Stability of Fronts in Gasless Combustion

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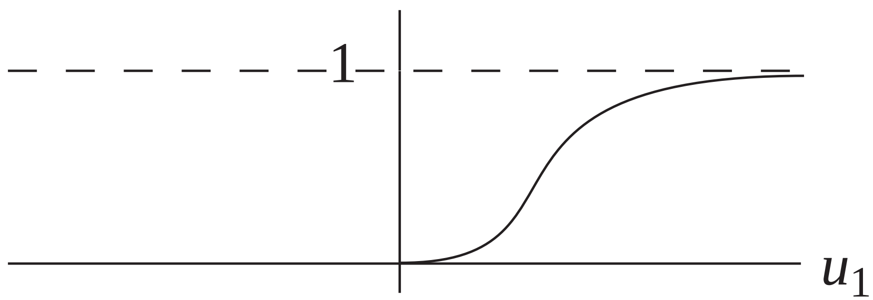
I. Introduction

A model for combustion of a solid fuel in one space dimension:

$$\begin{aligned}\partial_t u_1 &= \partial_{xx} u_1 + u_2 \rho(u_1),\\ \partial_t u_2 &= -\beta u_2 \rho(u_1),\end{aligned}$$

where

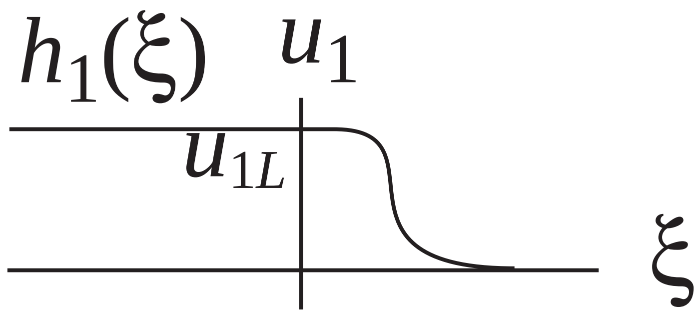
$$\rho(u_1) \begin{cases} 0 & \text{if } u_1 \leq 0, \\ e^{-\frac{1}{u_1}} & \text{if } u_1 > 0. \end{cases}.$$

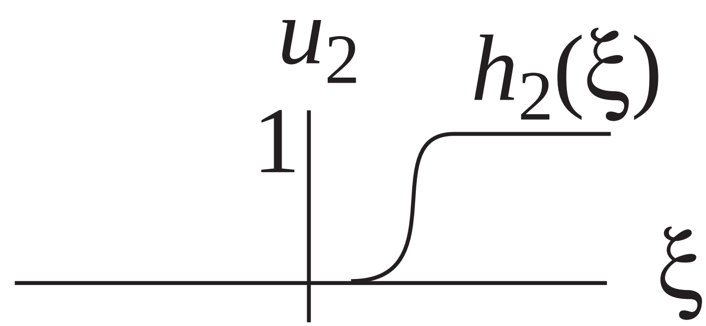


Graph of $\rho(u_1)$

- $u_1 =$ temperature.
- $u_2 =$ concentration of unburned fuel.
- $\rho =$ normalized reaction rate.
- $\beta > 0$ is the “exothermicity” parameter.
- $u_1 = 0$ is a background temperature at which the reaction does not take place.

We are interested in combustion fronts $H(\xi) = (h_1, h_2)(\xi)$, $\xi = x - \sigma t$.





- σ is the speed of the front.
- Without loss of generality, we take $\sigma > 0$.
- Behind the front: $(h_1, h_2) = (u_{1L}, 0)$.
- $u_{1L} > 0$ is the temperature of combustion, which is to be determined.
- Ahead of the front: $(h_1, h_2) = (0, u_{2R})$.
- $u_{2R} > 0$ is the concentration of fuel in the medium.
- We normalize so that $u_{2R} = 1$.

A combustion front is a traveling wave.

Stability of a traveling wave means that a small perturbation of it converges to one of its translates.

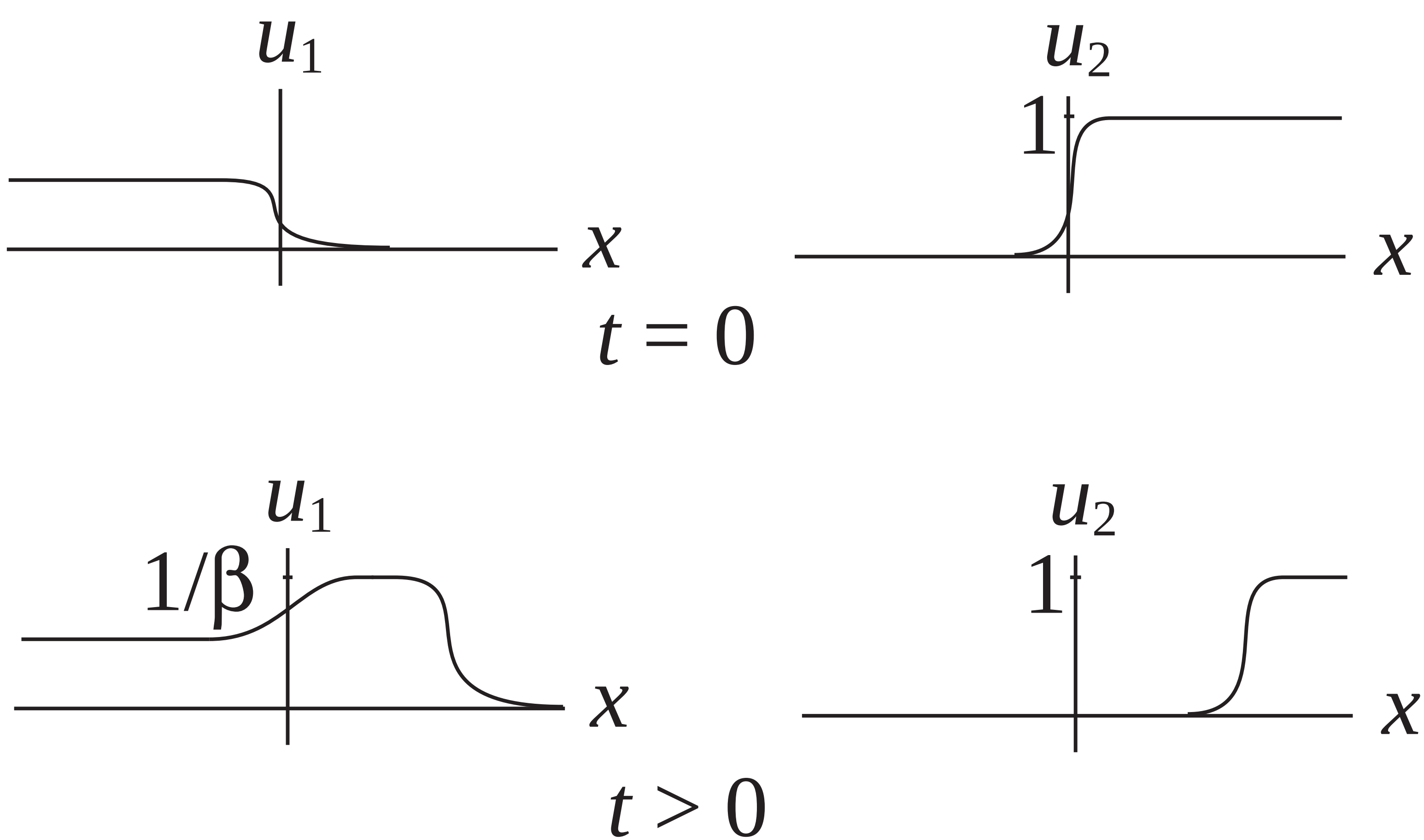
What's in the literature?

(1) $u_{1L} = \frac{1}{\beta}$.
(2) There is a **unique combustion front with positive speed that approaches both end states exponentially**, and a family of combustion fronts with faster wave speeds that approach the burned end state $(u, y) = (\frac{1}{\beta}, 0)$ exponentially and the unburned end state $(u_1, u_2) = (0, 1)$ more slowly.
(3) **Only the combustion front that approaches both end states exponentially is "physical."**
(4) Numerical simulations indicate that as β increases, the combustion front loses stability due to a pair of complex eigenvalues crossing the imaginary axis.
(5) For the linearization of the PDE at the combustion front, there is a bound on the possible size of eigenvalues with $\text{Re}\lambda \geq 0$.
(6) **Numerical Evans function calculations indicate that the 0 eigenvalue (which traveling waves always have) is simple, and there are no positive real eigenvalues for any β.**

(1)–(3), (5) : Varas, F. and Vega, J., SIAM J. Appl. Math. 62 (2002), 1810–1822.
(4): Bayliss, A. and Matkowsky, B., SIAM J. Appl. Math. 50 (1990), 437–459.
(6): Balasuriya, S., Gottwald, G., Hornibrook, J., and Lafortune, S., SIAM J. Appl. Math. 67 (2007), 464–486.

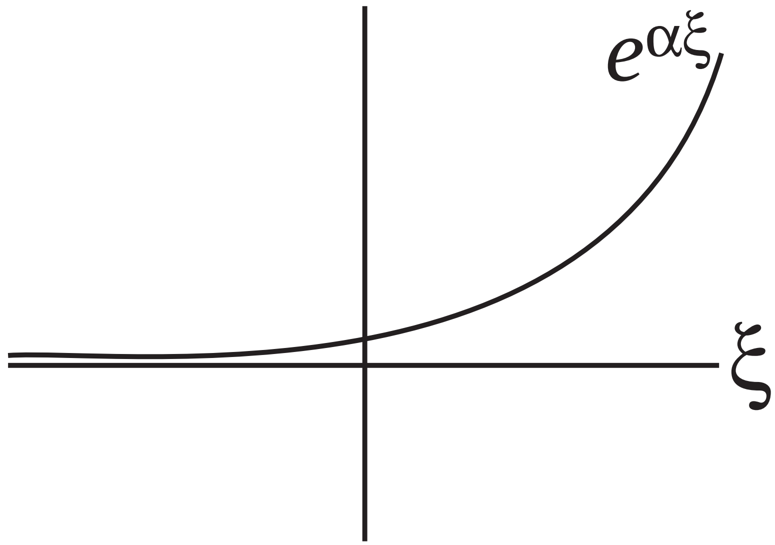
We'll only discuss the "physical" combustion front.

What kind of stability is it reasonable to expect?

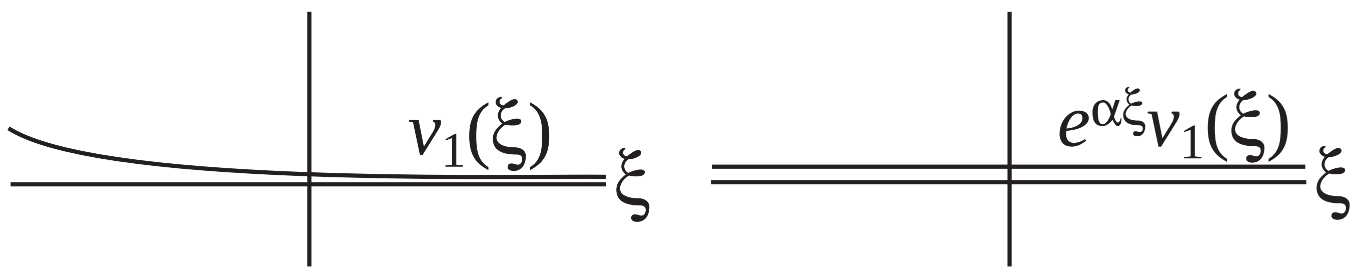


In a coordinate system moving with the speed of an exact traveling combustion front, our solution is very close to the exact front for $-a(t) < \xi < \infty$, where $a(t) \to \infty$ as $t \to \infty$.

The mathematical notion that captures this kind of stability is stability with respect to a norm with weight function $e^{\alpha\xi}$, $\alpha > 0$.



A perturbation of the combustion front that is small in this norm is exponentially close to the front at the right but may be far from it at the left.



For stability in this norm, as time increases, the solution with a perturbed initial condition must become very close to the combustion front at the right, but may continue to be far from the combustion front far to the left.

The linearization $\partial_t V = \mathcal{A}V$ of the PDE at the combustion front

The linearization of a PDE at a traveling wave $H(\xi)$ always has 0 has an eigenvalue. The eigenfunction is $H'(\xi)$.

Spectral stability of a traveling wave:

(1) 0 is a simple eigenvalue of $\mathcal{A}$, and (2) the rest of the spectrum of $\mathcal{A}$ lies in $\text{Re}\,\lambda < -\nu < 0$.

In our problem, in the space BUC^2 (BUC = bounded uniformly continuous functions with the sup norm), the essential spectrum includes the imaginary axis, hence no spectral stability.

Fortunately, **introducing a norm with weight function $e^{\alpha\xi}$, $\alpha > 0$, moves the essential spectrum to the left of the imaginary axis, hence there is the possibility of spectral stability**.

Linearized stability of a traveling wave:

$e^{\mathcal{A}t}$ has (1) a simple eigenvalue 1, and (2) a codimension-one invariant subspace on which $\|e^{\mathcal{A}t}\| \leq Ke^{-\nu t}$ for some $\nu > 0$.

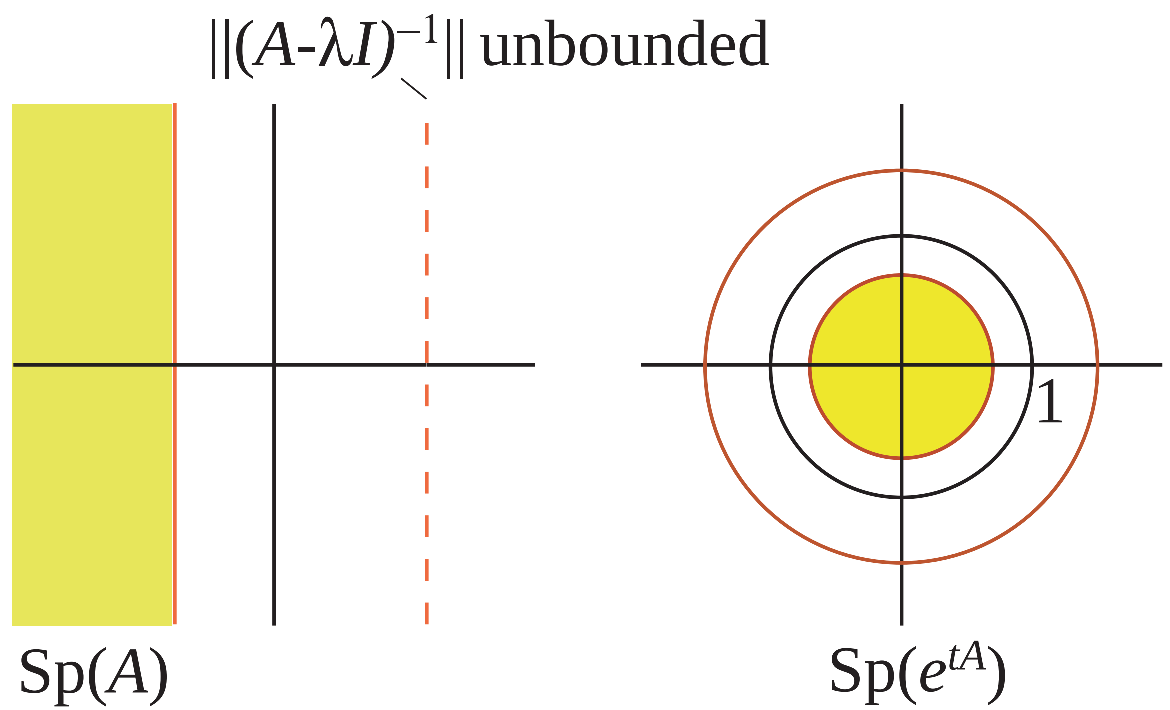
Spectral stability implies linearized stability for certain classes of operators, such as sectorial operators. Unfortunately, $\mathcal{A}$ is not sectorial, even after weighting the norm: **The essential spectrum includes a vertical line.**

This difficulty is typical of systems with no diffusion in some equations.

For a system with no diffusion in some equations, the linearized system $\partial_t V = \mathcal{A}V$ generates a C_0-semigroup, not an analytic semigroup. Linearized stability does not always follow from spectral stability.

For traveling *pulses* (left and right states are the same) in such systems, Evans showed that spectral stability does in fact imply linearized stability; his argument was simplified by Bates and Jones. However, their arguments do not work for traveling *fronts* (left and right states different).

How is it possible to have spectral stability without linearized stability?



Linearized stability of the combustion front

BUC has the norm

$$\|u\|_0 = \sup_{\xi \in \mathbb{R}} |u(\xi)|.$$

$BUC_\alpha = \{u : \mathbb{R} \to \mathbb{R} : e^{\alpha\xi} u(\xi) \in BUC\}$ has the norm

$$\|u\|_\alpha = \|e^{\alpha\xi} u(\xi)\|_0 = \sup_{\xi \in \mathbb{R}} e^{\alpha\xi} |u(\xi)|.$$

$\alpha > 0$ but not too big.

1. The eigenvalues of the linearization are the zeros of the Evans function $D(\lambda)$. We prove $D'(0) > 0$**, so the 0 eigenvalue is simple**.

2. We prove that $D(\lambda)$ **is positive for large positive real** λ. This is consistent with stability.

3. We prove that **in** BUC^2_α**, if the only eigenvalue in** $\mathrm{Re}\lambda \geq 0$ **is 0, then the combustion front is both spectrally stable and linearly stable**.

Verification that there are no eigenvalues in $\text{Re}\lambda \geq 0$ other than 0 must be done by a numerical Evans function calculation of a **winding number**, taking advantage of the fact that **the Evans function is analytic**. (Apparently true for small β, false for large β: combustion front loses stability in a Hopf bifurcation.)

Recall that there is a bound on the size of possible eigenvalues with $\text{Re}\lambda \geq 0$ (due to Varas and Vega).

Nonlinear stability of the combustion front

Unfortunately, the nonlinear terms in the PDE do not yield a map from BUC^2_α to itself.

Reason: consider

$$e^{\alpha\xi} v_2(\xi)\rho'(h_1(\xi))v_1(\xi).$$

- $\rho'(h_1(\xi))$ is bounded.
- $e^{\alpha\xi} v_2(\xi)$ is bounded if $v_2 \in BUC_\alpha$.
- However, $v_1(\xi)$ is not necessarily bounded.

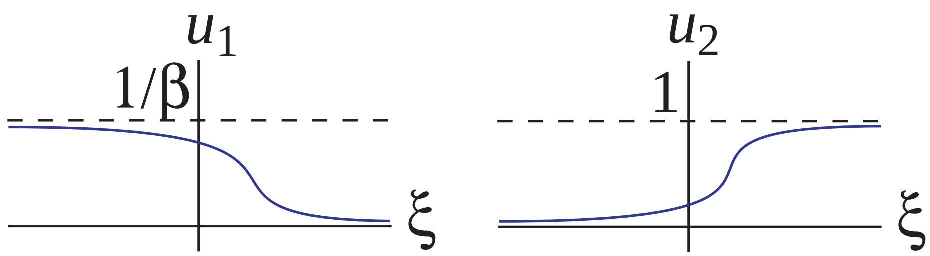
Let $BUC_m = BUC \cap BUC_\alpha$, with norm

$$\|u\|_m = \max(\|u\|_0, \|u\|_\alpha).$$

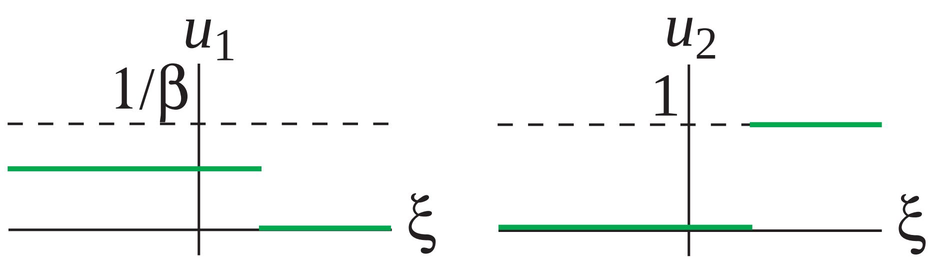
We prove that **if** $\|U_0 - H\|_m$ **is small, then there is a small number** q **such that** $\|U(t) - H(\xi - q)\|_\alpha \to 0$ **as** $t \to \infty$.

Why is the combustion front that approaches both end states exponentially the only one that's "physical"?

Combustion front:



Natural initial condition:



In our exponentially weighted norm, the natural initial condition is a small perturbation of the combustion front that approaches both end states exponentially, hence is (presumably) attracted to it.

Other combustion fronts may well attract sufficiently small perturbations of themselves! This happens for the n-degree Fisher-type equation $u_t = u_{xx} + u^n(1-u)$, $n > 1$: Wu, Y., Xing, X., and Ye, Q., Discrete Contin. Dyn. Syst. 16 (2006), 47–66.

Outline

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III. Linearization of the PDE at the Traveling Wave

IV. Spectral Stability and Linearized Stability

V. Nonlinear Stability

II. Traveling Waves

Notation:

$$\omega(u_1, u_2) = u_2 \rho(u_1).$$

In PDE let $\xi = x - \sigma t$:

$$\begin{aligned} \partial_t u_1 &= \partial_{\xi\xi} u_1 + \sigma \partial_\xi u_1 + \omega(u_1, u_2)), \\ \partial_t u_2 &= \sigma \partial_\xi u_2 - \beta \omega(u_1, u_2). \end{aligned}$$

A stationary solution of the PDE in moving coordinates is a traveling wave solution of the original PDE with speed σ.

Stationary solutions satisfy:

$$\begin{aligned} 0 &= \partial_{\xi\xi} u_1 + \sigma \partial_\xi u_1 + \omega(u_1, u_2)), \\ 0 &= \sigma \partial_\xi u_2 - \beta \omega(u_1, u_2). \end{aligned}$$

Boundary conditions:

$$(u_1, u_2, \partial_\xi u_1)(-\infty) = (u_{10}, 0, 0), \quad (u_1, u_2, \partial_\xi u_1)(\infty) = (0, 1, 0).$$

First-order traveling-wave system

$$
\begin{aligned}
\dot{u}_1 &= u_3, \\
\dot{u}_2 &= \frac{\beta}{\sigma}\omega(u_1, u_2), \\
\dot{u}_3 &= -\sigma u_3 - \omega(u_1, u_2).
\end{aligned}
$$

We want a solution that goes from an equilibrium $(u_{10}, 0, 0)$ **(each such point is an equilibrium) to the equilibrium** $(0, 1, 0)$**.**

Change of variables:

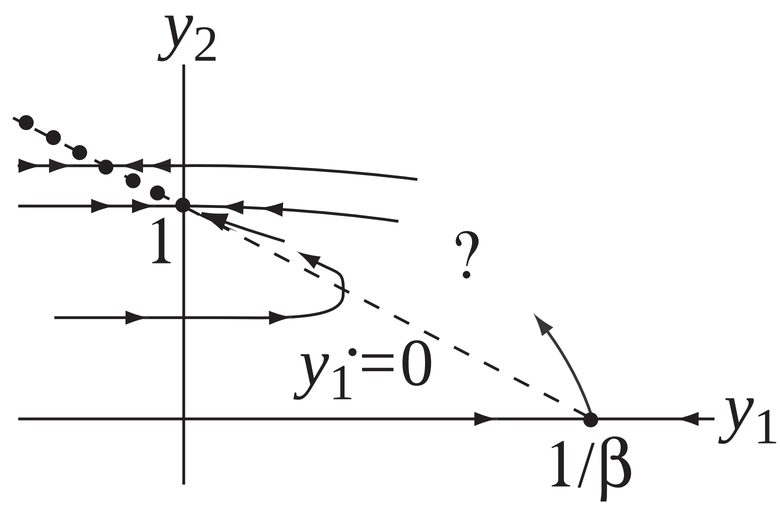
$$
\begin{aligned}
y_1 &= u_1, \\
y_2 &= u_2, \\
y_3 &= \sigma u_1 + \frac{\sigma}{\beta}u_2 + u_3.
\end{aligned}
$$

New system, equivalent but easier to study:

$$
\begin{aligned}
\dot{y}_1 &= -\sigma y_1 - \frac{\sigma}{\beta}y_2 + y_3, \\
\dot{y}_2 &= \frac{\beta}{\sigma}h(y_1, y_2), \\
\dot{y}_3 &= 0.
\end{aligned}
$$

Set $y_3 = \frac{\sigma}{\beta}$ so there will be an equilibrium $(y_1, y_2) = (0, 1)$:

$$\dot{y}_1 = g_1(y_1, y_2, \sigma) = -\sigma y_1 - \frac{\sigma}{\beta}(y_2 - 1),$$
$$\dot{y}_2 = g_2(y_1, y_2, \sigma) = \frac{\beta}{\sigma}\omega(y_1, y_2),$$



There is a unique $\sigma = c > 0$ for which there is a solution $(h_1, h_2)(\xi)$ that approaches $(\frac{1}{\beta}, 0)$ exponentially as $\xi \to -\infty$, and approaches $(0, 1)$ exponentially as $\xi \to \infty$.

The connection breaks in a regular manner as σ varies.

Melnikov integral:

Linearization along $(h_1, h_2)(\xi)$:

$$\begin{pmatrix} \dot v_1 \\ \dot v_2 \end{pmatrix} = \begin{pmatrix} -c & -\frac{c}{\beta} \\ \frac{\beta}{c}\partial_{u_1}\omega(h_1,h_2) & \frac{\beta}{c}\partial_{u_1}\omega(h_1,h_2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Up to scalar multiplication, the unique bounded solution of the adjoint equation is

$$\begin{pmatrix} \phi_1^*(\xi) & \phi_2^*(\xi) \end{pmatrix} = \exp\,(-\int_0^\xi a(\eta)\,d\eta)\begin{pmatrix} -\dot h_2(\xi) & \dot h_1(\xi) \end{pmatrix},$$

with $a(\xi) = -c + \frac{\beta}{c}\partial_{u_2}\omega(h_1,h_2)(\xi)$.

$$\begin{aligned} M &= \int_{-\infty}^{\infty} \begin{pmatrix} \phi_1^*(\xi) & \phi_2^*(\xi) \end{pmatrix} \begin{pmatrix} \partial_\sigma g_1(h_1(\xi),h_2(\xi),c) \\ \partial_\sigma g_2(h_1(\xi),h_2(\xi),c) \end{pmatrix} dt \\ &= \int_{-\infty}^{\infty} \exp\,(-\int_0^\xi a(\eta)\,d\eta)\begin{pmatrix} -\dot h_2(\xi) & \dot h_1(\xi) \end{pmatrix} \begin{pmatrix} -h_1(\xi) - \frac{1}{\beta}(h_2(\xi)-1) \\ -\frac{\beta}{c^2}\omega(h_1(\xi),h_2(\xi)) \end{pmatrix} dt \\ &= \int_{-\infty}^{\infty} \exp\,(-\int_0^\xi a(\eta)\,d\eta)\begin{pmatrix} -\dot h_2(\xi) & \dot h_1(\xi) \end{pmatrix} \begin{pmatrix} \frac{1}{c}\dot h_1(\xi) \\ -\frac{1}{c}\dot h_2(\xi) \end{pmatrix} dt \\ &= -\frac{2}{c}\int_{-\infty}^{\infty} \exp\,(-\int_0^\xi a(\eta)\,d\eta)\dot h_1(\xi)\dot h_2(\xi)\,dt > 0 \end{aligned}$$

because $\dot h_1(\xi) < 0$ and $\dot h_2(\xi) > 0$.

III. Linearization of the PDE at the Traveling Wave

Linearized PDE in moving coordinates

Notation: $U = (u_1, u_2)$, $\omega(U) = u_2\rho(u_1)$.

Linearize at $H(\xi)$:

$$\partial_t V = AV, \quad A = \begin{pmatrix} \partial_{\xi\xi} + c\partial_\xi + \partial_{u_1}\omega(H) & \partial_{u_2}\omega(H) \\ -\beta\partial_{u_1}\omega(H) & c\partial_\xi - \beta\partial_{u_2}\omega(H) \end{pmatrix}$$

Look for eigenvalue-eigenfunction pairs: solutions of the form $e^{\lambda t}V(\xi)$. They satisfy

$$\lambda V = AV.$$

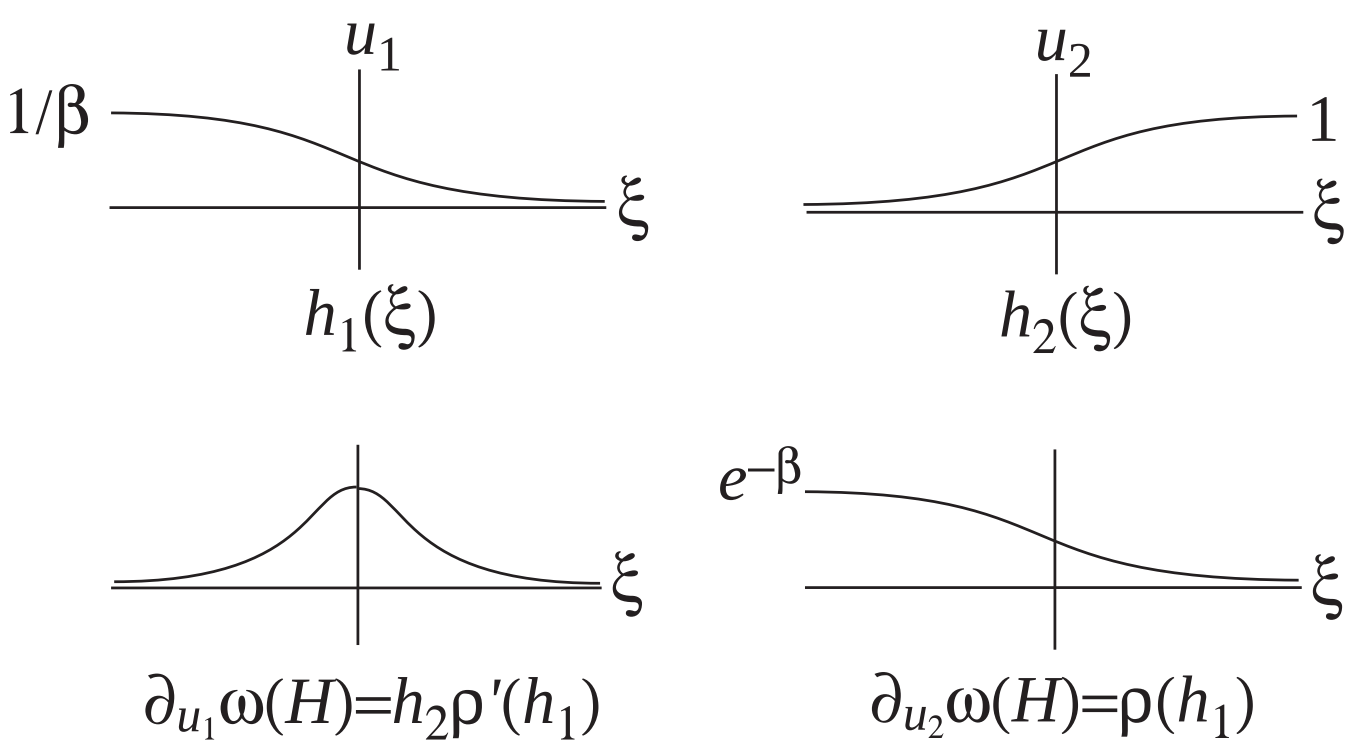
As a system:

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{\beta}{c}\partial_{u_1}\omega(H) & \frac{\beta}{c}\partial_{u_2}\omega(H) + \frac{\lambda}{c} & 0 \\ \lambda - \partial_{u_1}\omega(H) & -\partial_{u_1}\omega(H) & \end{pmatrix} \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{pmatrix}.$$

λ is an eigenvalue of the linearized PDE provided this Eigenvalue System has a nontrivial solution with appropriate behavior at $\xi = \pm\infty$.

Eigenvalue System:

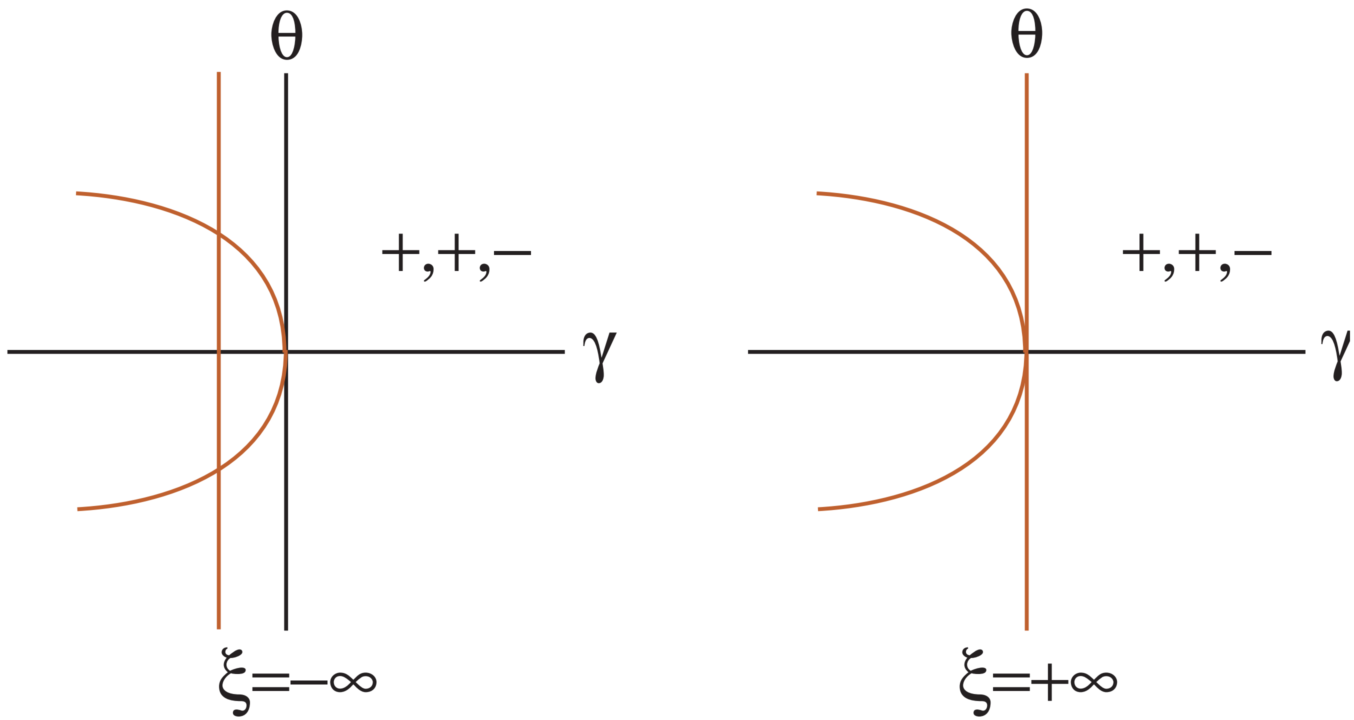
$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{\beta}{c}\partial_{u_1}\omega(H) & \frac{\beta}{c}\partial_{u_2}\omega(H) + \frac{\lambda}{c} & 0 \\ \lambda - \partial_{u_1}\omega(H) & -\partial_{u_1}\omega(H) & \end{pmatrix} \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{pmatrix}.$$



Write $\lambda = \gamma + i\theta$.

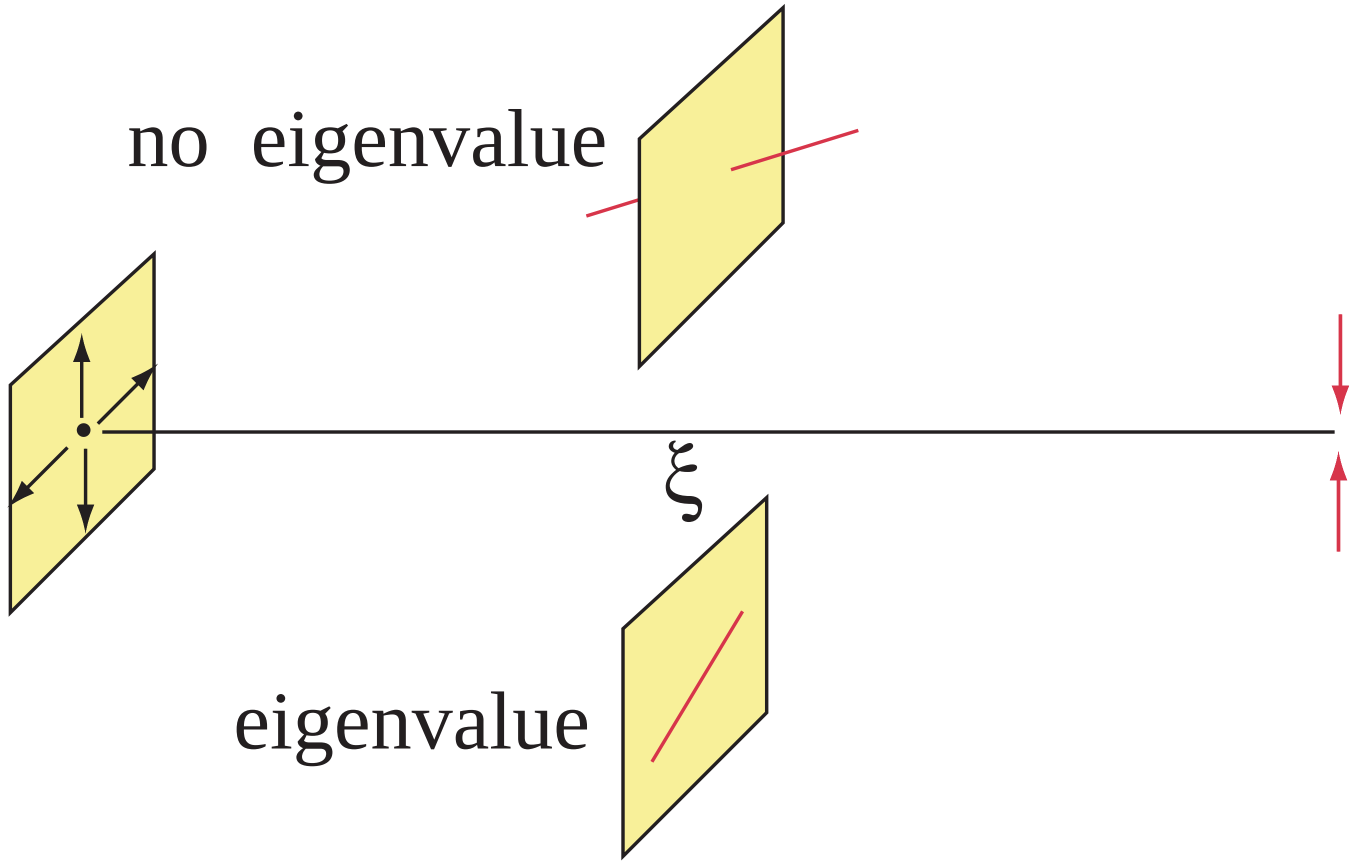
At $\xi = +\infty$, one eigenvalue has real part 0 if $\gamma = 0$ or $\gamma = -\frac{\theta^2}{c^2}$.

At $\xi = -\infty$, one eigenvalue has real part 0 if $\gamma = -\frac{\beta}{c}e^{-\beta}$ or $\gamma = -\frac{\theta^2}{c^2}$.



If we work in BUC^2, $\Omega_0 = \{\lambda : \text{Re } \lambda > 0\}$ is the "region of consistent splitting": at both $\xi = -\infty$ and $\xi = \infty$ the Eigenvalue System has two positive eigenvalues and one negative eigenvalue.

Eigenvalue System for $\lambda \in \Omega_0$:



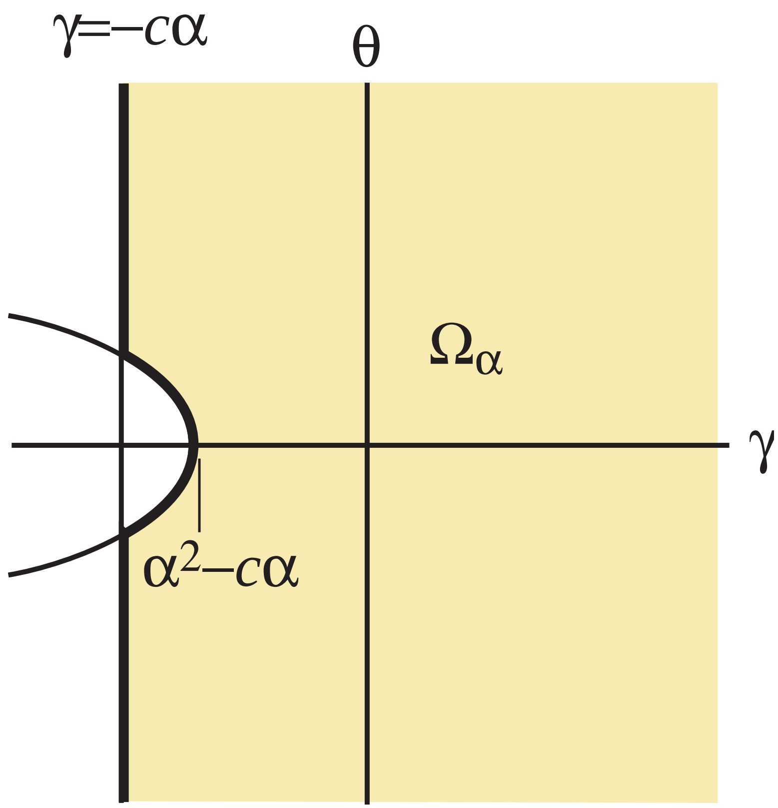
For λ in the region of consistent splitting, $\mathcal{A} - \lambda \mathcal{I}$ is Fredholm of index 0.

The boundary of the region of consistent splitting is in the essential spectrum.

If we work in BUC^2, the imaginary axis is in the essential spectrum: no spectral stability.

However:

Let $0 < \alpha < \frac{1}{2}c$. Let Ω_α denote the set of λ such that at both $\xi = -\infty$ and $\xi = \infty$ there are two eigenvalues greater than $-\alpha$ and one less than $-\alpha$. Ω_α is the region of consistent splitting when we work in BUC^2_α.



The parabola is $\gamma = (\alpha^2 - c\alpha) - \frac{\theta^2}{(c-2\alpha)^2}$.

The essential spectrum is to the left of the imaginary axis when we work in BUC^2_α.

Evans function

In the Eigenvalue System, let

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma & \frac{\sigma}{\beta} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

We obtain

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} -c & -\frac{c}{\beta} & 1 \\ \frac{\beta}{c}\partial_{u_1}\omega(h_1, h_2) & \frac{\beta}{c}\partial_{u_2}\omega(h_1, h_2) + \frac{\lambda}{c} & 0 \\ \lambda & \frac{\lambda}{\beta} & 0 \end{pmatrix} \begin{pmatrix} U \\ Y \\ W \end{pmatrix}.$$

This system is equivalent to the Eigenvalue System but is easier to study.

We'll write it $\dot{Z} = E(\xi, \lambda)Z$.

For $\lambda \in \Omega_\alpha$, there is a unique eigenvalue of $\dot{Z} = E(\xi, \lambda)Z$ at $\xi = \infty$ with real part less than $-\alpha$, call it $-\mu(\lambda) < -\alpha$. An eigenvector is

$$Z_+(\lambda) = \begin{pmatrix} -1 \\ 0 \\ -c + \mu(\lambda) \end{pmatrix}.$$

Let $Z_+(\xi, \lambda)$ be the unique solution of $\dot{Z} = E(\xi, \lambda)Z$ such that

$$\lim_{\xi \to \infty} e^{\mu(\lambda)\xi} Z_+(\xi, \lambda) = Z_+(\lambda).$$

$Z_+(\xi, 0)$ is a positive multiple of $(\dot{h}_1(\xi), \dot{h}_2(\xi), 0)$.

Note that $\dot{h}_1(\xi) < 0$; that's why we chose $Z_+(\lambda)$ to have its first component negative.

For $\lambda \in \Omega$, the unique eigenvalue of the adjoint system $\dot{\psi} = -\psi E(\xi, \bar{\lambda})$ at $\xi = -\infty$ with real part greater than α is $\overline{\mu(\lambda)}$. A corresponding left eigenvector is

$$\psi_-(\bar{\lambda}) = \begin{pmatrix} * & * & 1 \end{pmatrix}.$$

Let $\psi_-(\xi, \bar{\lambda})$ be the unique solution of $\dot{\psi} = -\psi E(\xi, \bar{\lambda})$ such that

$$\lim_{\xi \to -\infty} e^{-\overline{\mu(\lambda)}\xi} \psi_-(\xi, \bar{\lambda}) = \psi_+(\bar{\lambda}).$$

Let $\psi^*(\xi) = \psi_-(\xi, 0)$.

Recall $\begin{pmatrix} \phi_1^*(\xi) & \phi_2^*(\xi) \end{pmatrix}$ defined earlier, and define

$$\phi_3^*(\xi) = -\int_{-\infty}^{\xi} \phi_1^*(\eta)\, d\eta.$$

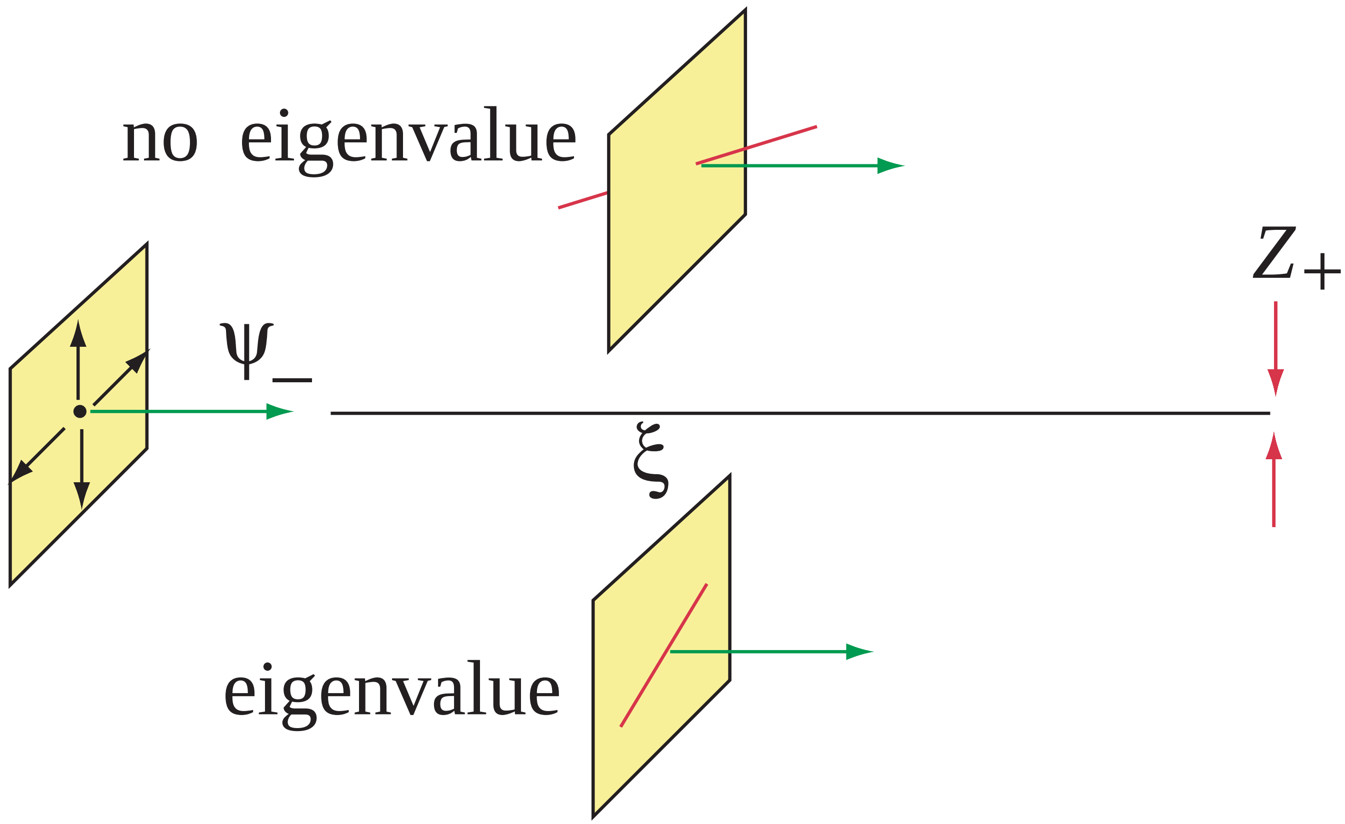
Proposition. As $\xi \to -\infty$, $\phi_3^*(\xi) \to 0$ like $e^{c\xi}$; and there is a number $d > 0$ such that as $\xi \to \infty$, $\phi_3^*(\xi) \to d$ exponentially. $\psi^*(\xi)$ is a positive multiple of $\begin{pmatrix} \phi_1^*(\xi) & \phi_2^*(\xi) & \phi_3^*(\xi) \end{pmatrix}$.

We define the Evans function

$$D(\lambda) = \bar{\psi}_-(\xi, \bar{\lambda}) Z_+(\xi, \lambda).$$

The product is independent of ξ and analytic in λ.

For $\lambda \in \Omega_\alpha$, λ is in the spectrum of the linearized PDE on BUC^2_α if and only if $D(\lambda) = 0$.



Of course, $D(0) = 0$.

Calculation of $D'(0)$

Sandstede gives the formula: up to multiplication by a positive number,

$$D'(0) = -\int_{-\infty}^{\infty} \psi^*(\xi) \frac{\partial E}{\partial \lambda}(\xi, 0) \dot{z}^*(\xi)\, d\xi.$$

Up to multiplication by a positive number, we calculate:

$$\begin{aligned}
D'(0) &= -\int_{-\infty}^{\infty} \psi^*(\xi) \frac{\partial E}{\partial \lambda}(\xi, 0) \dot{H}(\xi)\, d\xi \\
&= -\int_{-\infty}^{\infty} \begin{pmatrix} \psi_1^*(\xi) & \psi_2^*(\xi) & \psi_3^*(\xi) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 1 & \frac{1}{\beta} & 0 \end{pmatrix} \begin{pmatrix} \dot{h}_1(\xi) \\ \dot{h}_2(\xi) \\ 0 \end{pmatrix} d\xi \\
&= -\frac{1}{c}\int_{-\infty}^{\infty} \psi_2^*(\xi)\dot{h}_2(\xi)\, d\xi + \int_{-\infty}^{\infty} \psi_3^*(\xi)(\dot{h}_1(\xi) + \frac{1}{\beta}\dot{h}_2(\xi))\, d\xi. \\
&= -\frac{1}{c}\int_{-\infty}^{\infty} \psi_2^*(\xi)\dot{h}_2(\xi)\, d\xi - \frac{1}{c}\int_{-\infty}^{\infty} \psi_3^*(\xi)\ddot{h}_1(\xi)\, d\xi.
\end{aligned}$$

We integrate the second integral by parts:

$$\int_{-\infty}^{\infty} \psi_3^*(\xi)\ddot{h}_1(\xi)\,d\xi = \psi_3^*(\infty)\dot{h}_1(\infty) - \psi_3^*(-\infty)\dot{h}_1(-\infty) - \int_{-\infty}^{\infty} \dot{\psi}_3^*(\xi)\dot{h}_1(\xi)\,d\xi.$$

We have $\psi_3^*(\infty)$ finite, $\dot{h}_1(\infty) = 0$, $\psi_3^*(-\infty) = 0$, and $\dot{h}_1(-\infty) = 0$. Therefore the boundary terms vanish. We conclude that, up to multiplication by a positive number,

$$\begin{aligned} D'(0) = \frac{1}{c}\int_{-\infty}^{\infty} -\psi_2^*(\xi)\dot{h}_2(\xi)+\dot{\psi}_3^*(\xi)\dot{h}_1(\xi)\,d\xi &= \frac{1}{c}\int_{-\infty}^{\infty} -\psi_2^*(\xi)\dot{h}_2(\xi)+\psi_1^*(\xi)\dot{h}_1(\xi)\,d\xi \\ &= \frac{2}{c}\int_{-\infty}^{\infty} -\exp\left(-\int_0^{\xi} a(\eta)\,d\eta\right)\dot{h}_1(\xi)\dot{h}_2(\xi)\,d\xi > 0. \end{aligned}$$

IV. On BUC_α, spectral stability implies linearized stability

$$\partial_t V = AV, \quad A = \begin{pmatrix} \partial_{\xi\xi} + c\partial_\xi + \partial_{u_1}\omega(H) & \partial_{u_2}\omega(H) \\ -\beta\partial_{u_1}\omega(H) & c\partial_\xi - \beta\partial_{u_2}\omega(H) \end{pmatrix}$$

Let $\mathcal{A}$, $\mathcal{A}_\alpha$, and $\mathcal{A}_m$ be the linear operators on BUC^2, BUC^2_α, and BUC^2_m respectively given by $V \to AV$.

Each operator is closed and densely defined. If $V \in BUC^2_m$, then $\mathcal{A}V = \mathcal{A}_\alpha V = \mathcal{A}_m V$. Each operator generates a C_0 semigroup. If $V \in BUC^2_m$, then $e^{t\mathcal{A}}V = e^{t\mathcal{A}_\alpha}V = e^{t\mathcal{A}_m}V$.

$\mathcal{A}_0$ and $\mathcal{A}_m$ both have 0 in the essential spectrum.

$\mathcal{A}_\alpha$ is Fredholm with index zero (because 0 is in Ω_α), and that 0 is a simple eigenvalue. Therefore $\mathrm{R}(\mathcal{A}_\alpha)$ is a codimension-one closed subspace of BUC^2_α.

Let $\mathcal{P}^s_\alpha$ denote projection onto $\mathrm{R}(\mathcal{A}_\alpha)$ with kernel $\mathrm{N}(\mathcal{A}_\alpha)$. Let $\mathcal{P}^c_\alpha = \mathcal{I} - \mathcal{P}^s_\alpha$.

Theorem. Suppose the only eigenvalue of $\mathcal{A}_\alpha$ with nonnegative real part is 0. Then :

(1) The traveling wave is spectrally stable.
(2) The traveling wave is linearly stable. In particular, there are numbers $K > 0$ and $\nu > 0$ such that $\|e^{t\mathcal{A}_\alpha \mathcal{P}^s_\alpha}\| \le Ke^{-\nu t}$.

The proof of the theorem uses some notions from semigroup theory.

Let

$$W(\xi) = e^{\alpha\xi} V(\xi)$$

$V(t,\xi)$ is a solution of $\partial_t V = AV$ in BUC^2_α if and only if $W(t,\xi) = e^{\alpha\xi} V(t,\xi)$ is a solution of

$$W_t = \tilde{A} W, \quad \tilde{A} = \begin{pmatrix} \partial_{\xi\xi} + (c-2\alpha)\partial_\xi + \alpha^2 - c\alpha + \partial_{u_1}\omega(H) & \partial_{u_2}\omega(H) \\ -\beta\partial_{u_1}\omega(H) & c\partial_\xi - c\alpha - \beta\partial_{u_2}\omega(H) \end{pmatrix}$$

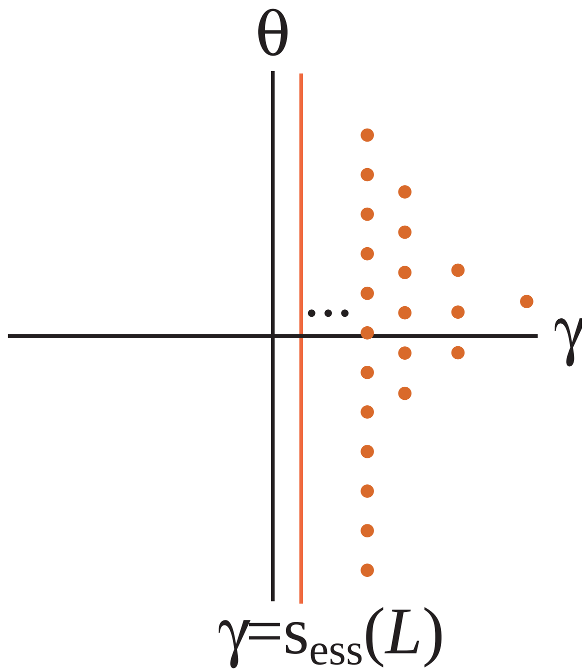
in BUC^2.

Let $\tilde{\mathcal{A}}$ be the linear operator on BUC^2 given by $W \to \tilde{A} W$.

Instead of considering $\mathcal{A}_\alpha$ on BUC^2_α we may consider $\tilde{\mathcal{A}}$ on BUC^2.

Spectral bounds

The **essential spectral bound** $s_{\mathrm{ess}}(\mathcal{L})$ is the infimum of all real ω such that the intersection $\mathrm{Sp}(\mathcal{L}) \cap \{\lambda : \mathrm{Re}\,\lambda \geq \omega\}$ is contained in the discrete spectrum of $\mathcal{L}$ and has only finitely many points.



For a *bounded* linear operator $T : \mathcal{Y} \to \mathcal{Y}$, define the seminorm

$$\|T\|_C = \inf_K \|T + K\|,$$

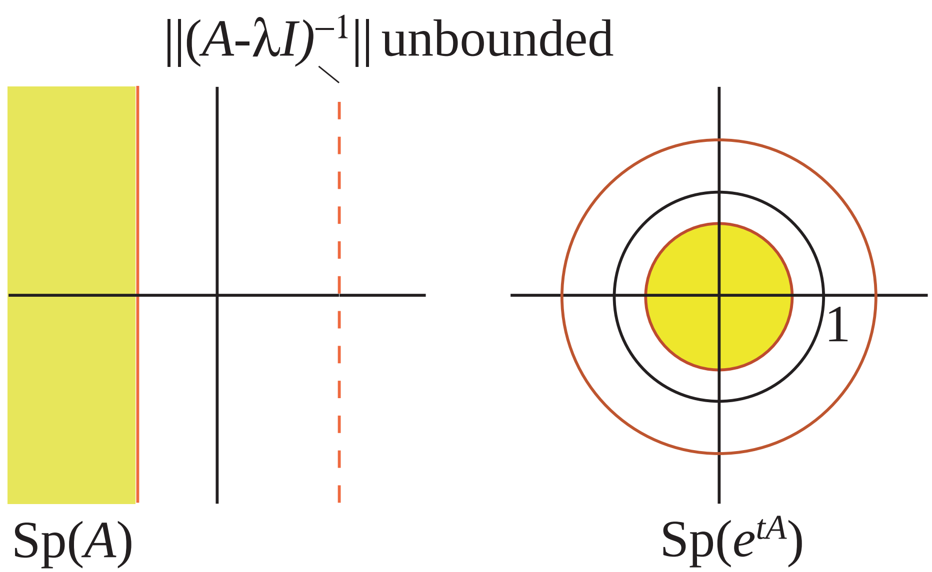
where the infimum is over the set of all compact operators $K : \mathcal{Y} \to \mathcal{Y}$.

If $\mathcal{L}$ generates a C_0-semigroup $e^{t\mathcal{L}}$, the **essential growth bound** $\omega_{\text{ess}}(\mathcal{L}) = \lim_{t\to\infty} t^{-1} \log \|e^{t\mathcal{L}}\|_C$.

In general:

$$\mathrm{s}_{\text{ess}}(\mathcal{L}) \le \omega_{\text{ess}}(\mathcal{L})$$

One kind of problem:



Facts about the essential growth bound

1. $e^{t\omega_{\text{ess}}(\mathcal{L})}$ is the radius of the essential spectrum of $e^{t\mathcal{L}}$ for any $t > 0$.

2. Let $\omega > \omega_{\text{ess}}(\mathcal{L})$ be a number such that no isolated eigenvalue of $\mathcal{L}$ has real part ω. Then there is a finite set $\{\lambda_1, \ldots, \lambda_k\} \subset \mathbb{C}$ such that

$$\operatorname{Sp}(\mathcal{L}) \cap \{\lambda : \operatorname{Re}\lambda \geq \omega\} = \operatorname{Sp}_{\text{d}}(\mathcal{L}) \cap \{\lambda : \operatorname{Re}\lambda \geq \omega\} = \{\lambda_1, \ldots, \lambda_k\}.$$

Let $E_1, \ldots, E_k$ be the generalized eigenspaces of $\lambda_1, \ldots, \lambda_k$ respectively; they are finite-dimensional. Then there is a closed subspace E_0 of $\mathcal{Y}$ such that $\mathcal{Y} = E_0 \oplus E_1 \oplus \cdots \oplus E_k$ and E_0 is invariant under $\mathcal{L}$. Moreover, there is a number $M > 0$ such that $\|e^{t\mathcal{L}}|E_0\| \leq Me^{\omega t}$.

Outline of proof: Consider $W_t = \tilde{\mathcal{A}} W$ on BUC^2.

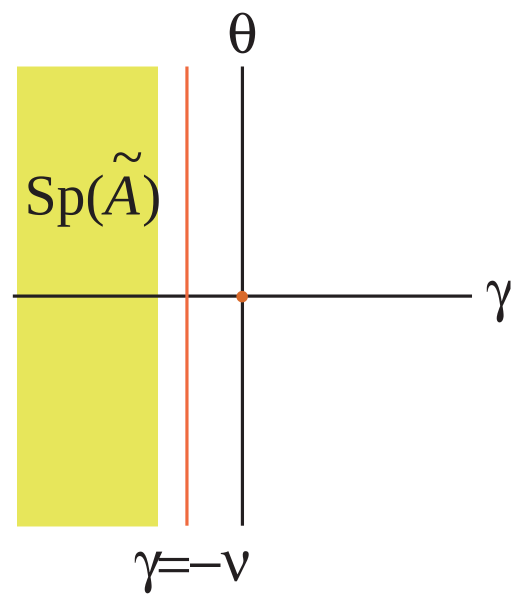
1. $\tilde{\mathcal{A}}$ generates a C_0-semigroup $e^{t\tilde{\mathcal{A}}}$.

2. The only eigenvalue of $\tilde{\mathcal{A}}$ with nonnegative real part is 0.

3. The eigenvalue 0 is simple.

4. $\omega_{\text{ess}}(\tilde{\mathcal{A}}) < 0$. (**The key point.** $\mathrm{s}_{\text{ess}}(\tilde{\mathcal{A}}) = \omega_{\text{ess}}(\tilde{\mathcal{A}}) = \alpha^2 - c\alpha < 0$.)

5. Therefore we can choose $-\nu < 0$ such that the only element of $\mathrm{Sp}(\tilde{\mathcal{A}})$ with real part greater than or equal to $-\nu$ is 0.



6. $BUC^2 = \mathrm{R}(\tilde{\mathcal{A}}) + \mathrm{N}(\tilde{\mathcal{A}})$ and $\|e^{t\tilde{\mathcal{A}}}|\mathrm{R}(\tilde{\mathcal{A}})\| \leq Ke^{-\nu t}$.

Outline of proof that $\omega_{\mathrm{ess}}(\tilde{\mathcal{A}}) < 0$:

Recall

$$
\begin{aligned}
\tilde{A} &= \begin{pmatrix} \partial_{\xi\xi} + (c-2\alpha)\partial_\xi + \alpha^2 - c\alpha + \partial_{u_1}\omega(H) & \partial_{u_2}\omega(H) \\ -\beta\partial_{u_1}\omega(H) & c\partial_\xi - c\alpha - \beta\partial_{u_2}\omega(H) \end{pmatrix} \\
&= \begin{pmatrix} C & \partial_{u_2}\omega(H) \\ -\beta\partial_{u_1}\omega(H) & G \end{pmatrix}.
\end{aligned}
$$

Let

$$
\mathcal{J}_1 = \begin{pmatrix} C & \partial_{u_2}\omega(H) \\ 0 & G \end{pmatrix}.
$$

1. C is a “localized” perturbation of the sectorial operator $\partial_{\xi\xi}+(c-2\alpha)\partial_\xi+\alpha^2-c\alpha$. Hence its essential spectrum has for its right boundary the parabola

$$
\{\lambda = \gamma + i\theta : \gamma = (\alpha^2 - c\alpha) - \frac{\theta^2}{(c-2\alpha)^2}\}.
$$

Also, $\mathrm{s}_{\mathrm{ess}}(\mathcal{A}) = \omega_{\mathrm{ess}}(\mathcal{A}) = \alpha^2 - c\alpha < 0$.

2. Associated with G are the two constant-coefficient operators $c\partial_\xi - c\alpha + \beta e^{-\frac{1}{\beta}}$ at $\xi = -\infty$ and $c\partial_\xi - c\alpha$ at $\xi = \infty$. Each has spectrum consisting of a single vertical line: $\operatorname{Re}\lambda = -c\alpha - \beta e^{-\frac{1}{\beta}}$ and $\operatorname{Re}\lambda = -c\alpha$ respectively. $\operatorname{Sp}(G) = \operatorname{Sp}_{\text{ess}}(G) = \{\lambda : -c\alpha - \beta e^{-\frac{1}{\beta}} \leq \operatorname{Re}\lambda \leq -c\alpha\}$. $\mathrm{s}_{\text{ess}}(G) = -c\alpha$.

3. It is known that G **has the spectral mapping property**, so $\mathrm{s}_{\text{ess}}(G) = \omega_{\text{ess}}(G)$.

4. **From triangularity of $\mathcal{J}_1$ and our understanding of the spectra of C and G, $\omega_{\text{ess}}(\mathcal{J}_1) \leq \max\{\omega_{\text{ess}}(C), \omega_{\text{ess}}(G)\} = \alpha^2 - c\alpha < 0$.**

5. **Since $\lim_{\xi\to\pm\infty} \partial_{u_1}\omega(H) = 0$, multiplication by $-\beta\partial_{u_1}\omega(H)$ is a compact operator.**

6. **From the variation of constants formula, $e^{t\tilde{\mathcal{A}}}$ is a compact perturbation of $e^{t\mathcal{J}_1}$**, so $\omega_{\text{ess}}(\tilde{\mathcal{A}}) = \omega_{\text{ess}}(\mathcal{J}_1)$ by definition.

V. Nonlinear stability

The PDE in moving coordinates:

$$\begin{aligned}\partial_\xi u_1 &= \partial_{\xi\xi} u_1 + c\partial_\xi u_1 + \omega(U)),\\ \partial_\xi u_2 &= c\partial_\xi u_2 - \beta\omega(U).\end{aligned}$$

Notation:

$$L = \begin{pmatrix} \partial_{\xi\xi} + c\partial_\xi & 0 \\ 0 & c\partial_\xi \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -\beta \end{pmatrix}.$$

The PDE becomes

$$U_t = LU + B\omega(U).$$

$$U_t = LU + B\omega(U).$$

Let $U = H + V$.

$$\begin{aligned}
\omega(U) &= u_2\rho(u_1), \\
\omega(H+V) &= \omega(H) + D\omega(H)V + \text{remainder}, \\
\text{remainder} &= h_2\rho_2(h_1, v_1)v_1^2 + \rho_1(h_1, v_1)v_1v_2 = n(H, V).
\end{aligned}$$

More notation:

$$R(\xi) = D\omega(H(\xi)) = \begin{pmatrix} h_2(\xi)\rho'(h_1(\xi)) & \rho(h_1(\xi)) \end{pmatrix},$$

$$A = L + BR(\xi).$$

The PDE becomes

$$V_t = AV + Bn(H, V).$$

We'll need a slightly different substitution:

$$U(\xi) = H(\xi - q) + V(\xi),$$

where q can change with time. The PDE becomes

$$-H'(\xi - q)\dot{q} + \partial_t V = (L + BR(\xi - q))V + Bn(H(\xi - q), V).$$

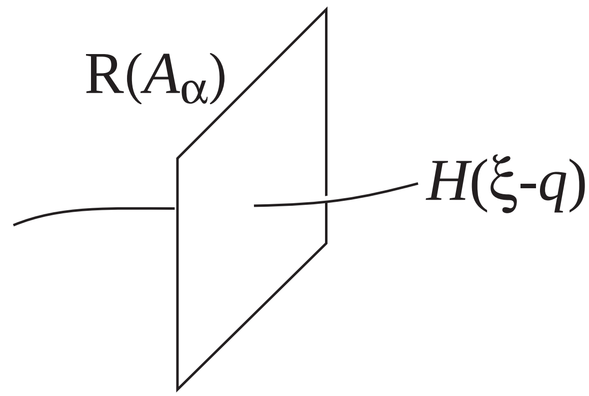
Let

$$S(\xi, q) = R(\xi - q) - R(\xi),$$

The PDE becomes

$$-H'(\xi - q)\dot{q} + \partial_t V = AV + BS(\xi, q)V + Bn(H(\xi - q), V).$$

$U(\xi) = H(\xi - q) + V(\xi)$. Assume $V \in \mathrm{R}(\mathcal{A}_\alpha)$.



Apply $\mathcal{P}_\alpha^s$ and $\mathcal{P}_\alpha^c$.

$$\partial_t V = AV + \mathcal{P}_\alpha^s \Big(BS(\xi, q)V + Bn(H(\xi - q), V) + H'(\xi - q)\dot{q} \Big),$$
$$-\mathcal{P}_\alpha^c H'(\xi - q)\dot{q} = \mathcal{P}_\alpha^c \Big(BS(\xi, q)V + Bn(H(\xi - q), V) \Big)$$

This step is formal: the nonlinear terms do not define a map from $\mathcal{E}_\alpha^2$ to itself.

Let

$$G(V, q) = BS(\xi, q)V + Bn(H(\xi - q), V),$$
$$\kappa(V, q) = \big(\mathcal{P}_\alpha^c H'(\xi - q)\big)^{-1} \mathcal{P}_\alpha^c G(V, q).$$

(Abuse of notation warning.) For q small, $\|\mathcal{P}_\alpha^c H'(\xi - q)\|$ is close to 1.

So formally we can rewrite our PDE as a system on $\mathrm{R}(\mathcal{A}_\alpha) \times \mathbb{R}$:

$$\begin{aligned} \partial_t V &= AV + G(V,q) + \kappa(V,q)H'(\xi - q), \\ \dot{q} &= \kappa(V,q) \end{aligned}$$

Proposition. The formulas for $G(V,q)$ and $\kappa(V,q)$ define mappings from $BUC^2_m \times \mathbb{R}$ to BUC_m and to $\mathbb{R}$ respectively. On any bounded neighborhood of $(0,0)$ in $BUC^2_m \times \mathbb{R}$, the mappings are Lipschitz, and there is a constant C such that:

(1) $\|G(V,q)\|_\alpha \leq C(|q| + \|V\|_0)\|V\|_\alpha$.

(2) $\|G(V,q)\|_m \leq C(|q| + \|V\|_m)\|V\|_m$.

(3) $|\kappa(V,q)| \leq C(|q| + \|V\|_0)\|V\|_\alpha$.

Reason: consider a term like $\rho_1(h_1, v_1)v_1v_2$

$$e^{\alpha\xi}|\rho_1(h_1(\xi), v_1(\xi))v_1(\xi)v_2(\xi)| \leq C\|v_1\|_0\|v_2\|_\alpha.$$

Therefore

$$\|\rho_1(h_1, v_1)v_1v_2\|_\alpha \leq C\|v_1\|_0\|v_2\|_\alpha.$$

Study of the system on the space $BUC^2_m \times \mathbb{R}$

1. Existence of solutions on $BUC^2_m \times \mathbb{R}$ and *a priori* bound

We shall study solutions of the system

$$\begin{aligned} \partial_t V &= AV + G(V,q) + \kappa(V,q)H'(\xi - q), \\ \dot{q} &= \kappa(V,q) \end{aligned}$$

Proposition 1. For each $\delta > 0$, if $\rho > 0$ is sufficiently small, then there exists $T_{\max}$, with $0 < T_{\max} \leq \infty$, such that the following is true: if $(V^0, q^0) \in BUC^2_m \times \mathbb{R}$ satisfies

$$\|(V^0, q^0)\|_{BUC^2_m \times \mathbb{R}} = \|V^0\|_m + |q^0| \leq \rho \tag{1}$$

and $0 \leq t < T_{\max}$, then $(V,q)(t, V^0, q^0)$ is defined and satisfies

$$\|V(t, V^0, q^0)\|_m + |q(t, V^0, q^0)| \leq \delta. \tag{2}$$

Let $T_{\max}(\delta, \rho)$ denote the supremum of all T such that (2) holds for all $0 \leq t < T$ whenever (1) is satisfied.

2. Decay of $\|V(t)\|_\alpha$

Proposition 2. Consider the solution given by Proposition 1. There is a number $C>0$ such that if (1) δ is sufficiently small and (2) $V^0 \in \mathrm{R}(\mathcal{P}^s_\alpha)\cap BUC^2_m$, then

(3) $\|V(t)\|_\alpha \leq Ke^{-\nu t/2}\|V^0\|_\alpha$ and $|q(t)-q^0| \leq C\|V^0\|_\alpha$ for $0\leq t< T_{\max}(\delta,\rho)$.

Moreover, if $T_{\max}(\delta,\rho)=\infty$, then there is $q^*\in\mathbb{R}$ such that

$$|q(t)-q^*| \leq Ce^{-\nu t/2}\|V^0\|_\alpha \text{ for all } t\geq 0. \tag{4}$$

3. Bounds for $\|V(t)\|_0$

Proposition 3. Consider the solution given by Proposition 1. There is a number $C>0$ such that if (1) δ is sufficiently small and (2) $V^0 \in \mathrm{R}(\mathcal{P}^s_\alpha)\cap BUC^2_m$, then for $0\leq t<T_{\max}(\delta,\rho)$):

$$\|v_1(t)\|_0 \leq C(|q^0|+\|V^0\|_m), \tag{5}$$

$$\|v_2(t)\|_0 \leq C(|q^0|+\|V^0\|_m)e^{-\nu t/2}. \tag{6}$$

This is the key step: the solution stays bounded in BUC^2, with a good bound.

$$\partial_t v_1 = (\partial_{\xi\xi} + c\partial_\xi)v_1 + \dots, \quad \partial_t v_2 = (c\partial_\xi + a(t,\xi))v_2 + \dots,$$

- $a(t,\xi) < -\nu$.
- Ignoring omitted terms, solutions of the second equation satisfy
$$\|v_2(t,\xi)\|_0 \le e^{-\nu t}\|v_2(0,\xi)\|_0.$$
- Ignoring omitted terms, the first equation generates a bounded semigroup in $BUC(\mathbb{R})$.
- When the omitted terms are included, the second equation can be solved first and the solution estimated, and the result can be used to estimate the solution of the first equation. Proposition 2 is also used.

- Idea–**bound the solution in a uniform norm in order to prove convergence in a weighted norm**–comes from R. Pego and M. Weinstein, *Asymptotic stability of solitary waves*, Comm. Math. Phys. **164** (1994), 305–349.
- In Pego and Weinstein, boundedness in the uniform norm follows from a Hamiltonian structure.
- In other papers, it is related to the stability of the bifurcating patterns that are connected by the front.

Why does

- the equation for v_1, the temperature perturbation, have solutions that are only bounded, but
- the equation for v_2, the fuel perturbation has solutions that decay?

Answer:

- At the left, the combustion front has temperature $\frac{1}{\beta}$ and 0 fuel.
- Increase the temperature at the left to $\frac{1}{\beta} + v_1$: it basically stays there.
- Increase the fuel at the left to v_2: it all burns!

4. Nonlinear stability

Lemma 1. Define $\mathcal{F} : (\mathrm{R}(\mathcal{P}^s_\alpha) \cap BUC^2_m) \times \mathbb{R} \to BUC^2_m$ by $\mathcal{F}(V,q) = V + H(\xi - q)$. Then $D\mathcal{F}(0,0)$ is an isomorphism, so $\mathcal{F}$ maps a neighborhood $\mathcal{V}$ of $(0,0)$ in $(\mathrm{R}(\mathcal{P}^s_\alpha) \cap BUC^2_m) \times \mathbb{R}$ diffeomorphically onto a neighborhood $\mathcal{U}$ of H in BUC^2_m.

Choose $\rho_{\mathcal{U}} > 0$ so that the ball of radius $\rho_{\mathcal{U}}$ about H in BUC_α is contained in $\mathcal{U}$.

Given $U^0 \in BUC^2_m$, let $U(t) = U(t, U^0)$ be the solution of our PDE in BUC^2_m with $U(0) = U^0$. If $\|U^0 - H\|_m \le \rho_{\mathcal{U}}$, we can use Lemma 1 to write

$$U^0 = V^0 + H(\xi - q^0) \text{ with } (V^0, q^0) \in (\mathrm{R}(\mathcal{P}^{\mathrm{s}}_\alpha) \cap \mathrm{BUC}^2_{\mathrm{m}}) \times \mathbb{R}. \tag{7}$$

If $\|U(t) - H\|_m \le \rho_{\mathcal{U}}$, we can use Lemma 1 to write

$$U(t) = V(t) + H(\xi - q(t)) \text{ with } (V(t), q(t)) \in (\mathrm{R}(\mathcal{P}^s_\alpha) \cap BUC^2_m) \times \mathbb{R}. \tag{8}$$

Nonlinear Stability Theorem There is a constant $C > 0$ such that for each sufficiently small $\delta > 0$, there exists ρ with $0 < \rho \leq \rho_{\mathcal{U}}$ such that the following is true. Let $U^0 \in BUC^2_m$ with $\|U^0 - H\|_m < \rho$, and let $(V^0, q^0) = \mathcal{F}^{-1}(U^0)$. Let $U(t)$ be the solution of our PDE in $\mathcal{E}^2$ with $U(0) = U^0$. Then:

(1) $U(t)$ is defined for all $t \geq 0$.

(2) For all $t \geq 0$, $U(t) \in \mathcal{U}$, so we can define $(V(t), q(t)) = \mathcal{F}^{-1}(U(t))$.

(3) $\|V(t)\|_m + |q(t)| < \delta$.

(4) $\|V(t)\|_\alpha \leq Ke^{-\nu t/2}\|V^0\|_\alpha$.

(5) There exists q^* such that $|q(t) - q^*| \leq Ce^{-\nu t/2}\|V^0\|_\alpha$.

(6) $\|v_1(t)\|_0 \leq C(|q^0| + \|V^0\|_m)$.

(7) $\|v_2(t)\|_0 \leq C(|q^0| + \|V^0\|_m)e^{-\nu t/2}$.