bar{H}$ depends only on U_0 and F, so that the Riemann solution is a U_L-boundary.

In case (b), the forward wave curve depends solely on U_0, F, and the parameter σ, which is determined by U_ℓ; thus the function H depends solely on (U_0, U_ℓ, F). By the same reasoning as in case (a), the Riemann solution is a U_L-boundary if w_ℓ^* occurs in a sequence $(w_1^*, ..., w_p^*)$ such that $\sum_{i=1}^{p} \rho(T_i^*) = 0$.

In case (c), the function H depends solely on $(s_{\ell-1}, s_\ell, F)$. Again the Riemann solution is a U_L-boundary if w_ℓ^* occurs in a sequence $(w_1^*, ..., w_p^*)$ such that $\sum_{i=1}^{p} \rho(T_i^*) = 0$.

Definition 6.1. Consider one of the conjectured codimension-one Riemann solutions.

(a) Suppose that the degeneracy occurs in a shock or rarefaction wave w_ℓ^*. Then this degeneracy is said to *precede* each wave w_m^* for which $m > \ell$, and it is said to *follow* each wave w_m^* for which $m < \ell$.

(b) Suppose that the forward wave curve $U_\ell^f(U_0^*, F^*, \sigma)$ is tangent to the line L. Then this degeneracy is said to *precede* each wave w_m^* for which $m \geqslant \ell + 1$, and it is said to *follow* each wave w_m^* for which $m \leqslant \ell$. (We regard the degeneracy as occurring at U_ℓ^*.)

(c) Suppose that the shock waves $w_{\ell-1}^*$ and w_ℓ^* have equal speeds. Then this degeneracy is said to *precede* each wave w_m^* for which $m \geqslant \ell$, and it is said to *follow* each wave w_m^* for which $m \leqslant \ell - 1$. (We regard the degeneracy as occurring at $U_{\ell-1}^*$, the state common to $w_{\ell-1}^*$ and w_ℓ^*.)

Lemma 6.2. *For the conjectured codimension-one Riemann solutions, described as above, w_ℓ^* occurs in a sequence $(w_1^*, ..., w_p^*)$ such that $\sum_{i=1}^{p} \rho(T_i^*) = 0$ if the degeneracy precedes a 1-family rarefaction wave.*

Proof. Suppose that the degeneracy precedes a wave of type R_1. This wave occurs in a composite 1-wave group (2.28) or a slow composite transitional wave group (2.29). Let w_m^* denote the last wave of type R_1 (the one that is not optional) in this wave sequence. Then Theorem 2.2 implies that $\sum_{i=1}^{m-1} \rho(T_i^*) = 0$. Since the degeneracy precedes the 1-family rarefaction wave, $\ell < m$. [In case (c), notice that w_ℓ^* is a shock wave, so that $m \neq \ell$.] Therefore we can take $p = m - 1$. □

This lemma inspires the following conjecture. Under appropriate nondegeneracy conditions,

(1) if the degeneracy follows all waves of type R_1 and precedes all waves of type R_2, it is an intermediate boundary;

(2) if the degeneracy precedes at least one wave of type R_1 and precedes all waves of type R_2, it is a U_L-boundary, and the case where the degeneracy follows all waves of type R_1 and follows at least one wave of type R_2 is dual;

(3) if the degeneracy precedes at least one wave of type R_1 and follows at least one wave of type R_2, it is an F-boundary.

If this conjecture is true, then it is location in the wave sequence, not wave type, that determines whether a degeneracy is an intermediate boundary, a U_L-boundary or dual, or an F-boundary. For example, any degeneracy that occurs between two $SA \cdot RS$ waves is an F-boundary. However, in most cases, the type of degeneracy limits its possible locations in the wave sequence. For example, a degeneracy of type $R \cdot S \leftrightarrow S \cdot A$ is an intermediate boundary, one in an $SA \cdot RS$ wave is an F-boundary, and if there are no doubly sonic transitional waves, then a degeneracy of type $R \cdot S$ wave is a U_L- or intermediate boundary, one in an $R \cdot RS$ wave is a U_L-boundary, and one in an $RS \cdot RS$ or $S \cdot RS$ wave is an F- or U_L-boundary. On the other hand, a degeneracy in an $RS \cdot S$ or $S \cdot S$ wave can be any of the four kinds of boundaries.

7. WAVE CURVES

In this section, we discuss codimension-one Riemann solutions from the perspective of wave curves. Consider the structurally stable Riemann solution (2.9) for $U_t + F^*(U)_x = 0$. As in Section 2.9, we can choose a wave w_ℓ^* that is the last wave in a 1-wave or transitional wave group and define associated maps s_i^f and U_i^f, $i = 1,..., \ell$. If $U_0 = U_0^*$ and $F = F^*$ are held fixed and σ is varied, the state $U_\ell^f(U_0^*, F^*, \sigma)$ traces a portion of the forward wave curve associated with U_0^* (and the choice w_ℓ^*). Dually, we can define the backward wave curves. A smooth portion of a wave curve can be extended until the associated Riemann solution becomes structurally unstable. If such a point of degeneracy represents a codimension-one Riemann solution that is a join, then in some circumstances the point can be regarded as a *junction point* at which two smooth parts of a wave curve, corresponding to different types of Riemann solutions, join to form a continuous curve. For some types of degeneracies, the wave curve is even C^1 at the junction points.

For definiteness we consider a forward wave curve. Let

$$(w_1^*,..., w_\ell^*)\colon U_0^* \xrightarrow{s_1^*} \cdots \xrightarrow{s_\ell^*} U_\ell^* \tag{7.1}$$

be a wave sequence of type $(T_1^*,..., T_\ell^*)$ for $U_t + F^*(U)_x = 0$ consisting of a 1-wave group and zero or more transitional wave groups, with G^* denoting its defining map. Suppose that this sequence is part of a codimension-one Riemann solution, with extra equation $H^* = 0$. Assume that the degeneracy occurs within the sequence (7.1) and that it follows all R_1 waves, so that, according to Section 6, it does not lead to a U_L-boundary or to an F-boundary. Suppose, further, that the degeneracy is one of the 27 joins in Tables VIII–XI. Thus the Riemann solution in which the wave sequence (7.1) appears is equivalent to another codimension-one Riemann solution, and the sequence (7.1) corresponds to a wave sequence

$$(w_1^\#,..., w_k^\#)\colon U_0^\# \xrightarrow{s_1^\#} \cdots \xrightarrow{s_k^\#} U_k^\# \tag{7.2}$$

of type $(T_1^\#,..., T_k^\#)$. Let $G^\#$ be the defining map for the sequence (7.2), and let $H^\# = 0$ be the extra equation defining its degeneracy. Then we expect the following statements to hold under appropriate nondegeneracy conditions:

For suitable open neighborhoods $\mathscr{U}_0$ of U_0^* and $\mathscr{F}$ of F^*, and an $\varepsilon > 0$:

- There exist smooth mappings $\hat\sigma(U_0)$, defined on $\mathscr{U}_0$, and $s_i^f(U_0, F, \sigma)$ and $U_i^f(U_0, F, \sigma)$, defined for $U_0 \in \mathscr{U}_0$, $F \in \mathscr{F}$, $\sigma \in [\hat\sigma(U_0), \sigma^* + \varepsilon)$, and $i = 1,..., \ell$, with $\hat\sigma(U_0^*) = \sigma^*$, $s_i^f(U_0^*, F^*, \sigma^*) = s_i^*$, and $U_i^f(U_0^*, F^*, \sigma^*) = U_i^*$, such that $G^* = 0$ and $H^* \geqslant 0$ when $s_i = s_i^f(U_0, F, \sigma)$ and $U_i = U_i^f(U_0, F, \sigma)$ for $i = 1,..., \ell$. Furthermore, $H^* = 0$ if and only if $\sigma = \hat\sigma(U_0)$. In particular, there exists a family

$$U_0 \xrightarrow{s_1^f(U_0, F, \sigma)} \cdots \xrightarrow{s_\ell^f(U_0, F, \sigma)} U_\ell^f(U_0, F, \sigma) \tag{7.3}$$

  of wave sequence for $U_t + F(U)_x = 0$ of type $(T_1^*,..., T_\ell^*)$.

- There exist smooth mappings $s_i^\#(U_0, F, \sigma)$ and $U_i^\#(U_0, F, \sigma)$, defined for $U_0 \in \mathscr{U}_0$, $F \in \mathscr{F}$, $\sigma \in (\sigma^* - \varepsilon, \hat\sigma(U_0)]$, and $i = 1,..., k$, with $s_i^\#(U_0^*, F^*, \sigma^*) = s_i^*$ and $U_i^\#(U_0^*, F^*, \sigma^*) = U_i^*$, such that $G^\# = 0$ and $H^\# \geqslant 0$ when $s_i = s_i^\#(U_0, F, \sigma)$ and $U_i = U_i^\#(U_0, F, \sigma)$ for $i = 1,..., k$. Furthermore, $H^\# = 0$ if and only if $\sigma = \hat\sigma(U_0)$. In particular, there exists a family

$$U_0 \xrightarrow{s_1^\#(U_0, F, \sigma)} \cdots \xrightarrow{s_k^\#(U_0, F, \sigma)} U_k^\#(U_0, F, \sigma) \tag{7.4}$$

  of wave sequence for $U_t + F(U)_x = 0$ of type $(T_1^\#,..., T_k^\#)$.

Table XII. Changes from an $R \cdot S$ Shock Wave to a Composite 1-Wave Group

T_i	Degeneracy	Matching degeneracy	Replacement for T_i in $(T_1^\sharp, ..., T_k^\sharp)$
$R \cdot S$	$R \cdot S(1)$	0	R_1
$R \cdot S$	$R \cdot S(2)$	$0 \; RS \cdot S$	$R_1 \; RS \cdot S$
$R \cdot S$	$R \cdot S(4)$	$R \cdot RS \; 0$	$R \cdot RS \; R_1$
$R \cdot S$	$R \cdot S(5)$	$R \cdot RS \; 0 \; RS \cdot S$	$R \cdot RS \; R_1 \; RS \cdot S$

- For $\sigma = \hat{\sigma}(U_0)$, the wave sequences (7.3) and (7.4) are equivalent.

Thus the curves $U_\ell^f(U_0, F, \sigma)$ for $\sigma \geqslant \hat{\sigma}(U_0)$ and $U_k^\sharp(U_0, F, \sigma)$ for $\sigma \leqslant \hat{\sigma}(U_0)$ fit together to form a continuous curve. In this situation, we consider the forward wave curve associated with U_0 to comprise not only the states $U_\ell^f(U_0, F, \sigma)$ but also $U_k^\sharp(U_0, F, \sigma)$ as σ is varied; in addition, the point $U_l^f(U_0, F, \hat{\sigma}(U_0)) = U_k^\sharp(U_0, F, \hat{\sigma}(U_0))$ is called a *junction point.*

From Tables VIII–XI we find 15 codimension-one degeneracies that cause junction points in forward wave curves. Indeed, the assumption that the degeneracy should follow all R_1 waves (and thus is not a U_L- or F-boundary) rules out 12 of the joins in these tables, namely, those that occur in $* \cdot RS$ shock waves. Two more degeneracies, $R \cdot S(3)$ and $RS \cdot S(3)$, are overcompressive cases discussed in Section 5; at these degeneracies, the forward wave curve terminates.

Tables XII–XVIII indicate the changes in wave structure that occur at these 15 codimension-one junction points. In these tables, T_i denotes the type of the wave in the forward wave curve sequence (7.3) that degenerates at the junction point. In all cases, the sequence of types $(T_1^\sharp, ..., T_k^\sharp)$ is identical to $(T_1, ..., T_\ell)$ except that T_i has been replaced by a sequence of one, two, or three different wave types. Notice also that none of the wave types after T_i in $(T_1, ..., T_\ell)$ (or after the replacement for T_i in $(T_1^\sharp, ..., T_k^\sharp)$) can be of type R_1.

Table XIII. Changes in the 1-Wave Group or in a Slow Transitional Wave Group

T_i	Degeneracy	Matching degeneracy	Replacement for T_i in $(T_1^\sharp, ..., T_k^\sharp)$
$RS \cdot S$	$RS \cdot S(2)$	$RS \cdot RS \; 0$	$RS \cdot RS \; R_1$
$RS \cdot S$	$RS \cdot S(5)$	$RS \cdot RS \; 0 \; RS \cdot S$	$RS \cdot RS \; R_1 \; RS \cdot S$
R_1	$R_1(2)$	$RS \cdot S(1)$	$R_1 \; RS \cdot S$

Table XIV. Change from an $S \cdot S$ Transitional Wave to a Composite Slow Transitional Wave Group

T_i	Degeneracy	Matching degeneracy	Replacement for T_i in $(T_1^\#, ..., T_k^\#)$
$S \cdot S$	$S \cdot S(2)$	$S \cdot RS\ \ 0$	$S \cdot RS\ \ R_1$
$S \cdot S$	$S \cdot S(3)$	$S \cdot RS\ \ 0\ \ RS \cdot S$	$S \cdot RS\ \ R_1\ \ RS \cdot S$

Table XV. Formation of an $S \cdot S$ Transitional Wave from an $R \cdot S$ Shock Wave

T_i	Degeneracy	Matching degeneracy	Replacement for T_i in $(T_1, ^\#..., T_k^\#)$
$R \cdot S$	$R \cdot S(6)$	$R \cdot S \leftrightarrow S \cdot S$	$R \cdot S\ \ S \cdot S$

Table XVI. Formation of an $S \cdot S$ Transitional Wave from a Composite 1-Wave Group or Composite Slow Transitional Wave Group

T_i	Degeneracy	Matching degeneracy	Replacement for T_i in $(T_1^\#, ..., T_k^\#)$
$RS \cdot S$	$RS \cdot S(4)$	$S \cdot S(1)$	$S \cdot S$
$RS \cdot S$	$RS \cdot S(6)$	$RS \cdot S \leftrightarrow S \cdot S$	$RS \cdot S\ \ S \cdot S$

Table XVII. Formation of an $S \cdot S$ Transitional Wave from Another Such Wave

T_i	Degeneracy	Matching degenracy	Replacement for T_i in $(T_1^\#, ..., T_k^\#)$
$S \cdot S$	$S \cdot S(4)$	$S \cdot S \leftrightarrow S \cdot S$	$S \cdot S\ \ S \cdot S$

Table XVIII. Formation of an $SA \cdot RS$ Wave from a Composite Fast Transitional Wave Group

T_i	Degeneracy	Matching degeneracy	Replacement for T_i in $(T_1^\#, ..., T_k^\#)$
$SA \cdot S$	Dual of $S \cdot RS(2)$	$SA \cdot RS\ \ 0$	$SA \cdot RS\ \ R_1$
$SA \cdot S$	Dual of $S \cdot RS(4)$	$SA \cdot RS\ \ 0\ \ RS \cdot S$	$SA \cdot RS\ \ R_1\ \ RS \cdot S$

Table XIX. Changes in a Composite Fast Transitional Wave Group

T_i	Degeneracy	Matching degeneracy	Replacement for T_i in $(T_1^\#,..., T_k^\#)$
R_2	Dual of $R_1(3)$	Dual of $RS \cdot RS(1)$	$SA \cdot SA$
$SA \cdot SA$	Dual of $RS \cdot RS(4)$	$SA \cdot SA\ 0\ SA \cdot SA$	$SA \cdot SA\ R_2\ SA \cdot SA$
$SA \cdot S$	Dual of $S \cdot RS(1)$	Dual of $RS \cdot RS(3)$	$SA \cdot SA$
$SA \cdot S$	Dual of $S \cdot RS(3)$	$SA \cdot SA\ 0\ SA \cdot S$	$SA \cdot SA\ R_2\ SA \cdot S$

We must also consider which of the duals of the 27 joins can give rise to junction points in forward wave curves. Of the 27 joins, 9 are of type $R \cdot *$. Their duals are of type $* \cdot A$ and hence cannot occur in forward wave curves. Four other joins are of type $SA \cdot *$. Their duals are of type $* \cdot RS$; hence they precede a wave of type R_1, and so cannot give rise to junction points in forward wave curves. One more join, $S \cdot S(4)$, is its own dual.

Of the remaining 13 joins, 7 are cases that appear in Tables XII–XVIII and have distinct duals that can give rise to junction points in forward wave curves. In fact, each entry in Table XIII has a dual that represents a change in a fast transitional wave group, each entry in Table XIV has a dual that represents a change from an $S \cdot S$ transitional wave to a composite fast transitional wave group, and each entry in Table XVI has a dual that represents the formation of an $S \cdot S$ transitional wave from a composite fast transitional wave group.

The remaining six joins are listed in Tables XIX–XXI.

The wave curve tables indicate how the structure of a Riemann solution can become complicated. Consider beginning with the left state U_L and following the forward wave curve. As in the Lax theory [10], the local wave curve consists of two branches, one corresponding to R_1 rarefaction waves and the other to $R \cdot S$ shock waves. These branches can be continued until a junction point is encountered, at which point the structure of the 1-wave group changes. For instance, along the shock branch there can be an $R \cdot RS\ 0$ degeneracy, after which the 1-wave group has the composite type $R \cdot RS\ R_1$, as seen from Table XII. Similarly, the rarefaction branch can intersect the inflection locus, leading to the composite wave group $R_1\ RS \cdot S$, as in Table XIII. In continuing the wave curve further, the

Table XX. Splitting of a Composite Fast Transitional Wave Group

T_i	Degeneracy	Matching degeneracy	Replacement for T_i in $(T_1^\#,..., T_k^\#)$
$SA \cdot SA$	Dual of $RS \cdot RS(5)$	$SA \cdot S \leftrightarrow S \cdot SA$	$SA \cdot S\ S \cdot SA$

Table XXI. Formation of an $S \cdot S$ Wave from a Composite Fast Transitional Wave Group

T_i	Degeneracy	Matching degeneracy	Replacement for T_i in $(T_1^{\#}, \ldots, T_k^{\#})$
$SA \cdot S$	Dual of $S \cdot RS(5)$	$SA \cdot S \leftrightarrow S \cdot S$	$SA \cdot S \;\; S \cdot S$

junction points listed in Tables XII and XIII lead to arbitrarily complicated 1-wave groups. These junction points all occur in the analysis of Liu [11].

More generally, the forward wave curve can comprise branches that are disconnected from the branch through U_L, and other junction points listed in the wave curve tables can occur. For instance, the shock branch (or a disconnected branch of the Hugoniot locus of U_L) can encounter the degeneracy $R \cdot S \leftrightarrow S \cdot S$ listed in Table XV, after which an $S \cdot S$ transitional wave is part of the Riemann solution. This phenomenon is observed in models with quadratic flux functions [19]. Thereafter, the slow transitional wave group can become arbitrarily complicated, and other such groups can be formed. Furthermore, the duals of the entries in Table XIV lead to the formation of fast transitional wave groups. Finally, Table XVIII indicates two mechanisms by which doubly sonic transitional waves can develop in the forward wave curve.

8. SUMMARY

In Ref. 17, we constructed sets of structurally stable Riemann solutions parameterized by manifolds. In the present paper, we have investigated codimension-one Riemann solutions, in which structural stability fails in a minimal way. We have formulated a definition for codimension-one Riemann solutions, which entails that structurally stable and codimension-one Riemann solutions together are parameterized by manifolds or manifolds-with-boundaries. Either a set of structurally stable solutions ends at certain codimension-one boundaries (frontiers) or it is continued past the boundary by another set (joins and folds). A comprehensive list of codimension-one Riemann solutions has been developed, and these solutions have been classified according to their geometric properties (frontiers, joins, and folds), their roles in solving Riemann problems (U_L-, F-, and intermediate boundaries), and their relationship to wave curves (junction points).

APPENDIX A. RAREFACTION WAVES AND CENTER MANIFOLDS

In this appendix, we prove Lemma 4.2. The proof is inspired by Ref. 23.

Proof. To simplify notation, let $\ell_i^* = \ell_i(U^*)$ and $r_i^* = r_i(U^*)$. Let $\check{F}(\tau)$ denote any function such that $\check{F}'(\tau) = DF(\check{U}(\tau))$. It is easily verified that for $i \geqslant 1$, $D^i\check{F}(0)$ depends only on the derivatives of F at U^* through order i and on $\check{U}'(0),..., D^{i-1}\check{U}(0)$.

Since $\check{U}(\tau)$ is an integral curve of the 1-family line field,

$$\check{F}'\check{U}' = \check{\lambda}_1 \check{U}' \tag{A.1}$$

Differentiating this relation, we obtain

$$\check{F}''\check{U}' + \check{F}'\check{U}'' = \check{\lambda}_1' \check{U}' + \check{\lambda}_1 \check{U}'' \tag{A.2}$$

Setting $\tau = 0$ and multiplying by ℓ_1^* and ℓ_2^* yields

$$\ell_1^* \check{F}''(0)\, \check{U}'(0) = \check{\lambda}_1'(0) \tag{A.3}$$

$$\ell_2^* \check{F}''(0)\, \check{U}'(0) + [\check{\lambda}_2(0) - \check{\lambda}_1(0)]\, \ell_2^* \check{U}''(0) = 0 \tag{A.4}$$

From Eq. (4.16) we conclude that

$$\ell_1^* D^i\check{U} = 0 \qquad \text{for all} \quad i \geqslant 2 \tag{A.5}$$

The function $g(\sigma)$ defining the differential equation (4.15) on the center manifold of U^* can be characterized as follows:

$$F(U_c(\sigma)) - F(U_-) - s^*[U_c(\sigma) - U_-] = g(\sigma)\, U_c'(\sigma) \tag{A.6}$$

Indeed, a solution $\hat{\sigma}(\xi)$ of Eq. (4.15) is such that $U_c(\hat{\sigma}(\xi))$ solves Eq. (4.10), so that

$$g(\hat{\sigma})\, U_c'(\hat{\sigma}) = \frac{d}{d\xi} U_c(\hat{\sigma}(\xi)) = F(U_c(\hat{\sigma})) - F(U_-) - s^*[U_c(\hat{\sigma}) - U_-] \tag{A.7}$$

Since the left-hand side of Eq. (A.6) vanishes when $\sigma = 0$, whereas $U_c'(0) = r_1^* \neq 0$, we must have

$$g(0) = 0 \tag{A.8}$$

Let $F_c(\sigma)$ denote any function such that $F_c'(\sigma) = DF(U_c(\sigma))$. Just as for $\check{F}(\tau)$, one verifies that if $i \geqslant 1$, then $D^iF_c(0)$ depends only on the derivatives of F at U^* through order i and on $U_c'(0),..., D^{i-1}U_c(0)$. In fact, if $\check{U}'(0) = U_c'(0),..., D^{i-1}\check{U}(0) = D^{i-1}U_c(0)$, then $D^i\check{F}(0) = D^iF_c(0)$.

Differentiating Eq. (A.6) yields

$$F'_c U'_c - s^* U'_c = g' U'_c + g U''_c \tag{A.9}$$

Setting $\sigma = 0$ and multiplying by ℓ_1^* yields

$$g'(0) = 0 \tag{A.10}$$

Differentiating Eq. (A.9) yields

$$F''_c U'_c + F'_c U''_c - s^* U''_c = g'' U'_c + 2g' U''_c + g U'''_c \tag{A.11}$$

Setting $\sigma = 0$ and multiplying by ℓ_1^* and ℓ_2^* yields

$$\ell_1^* F''_c(0)\ U'_c(0) = g''(0) \tag{A.12}$$

$$\ell_2^* F''_c(0)\ U'_c(0) + [\check{\lambda}_2(0) - \check{\lambda}_1(0)]\ \ell_2^* U''_c(0) = 0 \tag{A.13}$$

From Eq. (4.13) we find that

$$\ell_1^* D^i U_c = 0 \qquad \text{for all} \quad i \geqslant 2 \tag{A.14}$$

Since $\check{U}'(0) = U'_c(0) = r_1^*$, we have $\check{F}''(0) = F''_c(0)$.

Comparison of Eqs. (A.3)–(A.5) with Eqs. (A.12)–(A.14) shows that

$$\check{\lambda}'_1(0) = g''(0) \tag{A.15}$$

$$\check{U}''(0) = U''_c(0) \tag{A.16}$$

This completes the proof in the case $k = 1$, which is the first step in an inductive proof for general k.

Suppose that $k \geqslant 2$, and assume that $D^j \check{\lambda}_1(0) = 0$ for $j = 1,..., k-1$. As has just been proved, the equations

$$D^j \check{\lambda}_1(0) = D^{j+1} g(0) \qquad \text{for} \quad j = 1,..., i-1 \tag{A.17}$$

$$D^{j+1} \check{U}(0) = D^{j+1} U_c(0) \qquad \text{for} \quad j = 1,..., i-1 \tag{A.18}$$

hold if $i = 2$; proceeding by induction, suppose that they hold for an integer i such that $2 \leqslant i \leqslant k$. Differentiating Eq. (A.1) i times, we obtain

$$\sum_{j=1}^{i} \binom{i}{j} (D^{j+1} \check{F})(D^{i-j+1} \check{U}) + \check{F}' D^{i+1} \check{U}$$
$$= (D^i \check{\lambda}_1)\ \check{U}' + \sum_{j=1}^{i-1} \binom{i}{j} (D^j \check{\lambda}_1)(D^{i-j+1} \check{U}) + \check{\lambda}_1 D^{i+1} \check{U} \tag{A.19}$$

Setting $\tau = 0$ and multiplying by ℓ_1^* and ℓ_2^* yields

$$\ell_1^* \sum_{j=1}^{i} \binom{i}{j} D^{j+1}\check{F}(0)\, D^{i-j+1}\check{U}(0) = D^i\check{\lambda}_1(0) \tag{A.20}$$

$$\ell_2^* \sum_{j=1}^{i} \binom{i}{j} D^{j+1}\check{F}(0)\, D^{i-j+1}\check{U}(0) + [\check{\lambda}_2(0) - \check{\lambda}_1(0)]\, \ell_2^* D^{i+1}\check{U}(0) = 0 \tag{A.21}$$

Differentiating Eq. (A.6) $i+1$ times, we obtain

$$\sum_{j=1}^{i} \binom{i}{j} (D^{j+1}F_c)(D^{i-j+1}U_c) + F'_c D^{i+1}U_c - s^* D^{i+1}U_c$$
$$= (D^{i+1}g)\, U'_c + \sum_{j=1}^{i} \binom{i+1}{j} (D^j g)(D^{i-j+2}U_c) + gD^{i+2}U_c \tag{A.22}$$

Setting $\sigma = 0$ and multiplying by ℓ_1^* and ℓ_2^* yields

$$\ell_1^* \sum_{j=1}^{i} \binom{i}{j} D^{j+1}F_c(0)\, D^{i-j+1}U_c(0) = D^{i+1}g(0) \tag{A.23}$$

$$\ell_2^* \sum_{j=1}^{i} \binom{i}{j} D^{j+1}F_c(0)\, D^{i-j+1}U_c(0)$$
$$+ [\check{\lambda}_2(0) - \check{\lambda}_1(0)]\, \ell_2^* D^{i+1}U_c(0) = 0 \tag{A.24}$$

Equation (A.18) implies that

$$D^{j+1}\check{F}(0) = D^{j+1}F_c(0) \qquad \text{for} \quad j = 1,..., i \tag{A.25}$$

Therefore comparison of Eqs. (A.20), (A.21), and (A.5) with Eqs. (A.23), (A.24), and (A.14) yields the relations

$$D^i\check{\lambda}_1(0) = D^{i+1}g(0) \tag{A.26}$$

$$D^{i+1}\check{U}(0) = D^{i+1}U_c(0) \tag{A.27}$$

Continuing the induction to $i = k$ yields the result.

APPENDIX B. THE EXTRA EQUATION IN SEVEN CASES

B.1. The Case $S \cdot S(4)$

Consider a Riemann solution Eq. (2.9) of type $(T_1, \ldots, T_n)$, with $T_j = S \cdot S$ and $\sum_{i=1}^{n} \rho(T_i) = 2$, in which the jth wave $w_j^*: U_{j-1}^* \xrightarrow{s_j^*} U_j^*$ is a generalized shock wave. Assume that the phase portrait of the equation

$$\dot{U} = F(U) - F(U_{j-1}^*) - s_j^*(U - U_{j-1}^*) \tag{B.1}$$

has an $S \cdot S$ connection $\tilde{U}_1(\xi)$ from U_{j-1}^* to an equilibrium $\bar{U}^*$ and a second $S \cdot S$ connection $\tilde{U}_2(\xi)$ from $\bar{U}^*$ to U_j^*. As in Ref. 17, we can use line segments Σ_k through $\tilde{U}_k(0)$, transverse to $\tilde{U}_k(\xi)$, to define separation functions $S_k(U_{j-1}, s_j)$ for $k = 1, 2$. Then $S_1(U_{j-1}, s_j) = 0$ if and only if the equation

$$\dot{U} = F(U) - F(U_{j-1}) - s_j(U - U_{j-1}) \tag{B.2}$$

has a connection from U_{j-1} to a saddle point near $\bar{U}^*$; similarly, $S_2(U_{j-1}, s_j) = 0$ if and only if Eq. (B.2) has a connection from a saddle point near $\bar{U}^*$ to one near U_j.

We assume the nondegeneracy conditions that the vectors $DS_k(U_{j-1}^*, s_j^*)$, $k = 1, 2$, are linearly independent, and that the hyperbolicity ratio

$$r(\bar{U}^*) = \left| \frac{\lambda_1(\bar{U}^*) - s_j^*}{\lambda_2(\bar{U}^*) - s_j^*} \right| \tag{B.3}$$

differs from 1. Then in (U_{j-1}, s_j)-space, which is the parameter space for Eq. (B.2), there are three smooth bifurcation surfaces M_1, M_2, and M through (U_{j-1}^*, s_j^*). The surfaces M_1 and M_2 are the zero sets of $S_1(U_{j-1}, s_j)$ and $S_2(U_{j-1}, s_j)$, respectively. Therefore, along M_1 (respectively, M_2), there are $S \cdot S$ connections from U_{j-1} to a saddle near $\bar{U}^*$ (resp., from a saddle near $\bar{U}^*$ to one near U_j^*). The surfaces M_1 and M_2 meet transversally along a curve C. The surface M corresponds to connections from U_{j-1} to a saddle near U_j^*. This surface has C as its boundary, and it is tangent to M_1 (respectively, M_2) along C if $r(\bar{U}^*)$ is less than 1 (resp., greater than 1) [8].

Let us consider the case in which M is tangent to M_2 (see Fig. B1); the other case is analogous. The transversal Σ_2 can be used to define a separation function $S(U_{j-1}, s_j)$ between the unstable manifold of U_{j-1} and the

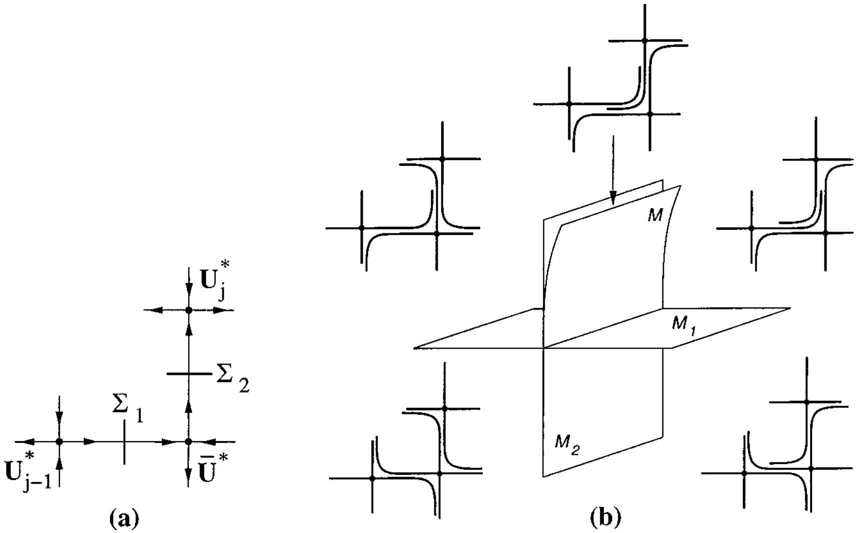

Fig. B1. The $S \cdot S(4)$ degeneracy: (a) phase portrait; (b) bifurcation diagram.

stable manifold of the saddle near U_j^*; S is defined on one side of M_1, say on $S_1 > 0$. We can extend S to M_1 by continuity; then the functions S and S_2 coincide along M_1. In fact, we can extend S to the region $S_1 < 0$ by setting it equal to S_2 there; the extended S will be C^1. The function S serves as one coordinate function of G, and for the extra equation $H = 0$ we can use $S_1 = 0$.

B.2. The Case $S \cdot S(3)$

Consider a Riemann solution Eq. (2.9) of type $(T_1,..., T_n)$, with $T_j = S \cdot S$ and $\sum_{i=1}^{n} \rho(T_i) = 2$, in which the jth wave $w_j^*: U_{j-1}^* \xrightarrow{s_j^*} U_j^*$ is a generalized shock wave. Assume that the phase portrait of Eq. (B.1) has an $S \cdot RS$ connection $\tilde{U}_1(\xi)$ from U_{j-1}^* to an equilibrium $\bar{U}^*$ and an $RS \cdot S$ connection $\tilde{U}_2(\xi)$ from $\bar{U}^*$ to U_j^*. There is a codimension-one surface N in (U_{j-1}, s_j)-space along which Eq. (B.2) has equilibria of type RS near $\bar{U}^*$; this equation undergoes a saddle-node bifurcation as N is crossed. There is a function $H(U_{j-1}, s_j)$ that has N as its zero set, has nonzero derivative, and is positive (respectively, negative) when Eq. (B.2) has two (resp., no) equilibria near $\bar{U}^*$.

The center manifold of $\bar{U}^*$ for Eq. (B.2) extends to a smooth family of invariant manifolds $W(U_{j-1}, s_j)$. Let Σ_1 be a line segment through $\tilde{U}_1(0)$, transverse to $\tilde{U}_1(\xi)$. Using Σ_1 we can define (1) a separation function $S_1(U_{j-1}, s_j)$ for the unstable manifold of U_{j-1} and the extended center manifold of $\bar{U}^*$ and (2) a separation function $S(U_{j-1}, s_j)$ for the unstable manifold of U_{j-1} and the stable manifold of the saddle near U_j^*. The function S is defined only when $H(U_{j-1}, s_j) < 0$.

Along the surface N, the functions S_1 and S coincide. In fact, we can extend S to the region $H > 0$ by setting it equal to S_1 there; the extended S will be as smooth as F. The function S serves as one coordinate function of G, and we can use $H = 0$ as the extra equation.

B.3. The Case $S \cdot RS(3)$

Consider a Riemann solution Eq. (2.9) of type $(T_1,..., T_n)$, with $T_j = S \cdot RS$ and $\sum_{i=1}^{n} \rho(T_i) = 2$, in which the jth wave $w_j^*: U_{j-1}^* \xrightarrow{s_j^*} U_j^*$ is a generalized shock wave. Assume that the phase portrait of Eq. (B.1) has an $S \cdot RS$ connection $\tilde{U}_1(\xi)$ from U_{j-1}^* to an equilibrium $\bar{U}^*$ and an $RS \cdot RS$ connection $\tilde{U}_2(\xi)$ from $\bar{U}^*$ to U_j^*. This case is similar to the case $S \cdot S(3)$, except that $S(U_{j-1}, s_j)$ is taken to be the separation function for the unstable manifold of U_{j-1} and the extended center manifold of U_j^*.

B.4. The Case $S \cdot RS(4)$

Dually, we may consider the following degeneracy. Consider a Riemann solution Eq. (2.9) of type $(T_1,..., T_n)$, with $T_j = SA \cdot S$ and $\sum_{i=1}^{n} \rho(T_i) = 2$, in which the jth wave $w_j^*: U_{j-1}^* \xrightarrow{s_j^*} U_j^*$ is a generalized shock wave. Assume that the phase portrait of Eq. (B.1) has an $SA \cdot RS$ connection $\tilde{U}_1(\xi)$ from U_{j-1}^* to an equilibrium $\bar{U}^*$ and an $RS \cdot S$ connection $\tilde{U}_2(\xi)$ from $\bar{U}^*$ to U_j^*. This case is also similar to $S \cdot S(3)$, except that $S_1(U_{j-1}, s_j)$ (respectively, $S(U_{j-1}, s_j)$) is taken to be the separation function for the extended center manifold of U_{j-1}^* and the extended center manifold of $\bar{U}^*$ (resp., the stable manifolds of saddles near U_j^*).

B.5. The Case $S \cdot RS(5)$

Consider a Riemann solution Eq. (2.9) of type $(T_1,..., T_n)$, with $T_i = S \cdot RS$ and $\sum_{i=1}^{n} \rho(T_i) = 2$, in which the jth wave $w_j^*: U_{j-1}^* \xrightarrow{s_j^*} U_j^*$ is a generalized shock wave. Assume that the phase portrait of Eq. (B.1) has an $S \cdot S$ connection $\tilde{U}_1(\xi)$ from U_{j-1}^* to an equilibrium $\bar{U}^*$, and an $S \cdot RS$ connection $\tilde{U}_2(\xi)$ from $\bar{U}^*$ to U_j^*. This case is similar to $S \cdot S(4)$, except that $S_2(U_{j-1}, s_j)$ (respectively, $S(U_{j-1}, s_j)$) is taken to be the separation function for the unstable manifolds of saddles near $\bar{U}^*$ (resp., the unstable manifold of U_{j-1}) and the extended center manifold of U_j^*.

B.6. The Cases $SA \cdot RS(4)$ and $SA \cdot RS(3)$

These cases are similar to $S \cdot RS(5)$ and $S \cdot RS(3)$ respectively, except that the unstable manifold of the saddle U_{j-1} is replaced by the extended center manifold of U_{j-1}^*.

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REFERENCES

1. Azevedo, A., Marchesin, D., Plohr, B., and Zumbrun, K. (1996). Nonuniqueness of non-classical solutions of Riemann problems. *Z. angew. Math. Phys.* **47**, 977–998.
2. Furtado, F. (1989). *Structural Stability of Nonlinear Waves for Conservation Laws*, Ph.D. thesis, New York University, New York.
3. Golubitsky, M., and Schaeffer, D. (1984). *Singularities and Groups in Bifurcation Theory*, Springer-Verlag, New York–Heidelberg–Berlin.
4. Isaacson, E., Marchesin, D., Palmeira, C. F., and Plohr, B. (1992). A global formalism for nonlinear waves in conservation laws. *Comm. Math. Phys.* **146**, 505–552.
5. Isaacson, E., Marchesin, D., Plohr, B., and Temple, J. B. (1992). Multiphase flow models with singular Riemann problems. *Mat. Aplic. Comput.* **11**, 147–166.
6. Keyfitz, B., and Kranzer, H. (1980). A system of non-strictly hyperbolic conservation laws arising in elasticity theory. *Arch. Rational Mech. Anal.* **72**, 219–241.
7. Keyfitz, B., and Kranzer, H. (1989). A viscosity approximation to a system of conservation laws with no classical Riemann solution. *Nonlinear Hyperbolic Problems*, Lect. Notes Math., Vol. 1402, Springer-Verlag, New York–Heidelberg–Berlin, pp. 185–197.
8. Kokubu, H. (1988). Homoclinic and heteroclinic bifurcations of vector fields. *Japan J. Appl. Math.* **5**, 455–501.

9. Korchinski, D. (1977). *Solution of a Riemann Problem for a 2 × 2 System of Conservation Laws Possessing no Classical Weak Solution*, Ph.D. thesis, Adelphi University, Garden City, NY.
10. Lax, P. (1957). Hyperbolic systems of conservation laws II. *Comm. Pure Appl. Math.* **10**, 537–566.
11. Liu, T.-P. (1974). The Riemann problem for general 2 × 2 conservation laws. *Trans. Am. Math. Soc.* **199**, 89–112.
12. Marchesin, D., and Paes-Leme, P. J. (1983). Riemann problems for inhomogeneous hyperbolic equations for fluids, *Proc. 14th Brazil. Colloq. Math.*, *Vol. 2*, IMPA, pp. 273–299.
13. Schaeffer, D., Schecter, S., and Shearer, M. (1993). Nonstrictly hyperbolic conservation laws with a parabolic line. *J. Diff. Eqs.* **103**, 94–126.
14. Schaeffer, D., and Shearer, M. (1987). The classification of 2 × 2 systems of non-strictly hyperbolic conservation laws, with application to oil recovery. *Comm. Pure Appl. Math.* **40**, 141–178.
15. Schaeffer, D., and Shearer, M. (1987). Riemann problems for nonstrictly hyperbolic 2 × 2 systems of conservation laws. *Trans. Am. Math. Soc.* **304**, 267–306.
16. Schecter, S. (1993). Numerical computation of saddle-node homoclinic bifurcation points. *SIAM J. Numer. Anal.* **30**, 1155–1178.
17. Schecter, S., Marchesin, D., and Plohr, B. (1996). Structurally stable Riemann solutions. *J. Diff. Eqs.* **126**, 303–354.
18. Shearer, M. (1986). Nonuniqueness of admissible solutions of Riemann initial value problems for a system of conservation laws of mixed type. *Arch. Rational Mech.* **93**, 45–59.
19. Shearer, M., Schaeffer, D., Marchesin, D., and Paes-Leme, P. (1987). Solution of the Riemann problem for a prototype 2 × 2 system of non-strictly hyperbolic conservation laws. *Arch. Rational Mech. Anal.* **97**, 299–320.
20. Smoller, J. (1969). On the solution of the Riemann problem with general step data for an extended class of hyperbolic systems. *Mich. Math. J.* **16**, 201–210.
21. Smoller, J. (1983). *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York–Heidelberg–Berlin.
22. Tan, D., Zhang, T., and Zheng, Y. (1994). Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws. *J. Diff. Eqs.* **112**, 1–32.
23. Wendroff, B. (1972). The Riemann problem for materials with non-convex equations of state: I. Isentropic flow; II. General flow. *J. Math. Anal. Appl.* **38**, 454–466, 640–658.