Traveling Waves in a Thin Film with Surfactant

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Sociological Background

Raleigh-Durham-Chapel Hill, North Carolina, USA

- North Carolina State University
- Duke University
- University of North Carolina at Chapel Hill
- SAMSI (Statistical and Applied Mathematical Sciences Institute)

"Dynamical Systems Working Group"

- People: Chris Jones (UNC-CH), Xiao-Biao Lin (NC State), S. (NC State), postdocs, graduate students, ...
- Topics: Geometric singular perturbation theory, traveling waves, Evans function, ...

Recently colleagues have begun presenting problems involving traveling waves.

This work grew out of a presentation by Michael Shearer (NC State) on work with Tom Witelski (Duke), Rachel Levy (Duke), Karen Daniels (NC State).

Scientific Background

Surfactant = **SURF**ace **ACT**ive **A**ge**NT** (1950): agent that reduces the surface tension of water by forming a molecular or atomic layer on the surface (surfactant is adsorbed rather than absorbed).

Surface tension is caused by attraction of adjacent water molecules. When it is locally reduced by addition of surfactant, the surfactant spreads (surfactant concentration gradient → surface tension gradient → Marangoni force). To see this, put a scrap of paper in a pan of water and add a drop of dishwashing detergent.

Our subject: **surfactant** added to **thin liquid film** on an **incline**.

One motivation: role of surfactants in breathing. Mammalian lungs contain alveoli, little cavities where gas is exchanged with blood. (Human lungs contain 300 million alveoli with radius 0.1 mm.)

Inside of alveoli is coated with liquid. During exhalation, air pressure in alveoli falls, and surface tension would try decrease volume of alveoli, leading to lung collapse. To avoid this, **lungs contain surfactant**, which reduces surface tension.

Infant respiratory distress syndrome: lungs don't produce enough surfactant. Leading cause of death in premature infants. Treatment involves adding surfactant through a breathing tube.

Experiment showing expansion of region of thin liquid film treated with surfactant. Appears to be a traveling front with stepped structure that is unstable to transverse perturbations.

From *Physics of Fluids* Gallery of Liquid Motion.

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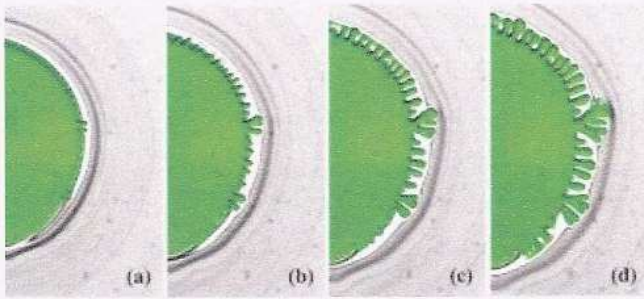


FIG. 1. A droplet of glycerol containing sodiumdodecylsulfate spreading on a glycerol film. The images were recorded at (a) $t=20$, (b) 28, (c) 36, and (d) 44 s after droplet deposition. Vertical image dimension=1 cm.

Marangoni Driven Structures in Thin Film Flows

Anton A. Darhuber and Sandra M. Troian, Princeton University

The spreading of a surfactant solution on a thin viscous film is dominated by Marangoni forces, which shear the liquid in proportion to the local surface concentration gradient. In conjunction with capillary forces, these tangential stresses create complex surface profiles whose fronts develop highly ramified patterns. Theoretical models indicate significant disturbance amplification and lateral undulations in regions of the film undergoing surfactant buildup by local pressure gradients or rapid film thinning.

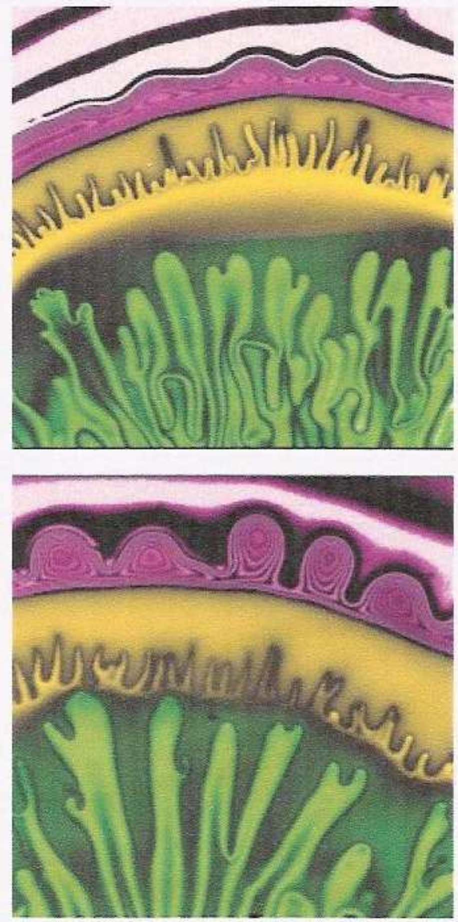
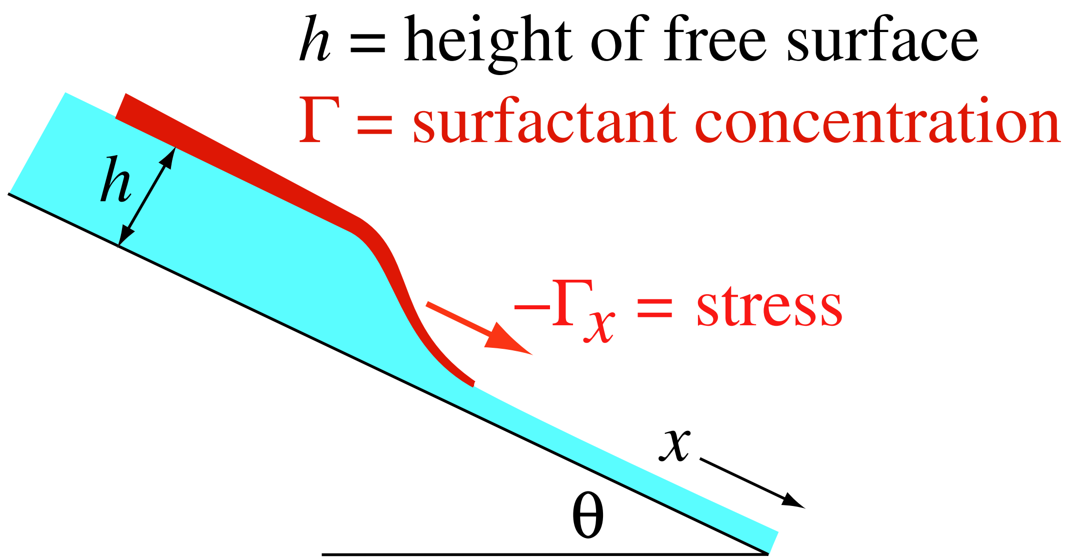


FIG. 2. A droplet of glycerol containing SDS spreading on a glycerol film. The images were recorded 5542 and 11 565 s after droplet deposition. Vertical image dimension=1 cm.

Governing Equations in One Space Dimension

From Edmonstone, Matar, and Craster, J. Engr. Math. 50 (2004):



$$h_t + \left(\frac{1}{3}Ch^3h_{xxx} - \frac{1}{3}G\cos\theta h^3h_x - \frac{1}{2}h^2\Gamma_x + \frac{1}{3}G\sin\theta h^3\right)_x = 0,$$

$$\Gamma_t + \left(\frac{1}{2}C\Gamma h^2h_{xxx} - \frac{1}{2}G\cos\theta\Gamma h^2h_x - h\Gamma\Gamma_x + \frac{1}{2}G\sin\theta h^2\Gamma - D\Gamma_x\right)_x = 0.$$

Parameters

- C = capillary number (surface tension)
- D^{-1} = Peclet number (surface diffusion)
- G = gravity coefficient

Traveling Wave System

$$h_t + \left(\frac{1}{3}Ch^3h_{xxx} - \frac{1}{3}G\cos\theta h^3h_x - \frac{1}{2}h^2\Gamma_x + \frac{1}{3}G\sin\theta h^3\right)_x = 0,$$

$$\Gamma_t + \left(\frac{1}{2}C\Gamma h^2h_{xxx} - \frac{1}{2}G\cos\theta\Gamma h^2h_x - h\Gamma\Gamma_x + \frac{1}{2}G\sin\theta h^2\Gamma - D\Gamma_x\right)_x = 0.$$

Let $\alpha = G\sin\theta$, $\beta = G\cos\theta$. **Fix $\alpha > 0$ and $\beta > 0$ (inclined plane).**

There is a traveling wave solution $(h, \Gamma)(x - st)$ of speed s provided

$$-sh' + \left(\frac{1}{3}Ch^3h''' - \frac{1}{3}\beta h^3h' - \frac{1}{2}h^2\Gamma' + \frac{1}{3}\alpha h^3\right)' = 0,$$

$$-s\Gamma' + \left(\frac{1}{2}C\Gamma h^2h''' - \frac{1}{2}\beta\Gamma h^2h' - h\Gamma\Gamma' + \frac{1}{2}\alpha h^2\Gamma - D\Gamma'\right)' = 0.$$

We'll work in the region $h > 0$ (wet surface) and $\Gamma \geq 0$.

$$-sh' + \left(\frac{1}{3}Ch^3h''' - \frac{1}{3}\beta h^3h' - \frac{1}{2}h^2\Gamma' + \frac{1}{3}\alpha h^3\right)' = 0,$$
$$-s\Gamma' + \left(\frac{1}{2}C\Gamma h^2h''' - \frac{1}{2}\beta\Gamma h^2h' - h\Gamma\Gamma' + \frac{1}{2}\alpha h^2\Gamma - D\Gamma'\right)' = 0.$$

We'll look for traveling waves with **a finite amount of surfactant**:

$$(h,\Gamma)(-\infty) = (h_L, 0), \quad (h,\Gamma)(\infty) = (h_R, 0), \quad h_L > h_R > 0.$$

Integrate:

$$-sh + \frac{1}{3}Ch^3h''' - \frac{1}{3}\beta h^3h' - \frac{1}{2}h^2\Gamma' + \frac{1}{3}\alpha h^3 = K_1,$$
$$-s\Gamma + \frac{1}{2}C\Gamma h^2h''' - \frac{1}{2}\beta\Gamma h^2h' - h\Gamma\Gamma' + \frac{1}{2}\alpha h^2\Gamma - D\Gamma' = 0.$$

The numbers s and K_1 are chosen so that

$$-sh_L + \frac{1}{3}\alpha h_L^3 = -sh_R + \frac{1}{3}\alpha h_R^3 = K_1.$$

Therefore

$$s = \frac{1}{3}\alpha(h_L^2 + h_Lh_R + h_R^2) > 0, \quad K_1 = -\frac{1}{3}\alpha h_Lh_R(h_L + h_R) < 0.$$

Rewrite the traveling wave ODEs:

$$\Gamma' = \frac{2\Gamma}{h}\frac{sh + 3K_1}{h\Gamma + 4D},$$
$$\beta h' - Ch''' = \frac{1}{h^3}\left(\alpha h^3 - 3sh - 3K_1 - 3h\Gamma\frac{sh + 3K_1}{h\Gamma + 4D}\right).$$

Rewrite as first-order ODEs:

$$\Gamma' = \frac{2\Gamma}{h}\frac{sh + 3K_1}{h\Gamma + 4D},$$
$$h' = k,$$
$$\sqrt{C}k' = l,$$
$$\sqrt{C}l' = Ck'' = Ch''' = \beta k - \frac{1}{h^3}\left(\alpha h^3 - 3sh - 3K_1 - 3h\Gamma\frac{sh + 3K_1}{h\Gamma + 4D}\right).$$

Multiply by $h\Gamma + 4D$ and set $\epsilon = \sqrt{C}$:

$$\begin{aligned}
\Gamma' &= \frac{2\Gamma}{h}(sh + 3K_1), \\
h' &= k(h\Gamma + 4D), \\
\epsilon k' &= l(h\Gamma + 4D), \\
\epsilon l' &= \left(\beta k - \frac{1}{h^3}\left(\alpha h^3 - 3sh - 3K_1\right)\right)(h\Gamma + 4D) + \frac{3}{h^2}\Gamma(sh + 3K_1).
\end{aligned}$$

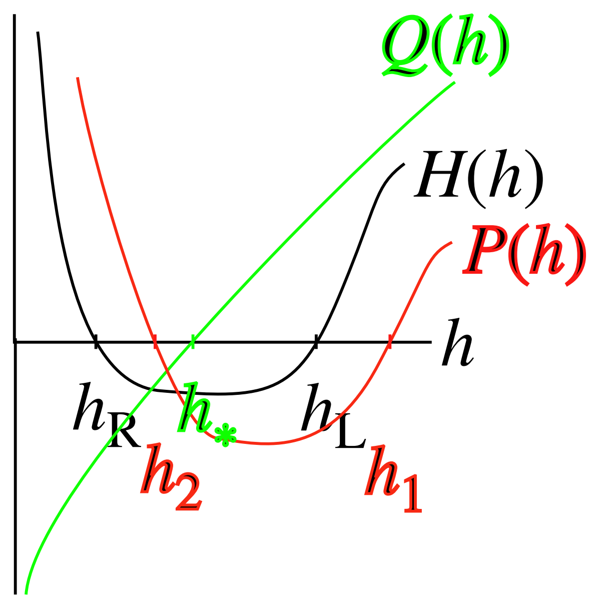
Traveling Wave System (rescale time):

$$\begin{aligned}
\dot{\Gamma} &= \epsilon\frac{2\Gamma}{h}(sh + 3K_1), \\
\dot{h} &= \epsilon k(h\Gamma + 4D), \\
\dot{k} &= l(h\Gamma + 4D), \\
\dot{l} &= \left(\beta k - \frac{1}{h^3}\left(\alpha h^3 - 3sh - 3K_1\right)\right)(h\Gamma + 4D) + \frac{3}{h^2}\Gamma(sh + 3K_1).
\end{aligned}$$

Let

$$H(h) = \frac{1}{h^3}\left(\alpha h^3 - 3sh - 3K_1\right), \quad Q(h) = \frac{2}{h}(sh + 3K_1),$$
$$P(h) = \frac{1}{h^3}\left(\alpha h^3 - 6sh - 12K_1\right).$$

$$H(h) = \frac{1}{h^3}\left(\alpha h^3 - 3sh - 3K_1\right), \quad Q(h) = \frac{2}{h}(sh + 3K_1),$$
$$P(h) = \frac{1}{h^3}\left(\alpha h^3 - 6sh - 12K_1\right).$$



(Figure shows signs and roots of H, P, Q for $\frac{h_R}{h_L} < \frac{1}{2}(\sqrt{3} - 1)$.) Then we have

(1) $$\dot{\Gamma} = \epsilon \Gamma Q(h),$$

(2) $$\dot{h} = \epsilon k(h\Gamma + 4D),$$

(3) $$\dot{k} = l(h\Gamma + 4D),$$

(4) $$\dot{l} = 4D\left(\beta k - H(h)\right) + h\Gamma\left(\beta k - P(h)\right).$$

Equilibria and Invariant Spaces

$$\dot{\Gamma} = \epsilon \Gamma Q(h),$$
$$\dot{h} = \epsilon k(h\Gamma + 4D),$$
$$\dot{k} = l(h\Gamma + 4D),$$
$$\dot{l} = 4D\left(\beta k - H(h)\right) + h\Gamma\left(\beta k - P(h)\right).$$

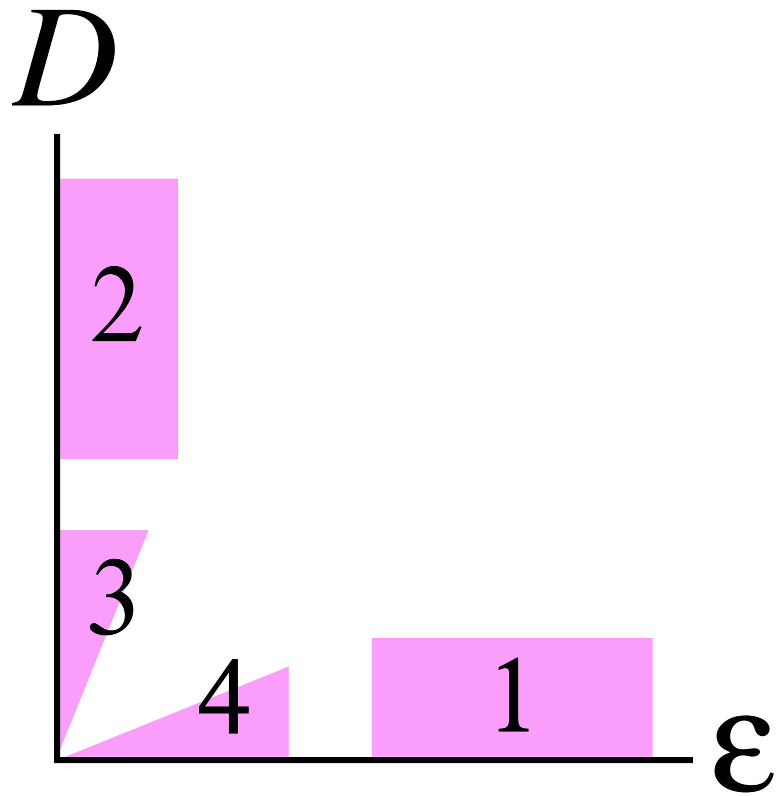
The space $\Gamma = 0$ is invariant.

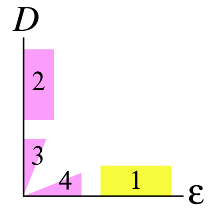
For every (ϵ, D) there are equilibria at $(0, h_L, 0, 0)$ and $(0, h_R, 0, 0)$.

For $\epsilon > 0$ and $D > 0$:

(1) These are the only equilibria.
(2) **At** $(0, h_L, 0, 0)$ **there are three eigenvalues with positive real part and one with negative real part.** For for fixed D, two of the eigenvalues with positive real part are complex if ϵ is sufficiently large.
(3) **At** $(0, h_R, 0, 0)$ **there are one eigenvalue with positive real part and three with negative real part.** For for fixed D, two of the eigenvalues with negative real part are complex if ϵ is sufficiently large.

Thus for $\epsilon > 0$ **and** $D > 0$**,** $W^u(0, h_L, 0, 0)$ **and** $W^s(0, h_R, 0, 0)$ **are 3-dimensional. If they are transverse, the intersection is 2-dimensional.**


D
2
3
4
1
ε



In the Traveling Wave System, **fix $\epsilon > 0$ and set $D = 0$ (fast system)**:

$$\begin{aligned}
\dot{\Gamma} &= \epsilon \Gamma Q(h), \\
\dot{h} &= \epsilon k h \Gamma, \\
\dot{k} &= l h \Gamma, \\
\dot{l} &= h \Gamma \left(\beta k - P(h) \right).
\end{aligned}$$

$\Gamma = 0$ is a manifold of equilibria that is normally hyperbolic for $h \neq h_*$.

$\Gamma = 0$ remains invariant for $D > 0$.

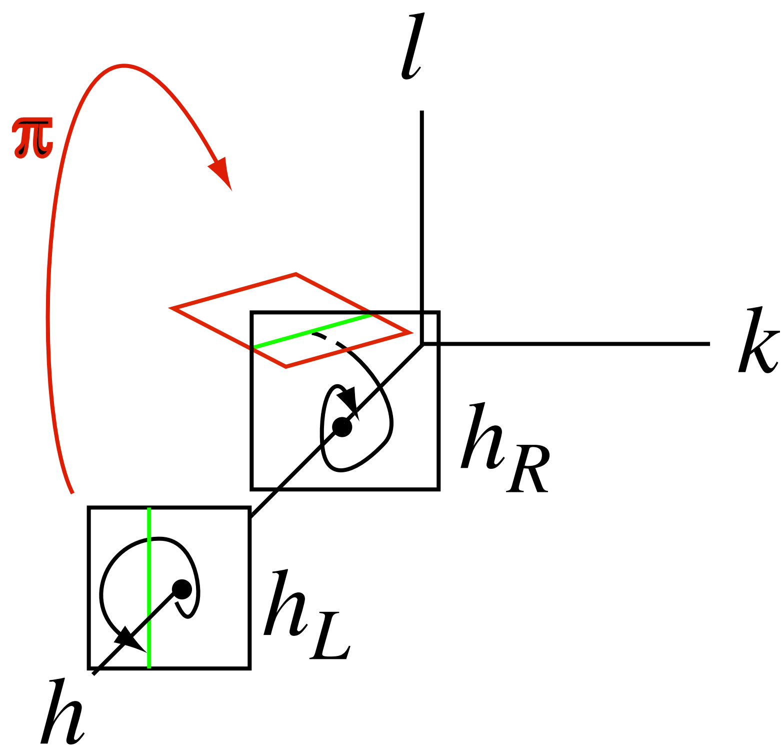
System on $\Gamma = 0$ for $D > 0$ after dividing by $4D$ (slow system):

$$\begin{aligned}
h' &= \epsilon k, \\
k' &= l, \\
l' &= \beta k - H(h).
\end{aligned}$$

Equilibria: $(h_L, 0, 0)$ and $(h_R, 0, 0)$ with 2-dimensional unstable manifold and 2-dimensional stable manifold respectively.

Singular connecting orbit:

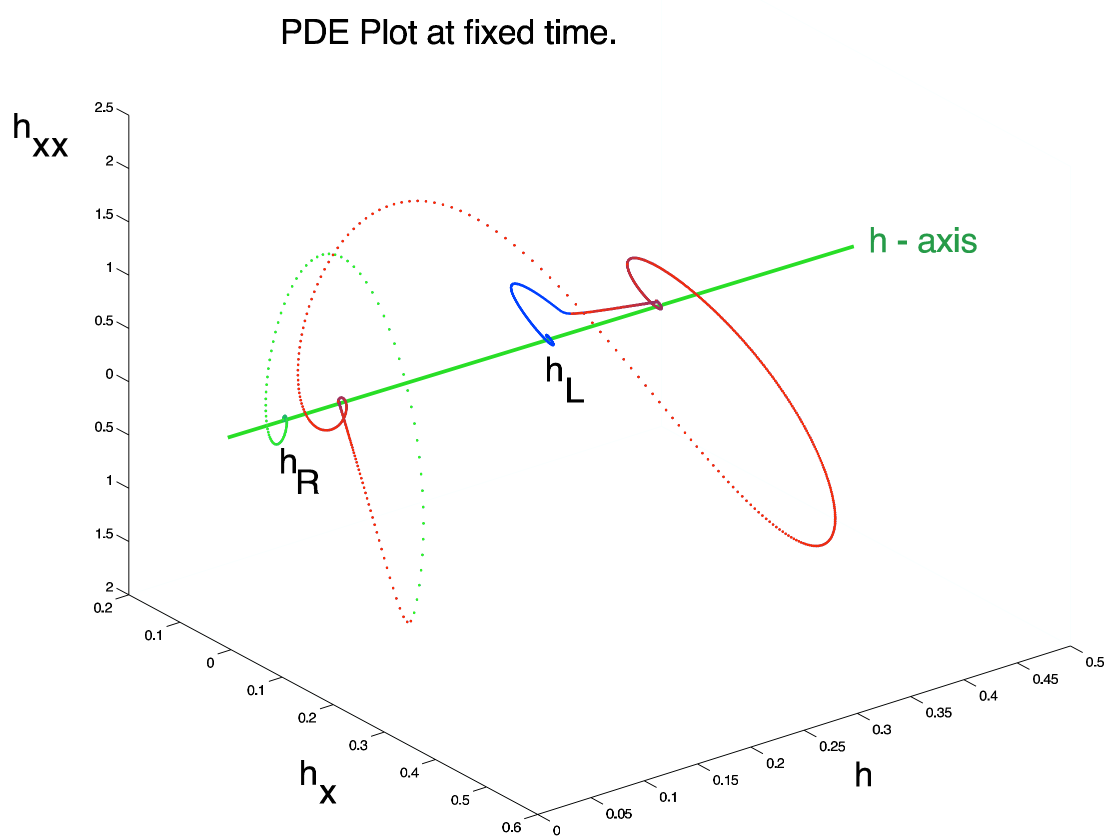
(1) Solution of the slow system from $(h_L, 0, 0)$ to a point in $W^u(h_L, 0, 0)$, the 2-dimensional unstable manifold of $(h_L, 0, 0)$ for the slow system.
(2) Connecting orbit of the fast system to a point in $W^s(h_R, 0, 0)$, the 2-dimensional stable manifold of $(h_R, 0, 0)$ for the slow system.
(3) Solution of the slow system to $(h_R, 0, 0)$.



To understand fast connecting orbits, divide fast system by $h\Gamma$ and ignore $\dot{\Gamma}$:

$$\begin{aligned}\dot{h} &= \epsilon k,\\ \dot{k} &= l,\\ \dot{l} &= \beta k - P(h).\end{aligned}$$

Equilibria: $(h_1, 0, 0)$ and $(h_2, 0, 0)$ with 2-dimension unstable manifold and 2-dimensional stable manifold respectively.



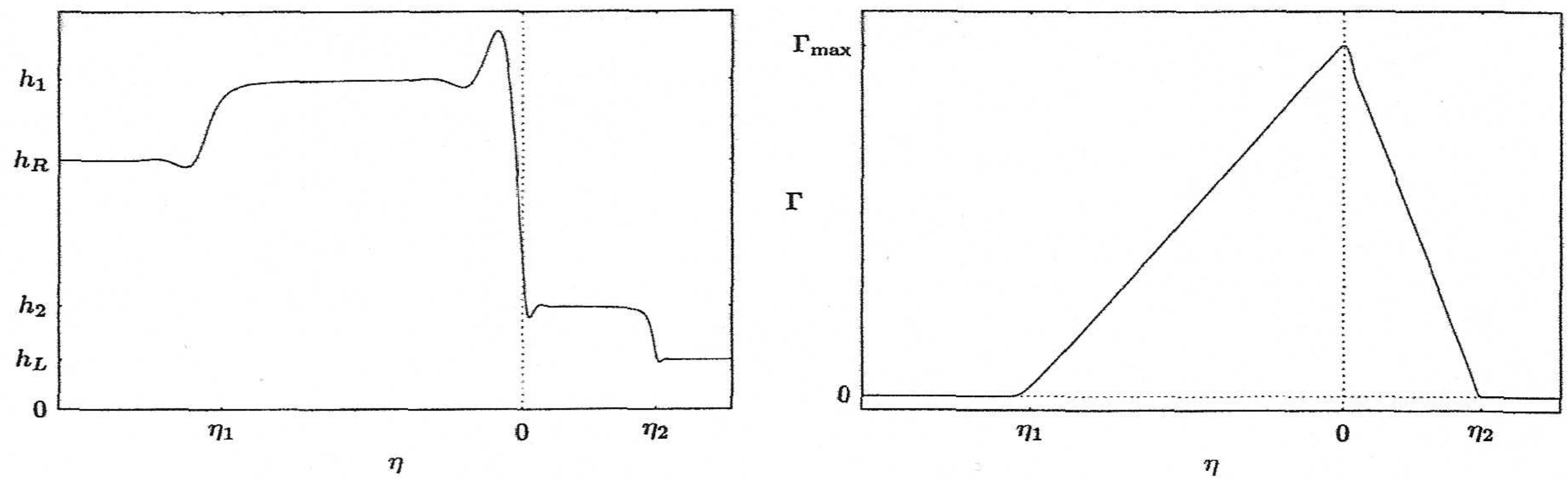
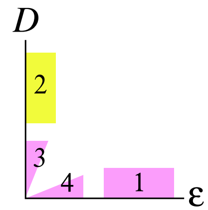


FIGURE 15. The effect of capillarity: numerical solutions with $\kappa > 0$, and β, δ both small.



In the Traveling Wave System, **fix** $D > 0$ **and set** $\epsilon = 0$ **(fast system)**:

$$\begin{aligned}
\dot{\Gamma} &= 0, \\
\dot{h} &= 0, \\
\dot{k} &= l(h\Gamma + 4D), \\
\dot{l} &= 4D\left(\beta k - H(h)\right) + h\Gamma\left(\beta k - P(h)\right).
\end{aligned}$$

The set

$$k = \frac{4DH(h) + h\Gamma P(h)}{\beta(4D + h\Gamma)}, \quad l = 0$$

is a normally hyperbolic manifold of equilibria of dimension 2: one positive eigenvalue, one negative eigenvalue. Note:

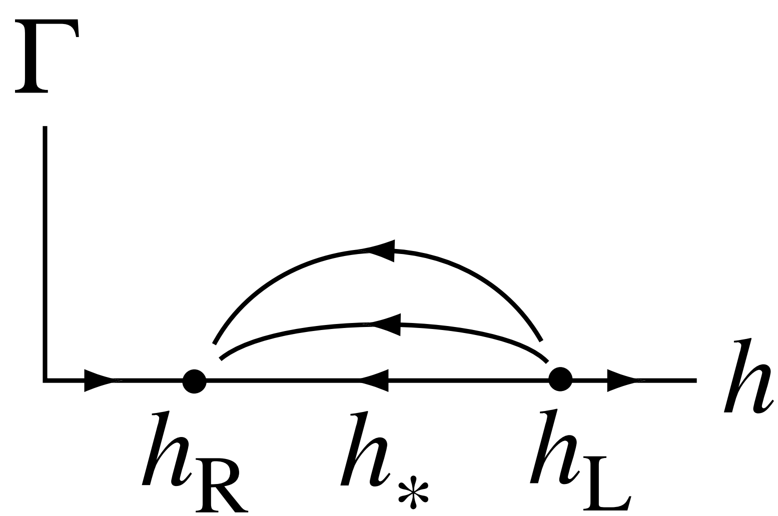
$$\begin{pmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial l} \\ \frac{\partial \dot{l}}{\partial k} & \frac{\partial \dot{l}}{\partial l} \end{pmatrix} = \begin{pmatrix} 0 & h\Gamma + 4D \\ (4D + h\Gamma)\beta & 0 \end{pmatrix}$$

Flow on perturbed normally hyperbolic invariant manifold:

$$\dot{\Gamma} = \epsilon \Gamma Q(h),$$
$$\dot{h} = \frac{\epsilon}{\beta}(4DH(h) + h\Gamma P(h)) + O(\epsilon^2).$$

In slow time:

$$\Gamma' = \Gamma Q(h),$$
$$h' = \frac{1}{\beta}(4DH(h) + h\Gamma P(h)) + O(\epsilon),$$



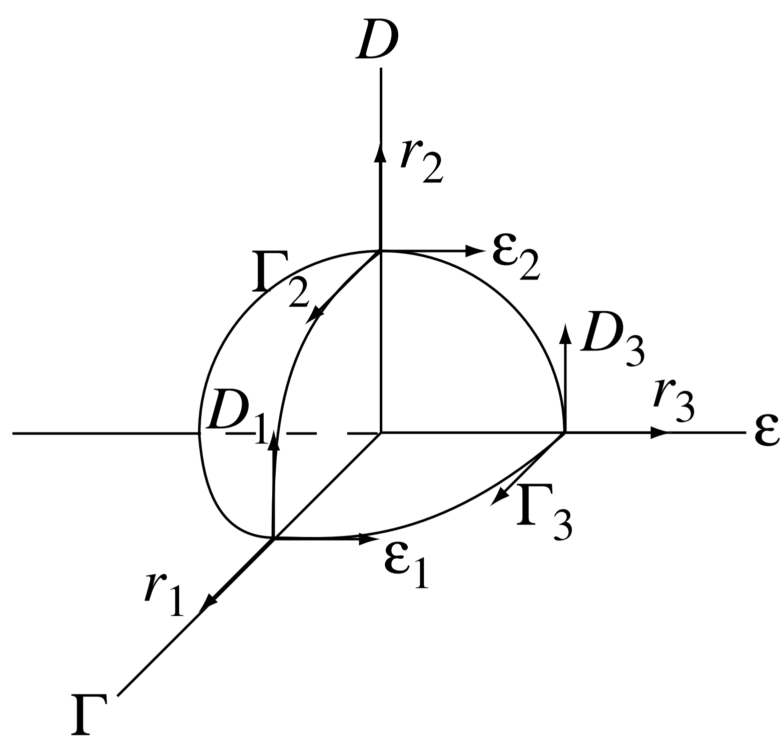
Blow-up

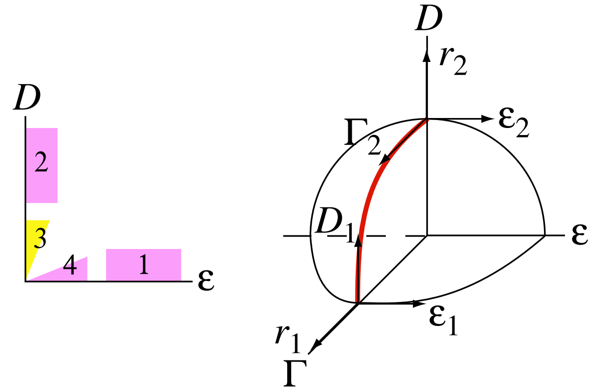
$$\Gamma = \bar{r}\bar{\Gamma}, \quad \epsilon = \bar{r}\bar{\epsilon}, \quad D = \bar{r}\bar{D}.$$

Blow-up space: $\{((\bar{\Gamma}, \bar{\epsilon}, \bar{D}), \bar{r}, h, k, l) : \bar{\Gamma}^2 + \bar{\epsilon}^2 + \bar{D}^2 = 1 \text{ and } \bar{r} \geq 0\}$.

Vector field: Original rewritten in these coordinates and divided by $\bar{r}$.

Three coordinate systems:





The region $\bar{\Gamma} > 0$:

$$\Gamma = r_1, \quad \epsilon = r_1 \epsilon_1, \quad D = r_1 D_1.$$

$$\begin{aligned}
\dot{r}_1 &= r_1 \epsilon_1 Q(h), \\
\dot{h} &= r_1 \epsilon_1 k(h + 4D_1), \\
\dot{k} &= l(h + 4D_1), \\
\dot{l} &= 4D_1 \left(\beta k - H(h)\right) + h \left(\beta k - P(h)\right), \\
\dot{\epsilon_1} &= -\epsilon_1^2 Q(h), \\
\dot{D}_1 &= -\epsilon_1 D_1 Q(h).
\end{aligned}$$

Three-dimensional manifold of equilibria:

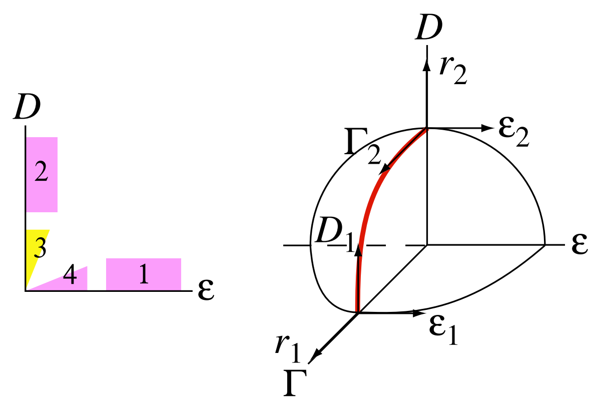
$$k = \frac{4D_1 H(h) + hP(h)}{\beta(4D_1 + h)}, \quad l = 0, \quad \epsilon_1 = 0.$$

Normal to this manifold, each equilibrium has one positive eigenvalue, one negative eigenvalue, and one zero eigenvalue.

Part of four-dimensional normally hyperbolic invariant manifold M:

$$k = K(r_1, h, \epsilon_1, D_1), \quad l = L(r_1, h, \epsilon_1, D_1).$$

K and L are initially only defined for ϵ_1 small.



System on M:

$$\begin{aligned}
\dot{r}_1 &= r_1 \epsilon_1 Q(h), \\
\dot{h} &= r_1 \epsilon_1 K(r_1, h, \epsilon_1, D_1)(h + 4D_1), \\
\dot{\epsilon}_1 &= -\epsilon_1^2 Q(h), \\
\dot{D}_1 &= -\epsilon_1 D_1 Q(h).
\end{aligned}$$

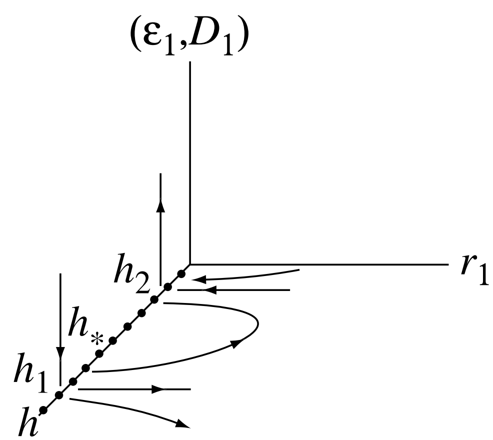
Divide by ϵ_1:

$$\begin{aligned}\dot{r}_1 &= r_1 Q(h),\\ \dot{h} &= r_1 K(r_1, h, \epsilon_1, D_1)(h + 4D_1),\\ \dot{\epsilon}_1 &= -\epsilon_1 Q(h),\\ \dot{D}_1 &= -D_1 Q(h).\end{aligned}$$

Equilibria: h-axis. Normally hyperbolic for $h \neq h_*$: one eigenvalue $Q(h)$, with eigenvector in the r_1-direction; two eigenvalues $-Q(h)$, eigenspace is $\epsilon_1 D_1$-space.

Spaces $r_1 = 0$, $\epsilon_1 = 0$, and $D_1 = 0$ are invariant. In $r_1 h$-space:

$$\begin{aligned}\dot{r}_1 &= r_1 Q(h),\\ \dot{h} &= \frac{r_1}{\beta} h P(h).\end{aligned}$$



The region $\bar{D} > 0$:

$$\Gamma = r_2\Gamma_2, \quad \epsilon = r_2\epsilon_2, \quad D = r_2.$$

$$\begin{aligned}
\dot{\Gamma}_2 &= \epsilon_2\Gamma_2 Q(h), \\
\dot{h} &= r_2\epsilon_2 k(h\Gamma_2 + 4), \\
\dot{k} &= l(h\Gamma_2 + 4), \\
\dot{l} &= 4\left(\beta k - H(h)\right) + h\Gamma_2\left(\beta k - P(h)\right), \\
\dot{\epsilon_2} &= 0, \\
\dot{r_2} &= 0.
\end{aligned}$$

Three-dimensional manifold of equilibria:

$$k = \frac{4H(h) + h\Gamma_2 P(h)}{\beta(4 + h\Gamma_2)}, \quad l = 0, \quad \epsilon_2 = 0.$$

Normal to this manifold, each equilibrium has one positive eigenvalue, one negative eigenvalue, and one zero eigenvalue.

Part of four-dimensional normally hyperbolic invariant manifold M:

$$k = K(\Gamma_2, h, \epsilon_2, r_2), \quad l = L(\Gamma_2, h, \epsilon_2, r_2),$$

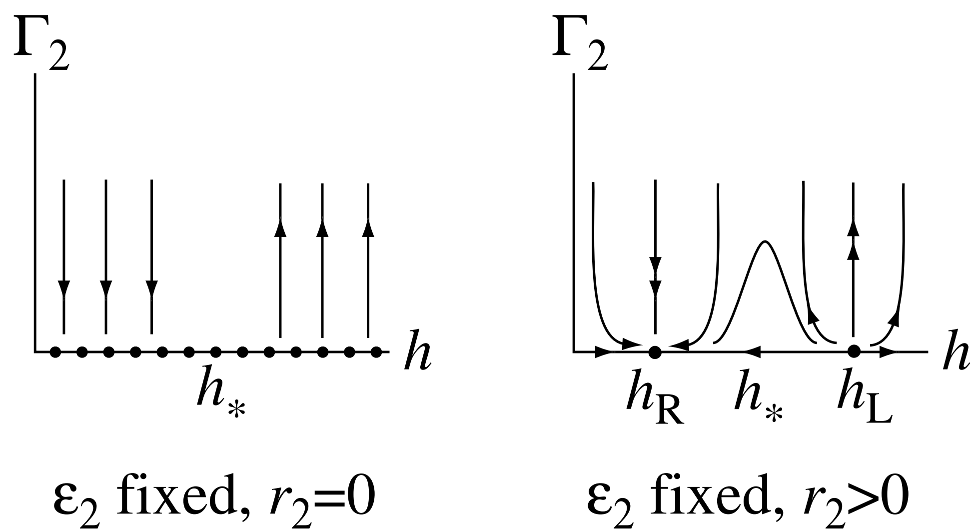
K and L are only defined for ϵ_2 small.

System on M:

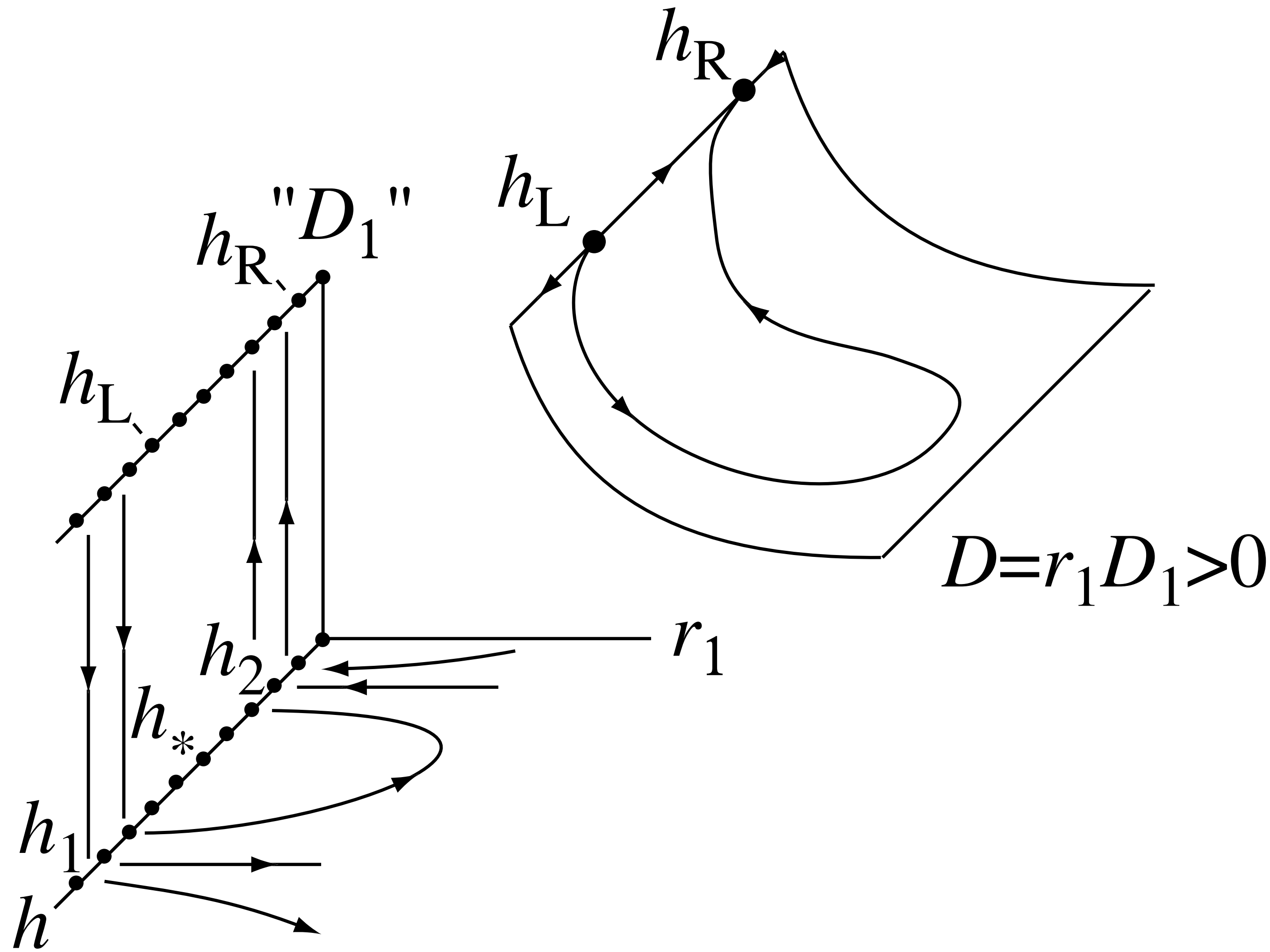
$$\begin{aligned} \dot{\Gamma_2} &= \epsilon_2\Gamma_2 Q(h), \\ \dot{h} &= r_2\epsilon_2 K(\Gamma_2, h, \epsilon_2, r_2)(h\Gamma_2 + 4), \\ \dot{\epsilon_2} &= 0, \\ \dot{r_2} &= 0. \end{aligned}$$

Divide by ϵ_2:

$$\begin{aligned} \dot{\Gamma_2} &= \Gamma_2 Q(h), \\ \dot{h} &= r_2 K(\Gamma_2, h, \epsilon_2, r_2)(h\Gamma_2 + 4), \\ \dot{\epsilon_2} &= 0, \\ \dot{r_2} &= 0. \end{aligned}$$

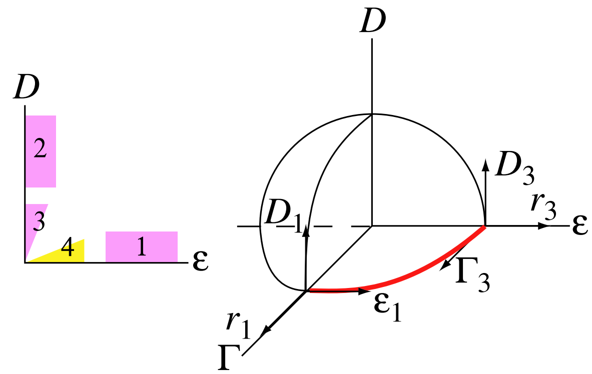


Combine the coordinate patches and use "Corner Lemma":



We get traveling waves with three steps.

(To see all this in one coordinate patch, replace Γ with $\sigma = 4D + h\Gamma$.)



The region $\bar{\Gamma} > 0$ Revisited:

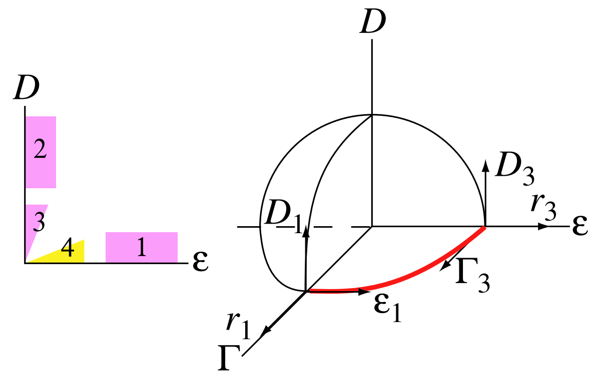
$$\Gamma = r_1, \quad \epsilon = r_1\epsilon_1, \quad D = r_1 D_1.$$

$$\begin{aligned}
\dot{r}_1 &= r_1\epsilon_1 Q(h), \\
\dot{h} &= r_1\epsilon_1 k(h + 4D_1), \\
\dot{k} &= l(h + 4D_1), \\
\dot{l} &= 4D_1\left(\beta k - H(h)\right) + h\left(\beta k - P(h)\right), \\
\dot{\epsilon}_1 &= -\epsilon_1^2 Q(h), \\
\dot{D}_1 &= -\epsilon_1 D_1 Q(h).
\end{aligned}$$

Four-dimensional normally hyperbolic invariant manifold M:

$$k = K(r_1, h, \epsilon_1, D_1), \quad l = L(r_1, h, \epsilon_1, D_1).$$

K and L are initially only defined for ϵ_1 small. However, M and its stable and unstable manifolds can be extended by the flow.



In particular, within the invariant space $r_1 = D_1 = 0$, the system divided by h reduces to

$$\begin{aligned} \dot{h} &= 0, \\ \dot{k} &= l, \\ \dot{l} &= (\beta k - P(h)), \\ \dot{\epsilon_1} &= -\epsilon_1^2 \frac{Q(h)}{h}. \end{aligned}$$

Therefore, within the space $r_1 = D_1 = 0$, M extends in the ϵ_1 direction to

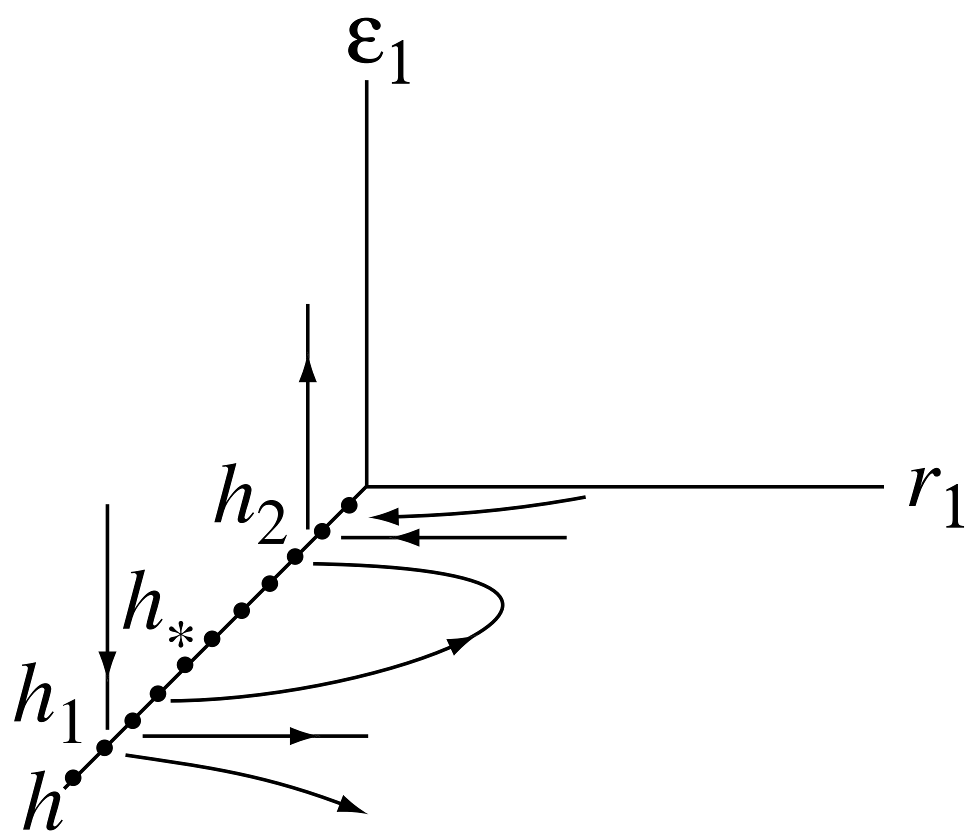
$$k = \frac{1}{\beta} P(h), \quad l = 0, \quad h \text{ and } \epsilon_1 \text{ arbitrary.}$$

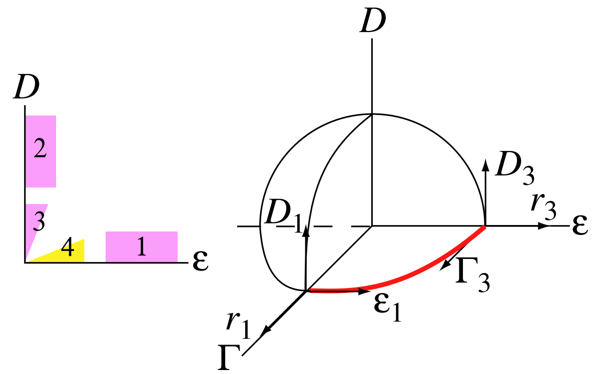
Within the space $r_1 = D_1 = 0$, $W^u(M)$ and $W^s(M)$ are given by

$$l = \sqrt{\beta}(k - \frac{1}{\beta}P(h)), \quad h,\ k, \text{ and } \epsilon_1 \text{ arbitrary},$$

$$l = -\sqrt{\beta}(k - \frac{1}{\beta}P(h)), \quad h,\ k, \text{ and } \epsilon_1 \text{ arbitrary}.$$

Flow in M:





The region $\bar{\epsilon} > 0$:

$$\Gamma = r_3\Gamma_3, \quad \epsilon = r_3, \quad D = r_3 D_3.$$

$$\begin{aligned}
\dot{\Gamma}_3 &= \Gamma_3 Q(h), \\
\dot{h} &= r_3 k(h\Gamma_3 + 4D_3), \\
\dot{k} &= l(h\Gamma_3 + 4D_3), \\
\dot{l} &= 4D_3\left(\beta k - H(h)\right) + h\Gamma_3\left(\beta k - P(h)\right), \\
\dot{r_3} &= 0, \\
\dot{D}_3 &= 0.
\end{aligned}$$

Four-dimensional invariant manifold M (previous one in new coordinates):

$$k = K(\Gamma_3, h, r_3, D_3), \quad l = L(\Gamma_3, h, r_3, D_3), \quad \Gamma_3 > 0,\ D_3 \text{ near } 0.$$

Normal hyperbolicity is lost for $(\Gamma_3, D_3) = (0, 0)$, which is the crucial set.

Within the invariant space $D_3 = r_3 = 0$, M is given by

$$k = \frac{1}{\beta}P(h), \quad l = 0, \quad \Gamma_3 > 0 \text{ and } h > 0,$$

$W^u(M)$ by

$$l = \sqrt{\beta}(k - \frac{1}{\beta}P(h)), \quad \Gamma_3 > 0,\ h > 0, \text{ and } k \text{ arbitrary},$$

and $W^s(M)$ by

$$l = -\sqrt{\beta}(k - \frac{1}{\beta}P(h)), \quad \Gamma_3 > 0,\ h > 0, \text{ and } k \text{ arbitrary}.$$

These manifolds extend smoothly to $\Gamma_3 = 0$, but normal hyperbolicity is lost there.

For $r_3 > 0$ and $D_3 > 0$, $(0, h_L, 0, 0)$ and $(0, h_R, 0, 0)$ have 3-dimensional unstable and stable manifolds respectively that extend smoothly to the boundary of $\{(r_3, D_3) : r_3 \geq 0$ and $D_3 \geq 0\}$.

Consider the 2-dimensional manifolds of equilibria

$$A = \{(0, h_L, 0, 0, r_3, D_3) : r_3,\ D_3 \text{ small}\},$$
$$B = \{(0, h_R, 0, 0, r_3, D_3) : r_3,\ D_3 \text{ small}\}.$$

$W^u(A)$ **(extended to $r_3 \geq 0$ and $D_3 \geq 0$)** is given by

$$l = L_u(\Gamma_3, h, k, r_3, D_3), \quad L_u(0, h, k, 0, D_3) = \sqrt{\beta}(k - \frac{1}{\beta}H(h)).$$

$W^s(B)$ **(extended to $r_3 \geq 0$ and $D_3 \geq 0$)** is given by

$$l = L_s(\Gamma_3, h, k, r_3, D_3), \quad L_s(0, h, k, 0, D_3) = -\sqrt{\beta}(k - \frac{1}{\beta}H(h)).$$

These explicit formulas show that for r_3 near 0 and D_3 near 0,

- $W^u(A)$ **is transverse to** $W^s(M)$**.**
- $W^u(M)$ **is transverse to** $W^s(B)$**.**

$$A = \{(0, h_L, 0, 0, r_3, D_3) : r_3,\ D_3 \text{ small}\},$$
$$B = \{(0, h_R, 0, 0, r_3, D_3) : r_3,\ D_3 \text{ small}\}.$$

- $W^u(A)$ is transverse to $W^s(M)$.
- $W^u(M)$ is transverse to $W^s(B)$.

Using the Corner Lemma, we can show that the 5-dimensional manifolds $W^u(A)$ and $W^s(B)$ meet transversally.

The solutions in their intersection trace solutions in M.