Stability of Patterns

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Overview

Motivation

Systems of viscous conservation laws and their variants are partial differential equations that govern physical situations in which quantities are transported but also subject to effects such as diffusion and surface tension.

Examples: gas dynamics, flows in porous media, thin film flows, traffic flow.

On a large scale and over a long time, these flows produce patterns that are not very sensitive to initial conditions.

Simplest example: traveling waves.

More typically, the asymptotic pattern contains different parts propagating with different speeds.

Main Difficulty

Asymptotic pattern, apparently a Riemann solution of the underlying system of conservation laws, is not a solution of the viscous conservation laws.

Approach

A related equation, the Dafermos regularization of a system of conservation laws, has smoothed Riemann solutions (Riemann-Dafermos solutions) as steady-states. Start by looking at them.

Outline

1. Viscous conservation laws
2. Conservation laws: rarefactions, shock waves, Riemann solutions
3. Dafermos regularization: Riemann-Dafermos solutions
4. Time-asymptotic stability
   (a) Riemann solutions of conservation laws
   (b) Traveling waves of viscous conservation laws
   (c) Riemann-Dafermos solutions of Dafermos regularization
5. Open problem: put the pieces together

1. Viscous conservation laws

System of viscous conservation laws

(VCL) $$u_T + f(u)_X = (B(u)u_X)_X$$

$$X \in \mathbb{R}, \quad u \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n$$

All eigenvalues of $B(u)$ have positive real part.

Boundary conditions

(BC) $$u(-\infty, T) = u^\ell, \quad u(\infty, T) = u^r$$

Initial condition

(IC) $$u(X, 0) = u_0(X)$$

We are interested in the typical behavior of solutions as $T \to \infty$.

Example

Bertozzi, Munch, Shearer, "Undercompressive shocks in thin film flows," Physica D, 1999.

$$h_T + (h^2 - h^3)_X = -(h^3 h_{XXX})_X$$

$$h(-\infty, T) = h_\infty, \quad h(\infty, T) = b, \quad 0 < b < h_\infty.$$

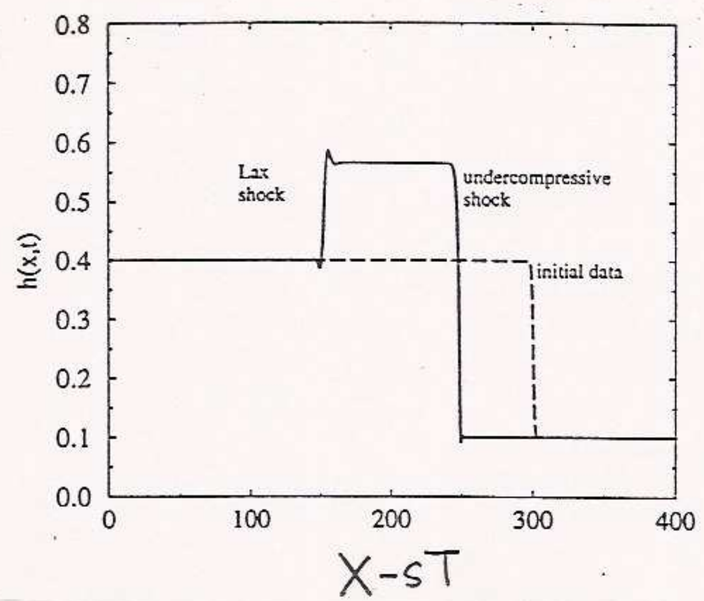


Fig. 8. Undercompressive double shock structure for $h_\infty = 0.4$ to $b = 0.1$ at time 4800.

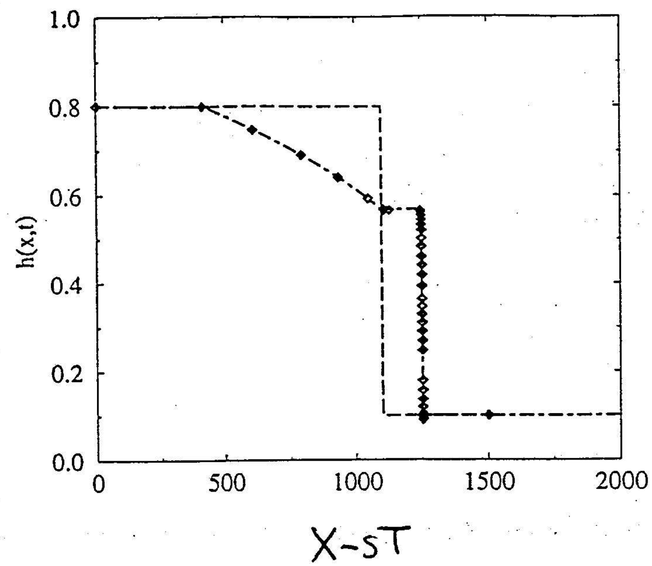


Fig. 9. A rarefaction undercompressive shock solution for smoothed Riemann initial data with $h_\infty = 0.8$ and $b = 0.1$. The initial condition (34) is shown as a dashed line and the numerical solution at time $t = 1400$ is shown as diamonds. The theoretical inviscid non-classical rarefaction-shock is shown as a dot-dashed line.

It is believed that as $T \to \infty$, solutions of intitial-boundary-value problems (VCL)–(BC)–(IC) typically approach a limiting state $u(x)$, $x = \frac{X}{T}$, that is **not** a solution of (VCL).

Instead $u(x)$ is a **Riemann solution** for the system of conservation laws

$$\text{(CL)} \qquad u_T + f(u)_X = 0$$

obtained from (VCL) by dropping the viscous term.

Roughly speaking, **Riemann solutions are to viscous conservation laws as equilibria are to ODEs**: They are the simplest asymptotic states.

Shock waves in the Riemann solution should satisfy the **viscous profile criterion** for $B(u)$.

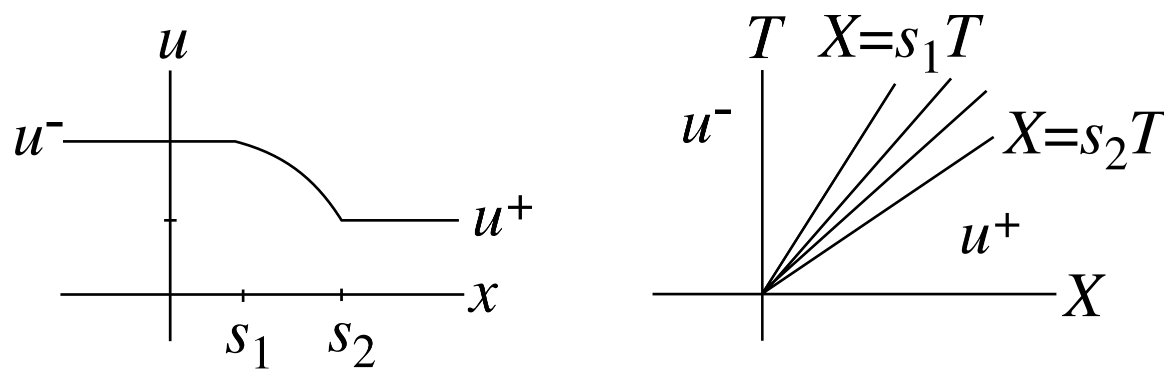
2. Systems of conservation laws

(CL) $$u_T + f(u)_X = 0$$

Solutions that are functions of $x = \frac{X}{T}$

(i) Constants

(ii) Rarefactions



A differentiable function $u(x)$, $x = \frac{X}{T}$, is a solution of (CL) provided

$$(Df(u) - xI)u_x = 0.$$

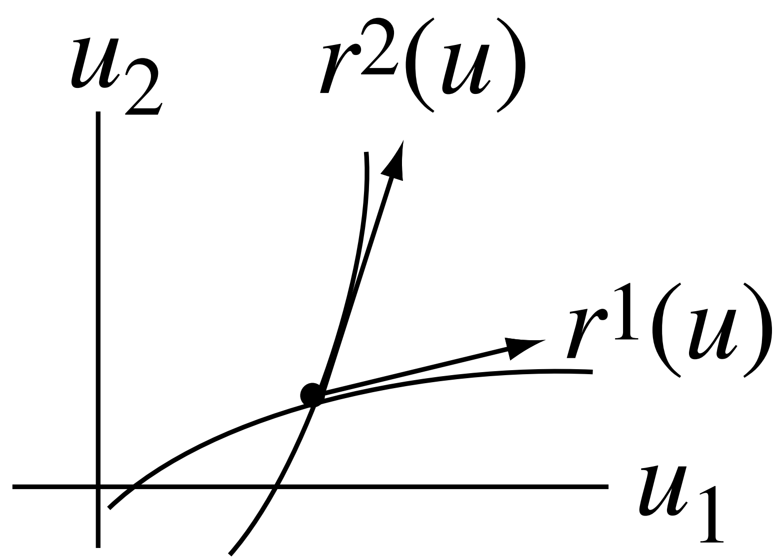
So either (1) $u_x = 0$, or (2) $u_x \neq 0$, x is an eigenvalue of $Df(u(x))$, and u_x is a corresponding eigenvector.

Assume

(1) **Strict hyperbolicity**: $Df(u)$ has n distinct real eigenvalues $\nu^1(u) < \ldots < \nu^n(u)$.

(2) **Genuine nonlinearity**: corresponding eigenvectors $r^i(u)$ can be chosen so that $D\nu^i(u)r^i(u) = 1$.

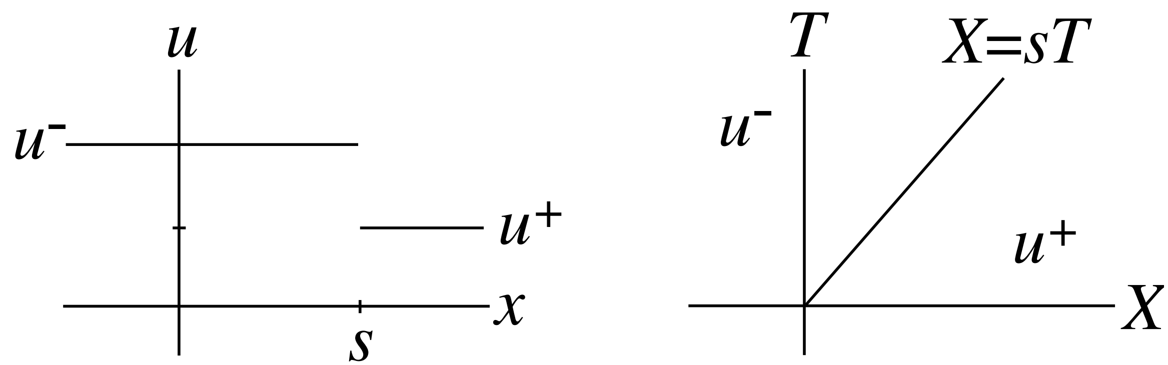
Then there are n rarefaction waves $u = u^i(x)$: $u^i(x)$ is a solution of the ODE $u_x = r^i(u)$, parameterized so that $\nu^i(u(x)) = x$.



(iii) Shock waves

These are discontinuous functions

(SW)
$$u(x) = \begin{cases} u^- & \text{for } x < s, \\ u^+ & \text{for } x > s. \end{cases}$$

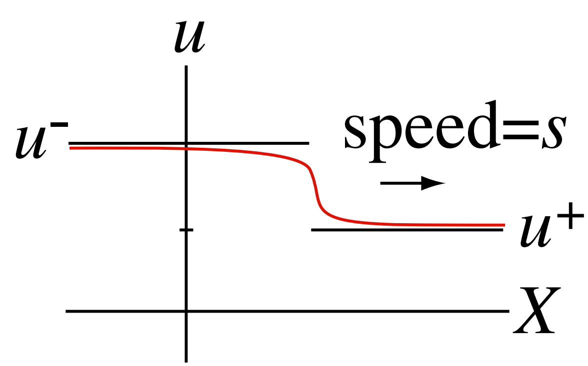


A shock wave satisfies the **viscous profile criterion** for the viscosity $(B(u)u_X)_X$ provided

(VCL)
$$u_T + f(u)_X = (B(u)u_X)_X$$

has a traveling wave solution $u(\xi)$, $\xi = X - sT$, with

(TWBC)
$$u(-\infty) = u^-, \quad u(+\infty) = u^+.$$



Such a traveling wave solution of (VCL) exists provided there is a solution $u(\xi)$ of the ODE BVP

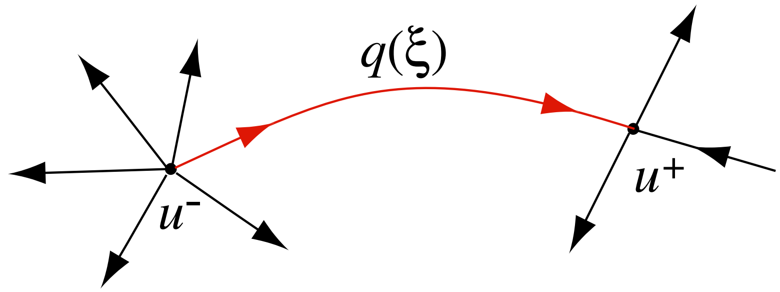
(TWODE) $$(Df(u) - sI)u_\xi = (B(u)u_\xi)_\xi,$$

(TWBC) $$u(-\infty) = u^-, \quad u(+\infty) = u^+.$$

Equivalently, the ODE

$$\dot{u} = B(u)^{-1}\left(f(u) - f(u^-) - s(u - u^-)\right)$$

has an equilibrium at u^+ (it automatically has one at u^-) and a connecting orbit $q(\xi)$ from u^- to u^+.



Note that the triple (u^-, s, u^+) must satisfy the **Rankine-Hugoniot condition**

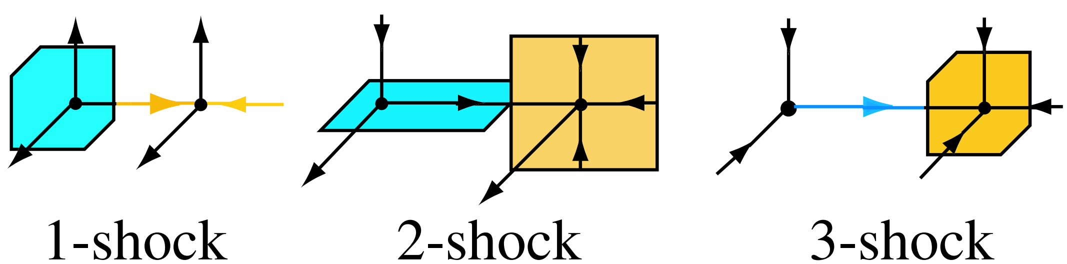
(RHC) $$f(u^+) - f(u^-) - s(u^+ - u^-) = 0,$$

which is a necessary condition for (SW) to be a weak solution of (CL).

Regular Lax i-shock

- $Df(u^+) - sI$ has i negative eigenvalues and $n - i$ positive eigenvalues.
- $Df(u^-) - sI$ has $i - 1$ negative eigenvalues and $n - i + 1$ positive eigenvalues.
- $W^u(u^-)$ and $W^s(u+)$ meet transversally along a solution $q(\xi)$.

Regular Lax i-shocks with $n = 3$:



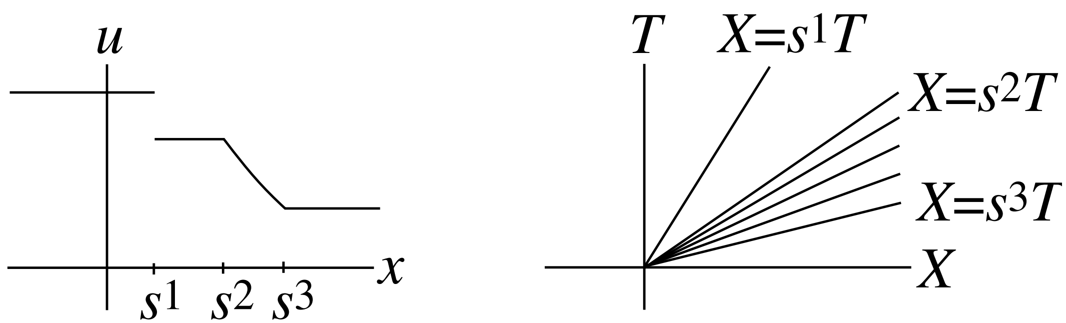
Riemann solutions

A Riemann solution of (CL) is a solution of the form $u(x)$, $x = \frac{X}{T}$, that approaches constants as $x \to \pm\infty$. $u(x)$ satisfies the "ODE BVP"

$$\text{(RODE)} \qquad (Df(u) - xI)u_x = 0, \quad -\infty < x < \infty,$$

$$\text{(ODEBC)} \qquad u(-\infty) = u^\ell, \quad u(\infty) = u^r.$$

Riemann solutions are composed of constant parts, rarefaction waves, and shock waves. They typically contain at least n waves, with different speeds.



Shock waves:

$$\lim_{x \to s-} u(x) = u^- \neq u^+ = \lim_{x \to s+} u(x)$$

The triple (u^-, s, u^+) is required to satisfy the viscous profile criterion for a given $(B(u)u_X)_X$.

3. Dafermos regularization

An approach to studying asympototic behavior of (VCL)

Since solutions of

$$\text{(VCL)} \qquad u_T + f(u)_X = (B(u)u_X)_X,$$

$$\text{(BC)} \qquad u(-\infty, T) = u^\ell, \quad u(+\infty, T) = u^r$$

are believed to approach Riemann solutions $u(x)$, $x = \frac{X}{T}$, we let

$$x = \frac{X}{T}, \quad t = \ln T.$$

(VCL)–(BC) becomes

$$\text{(VCL*)} \qquad u_t + (Df(u) - xI)u_x = e^{-t}(B(u)u_x)_x,$$

$$\text{(BC)} \qquad u(-\infty, t) = u^\ell, \quad u(+\infty, t) = u^r.$$

Thus in (x, t) variables (VCL) becomes **spatially dependent** and **nonautonomous**.

$$\text{(VCL*)} \qquad u_t + (Df(u) - xI)u_x = e^{-t}(B(u)u_x)_x,$$

$$\text{(BC)} \qquad u(-\infty, t) = u^\ell, \quad u(+\infty, t) = u^r.$$

As $t \to \infty$, solutions should approach a solution of

$$\text{(RODE)} \qquad (Df(u) - xI)u_x = 0, \quad -\infty < x < \infty,$$

$$\text{(ODEBC)} \qquad u(-\infty) = u^\ell, \quad u(\infty) = u^r.$$

This is the ODE BVP for Riemann solutions. Let $u_0(x)$ be a Riemann solution, with shock waves that satisfy the viscous profile criterion for $(B(u)u_x)_x$.

In studying nonautonomous systems like (VCL*), it is natural to first **freeze coefficients** and study the resulting **autonomous** system. So in (VCL*) let

$$\epsilon = e^{-t}.$$

For large t, ϵ is small. We obtain

$$\text{(DR)} \qquad u_t + (Df(u) - xI)u_x = \epsilon(B(u)u_x)_x.$$

(DR) $$u_t + (Df(u) - xI)u_x = \epsilon(B(u)u_x)_x$$

In (X, T) variables, (DR) is

(DR#) $$u_T + f(u)_X = \epsilon T(B(u)u_X)_X.$$

(DR#) is the **Dafermos regularization** of the system of conservation laws (CL) associated with the viscosity $B(u)$. It is usually regarded as an artificial, nonphysical equation because of the factor T in the viscous term.

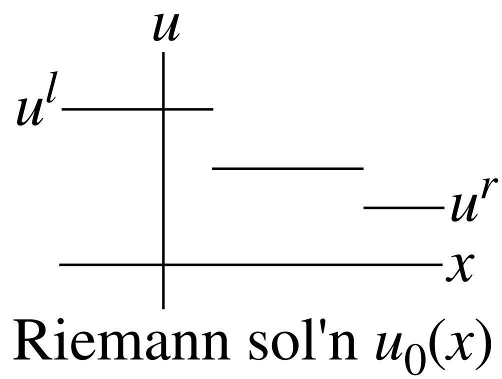
However, **if one is interested in the behavior of solutions of (VCL) for large T, and uses the appropriate variables (x, t), the Dafermos regularization in the form (DR) is a natural simplification of (VCL).**

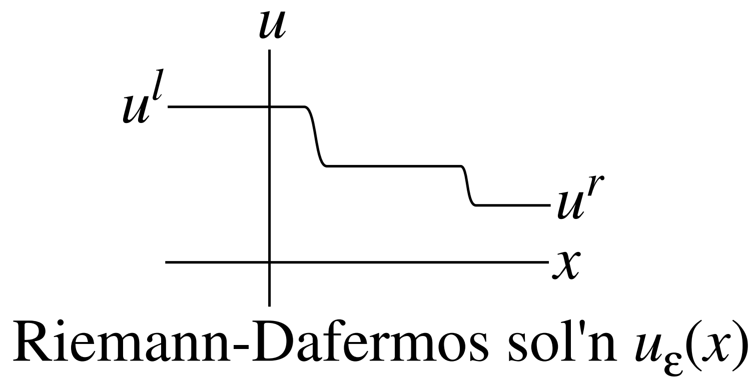
A stationary solution $u_\epsilon(x)$ of (DR) satisfies

(DODE) $$(Df(u) - xI)u_x = \epsilon(B(u)u_x)_x.$$

(ODEBC) $$u(-\infty) = u^\ell, \quad u(+\infty) = u^r.$$

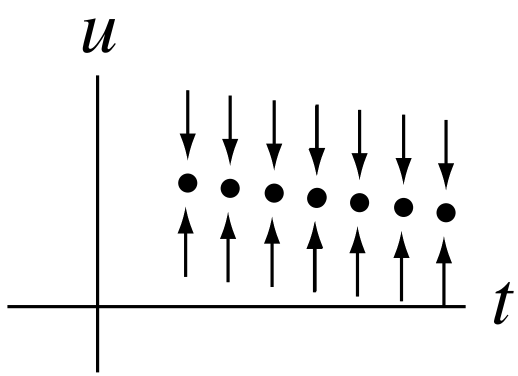
We call $u_\epsilon(x)$ a **Riemann-Dafermos solution**. This Dafermos ODE BVP is a viscous regularization of the Riemann "ODE BVP."

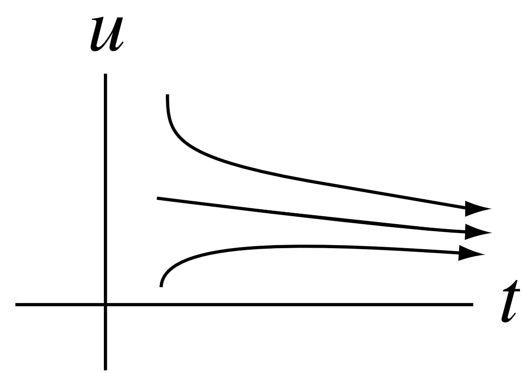




Idea:

(1) Show each $u_\epsilon(x)$ is linearly stable for (DR).
(2) Use "renormalization" (Keith Promislow) to show that near the function $u(t) = u_{e^{-t}}(t)$ is an asymptotically stable solution of (VCR).





Construction of Riemann-Dafermos solutions via geometric singular perturbation theory

Take $B(u) \equiv I$. (DR)–(BC) becomes

(DR) $$u_t + (Df(u) - xI)u_x = \epsilon u_{xx},$$

(BC) $$u(-\infty, t) = u^\ell, \quad u(+\infty, t) = u^r.$$

Assume strict hyperbolicity and genuine nonlinearity.

Stationary solutions $u_\epsilon(x)$ satisfy the Dafermos ODE BVP

(DODE) $$(Df(u) - xI)u_x = \epsilon u_{xx},$$

(ODEBC) $$u(-\infty) = u^\ell, \quad u(+\infty) = u^r.$$

Idea of Peter Szmolyan (1997). Start with the Dafermos ODE BVP:

$$
\begin{aligned}
&\text{(DODE)} && (Df(u) - xI)u_x = \epsilon u_{xx},\\
&\text{(ODEBC)} && u(-\infty) = u^\ell, \quad u(+\infty) = u^r.
\end{aligned}
$$

Write as a nonautonomous system:

$$
\begin{aligned}
\epsilon u_x &= v,\\
\epsilon v_x &= (Df(u) - xI)v.
\end{aligned}
$$

Let $x = x_0 + \epsilon\xi$, let a dot denote derivative with respect to ξ, write as an autonomous system:

$$
\begin{aligned}
\dot{u} &= v,\\
\dot{v} &= (Df(u) - xI)v,\\
\dot{x} &= \epsilon,
\end{aligned}
$$

$$
(u, v, x)(-\infty) = (u^\ell, 0, -\infty), \quad (u, v, x)(\infty) = (u^r, 0, \infty).
$$

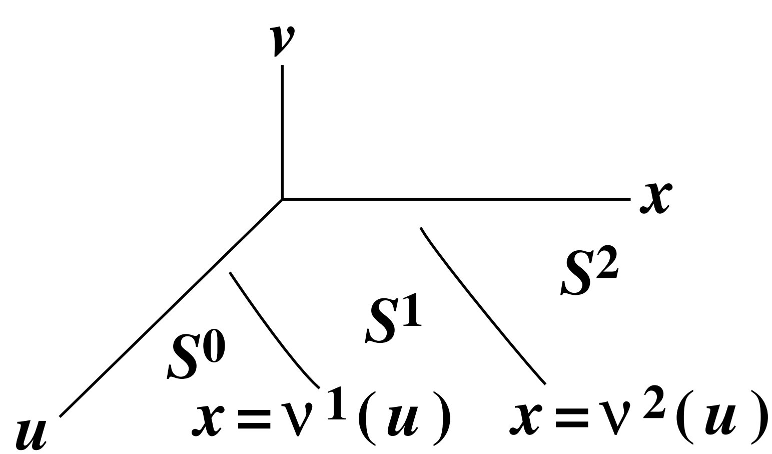
To begin analysis, let $\epsilon = 0$.

$\epsilon = 0$:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= (Df(u) - xI)v, \\ \dot{x} &= 0. \end{aligned}$$

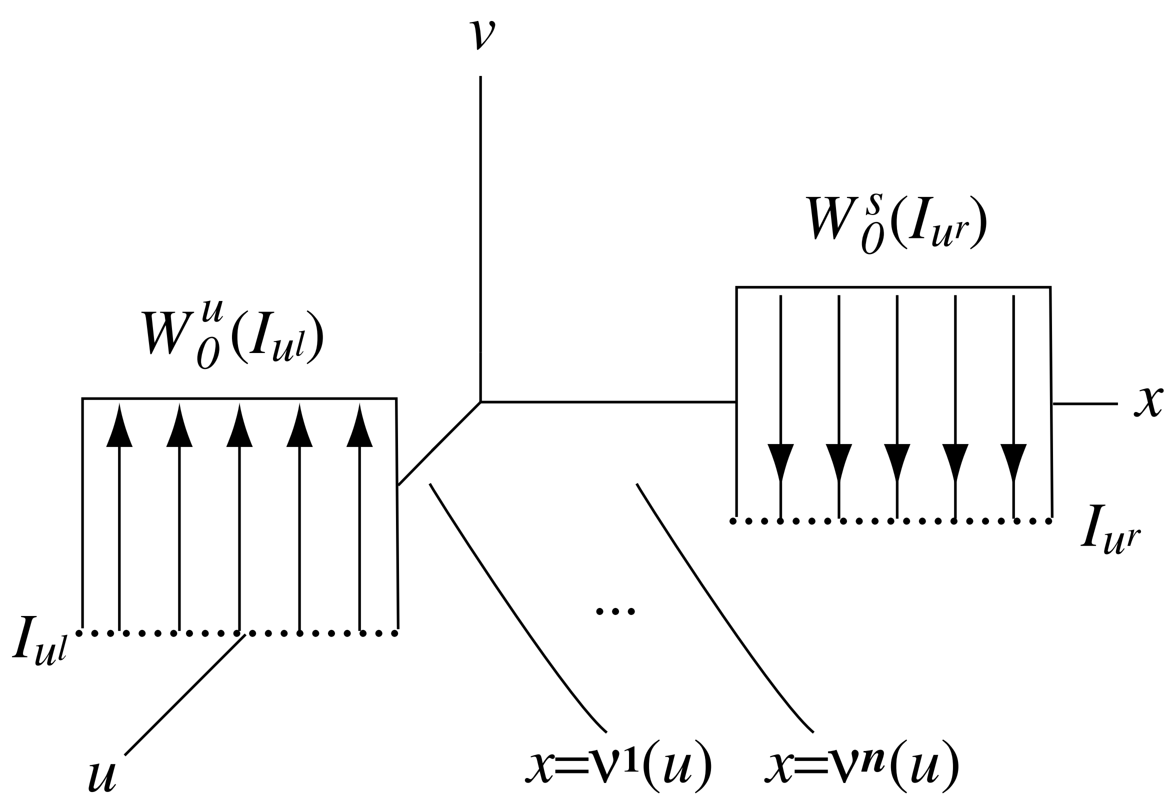
Solutions have x constant and satisfy $\ddot{u} = (Df(u) - xI)\dot{u}$, the traveling wave ODE.

- ux-space consists of equilibria.
- It is divided by codimension-one submanifolds
$$x = \nu^1(u), \ldots, x = \nu^n(u)$$
into open subsets $S^0, \ldots, S^n$.
- Equilibria in S^i have i negative eigenvalues and $n - i$ positive eigenvalues.
- The S^i are normally hyperbolic invariant manifolds.



$$
\begin{aligned}
I_{u^\ell} &= \{(u,v,x) : u = u^\ell, v = 0, x < \nu^1(u)\},\\
I_{u^r} &= \{(u,v,x) : u = u^r, v = 0, \nu^n(u) < x\}.
\end{aligned}
$$

- $W_0^u(I_{u^\ell})$ has dimension $n+1$.
- $W_0^s(I_{u^r})$ has dimension $n+1$.



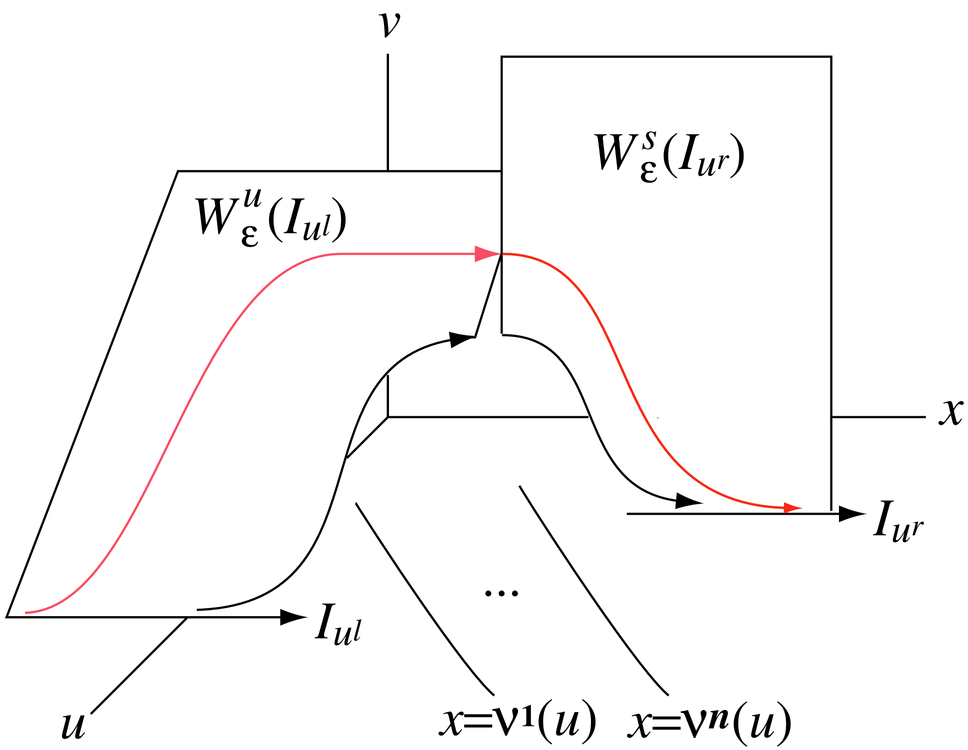
$\epsilon > 0$:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= (Df(u) - xI)v, \\ \dot{x} &= \epsilon, \end{aligned}$$

$$(u, v, x)(-\infty) = (u^\ell, 0, -\infty), \quad (u, v, x)(\infty) = (u^r, 0, \infty).$$

- ux-space remains invariant.
- On it, $\dot{u} = 0$, $\dot{x} = \epsilon$.

- $W^u_\epsilon(I_{u^\ell})$ has dimension $n + 1$.
- $W^s_\epsilon(I_{u^r})$ has dimension $n + 1$.
- It they intersect transversally they intersect in a curve.
- That curve is a solution of the Dafermos ODE BVP (a Riemann-Dafermos solution).



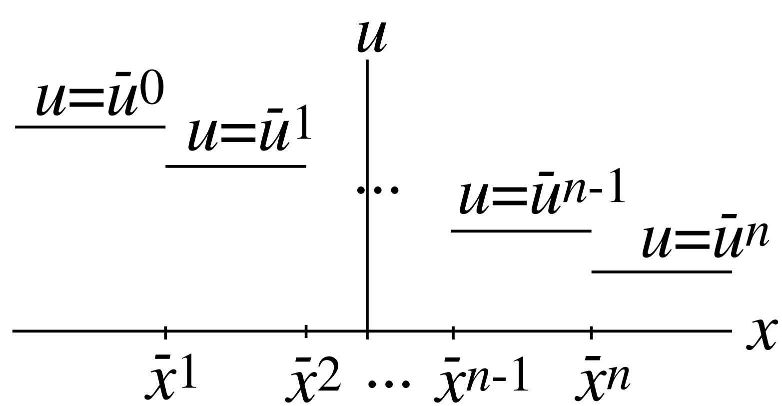
v
$W^s_\varepsilon(I_{u^r})$
$W^u_\varepsilon(I_{u^l})$
x
I_{u^r}
I_{u^l}
...
u
$x=\nu^1(u)$
$x=\nu^n(u)$

Riemann-Dafermos solutions near structurally stable Riemann solutions with n Lax shock waves

Suppose the Riemann solution $u_0(x)$ for (CL) with $u_0(-\infty) = u^\ell$ and $u_0(\infty) = u^r$ has the following properties:

- There are speeds $\bar{x}^1 < \bar{x}^2 < \ldots < \bar{x}^n$ such that

$$u_0(x) = \begin{cases} \bar{u}^0 & \text{for } x < \bar{x}^1, \\ \bar{u}^i & \text{for } \bar{x}^i < x < \bar{x}^{i+1},\ i = 1, \ldots, n-1, \\ \bar{u}^n & \text{for } x > \bar{x}^n. \end{cases}$$



- $\bar{u}^0 = u^\ell$ and $\bar{u}^n = u^r$.

- For each $i = 1, \ldots, n$, the triple $(\bar{u}^{i-1}, \bar{x}^i, \bar{u}^i)$ is a regular Lax i-shock with viscous profile $q^i(\xi)$.

- If we set $u^0 = \bar{u}^0$ and $u^n = \bar{u}^n$ in the system of equations

$$f(u^1) - f(u^0) - x^1(u^1 - u^0) = 0,$$
$$\vdots$$
$$f(u^n) - f(u^{n-1}) - x^n(u^n - u^{n-1}) = 0,,$$

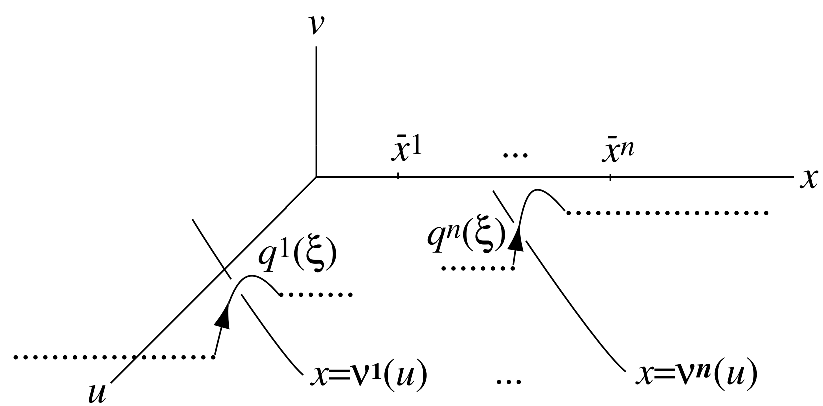
then the resulting system of n^2 equations in the n^2 variables $(x^1, u^1, \ldots, u^{n-1}, x^n)$ has $(\bar{x}^1, \bar{u}^1, \ldots, \bar{u}^{n-1}, \bar{x}^n)$ as a regular solution.

Let $\bar{x}^0 = -\infty$, $\bar{x}^n = \infty$. Let Γ denote the union of the $n+1$ line segments

$$\{(\bar{u}^i, 0, x) : \bar{x}^i < x < \bar{x}^{i+1}\}, \quad i = 0, \dots, n,$$

and the n connecting orbits

$$(q^i(\xi), \dot{q}^i(\xi), \bar{x}^i), \quad -\infty < \xi < \infty, \quad i = 1, \dots, n.$$



Γ is a **singular solution** ($\epsilon = 0$) of the boundary value problem

$$\begin{aligned}
\dot{u} &= v, \\
\dot{v} &= (Df(u) - xI)v, \\
\dot{x} &= \epsilon,
\end{aligned}$$
$$(u, v, x)(-\infty) = (u^\ell, 0, -\infty), \quad (u, v, x)(\infty) = (u^r, 0, \infty).$$

Theorem (Szmolyan). For small $\epsilon > 0$ there is a Riemann-Dafermos solution $u_\epsilon(x)$ near Γ.

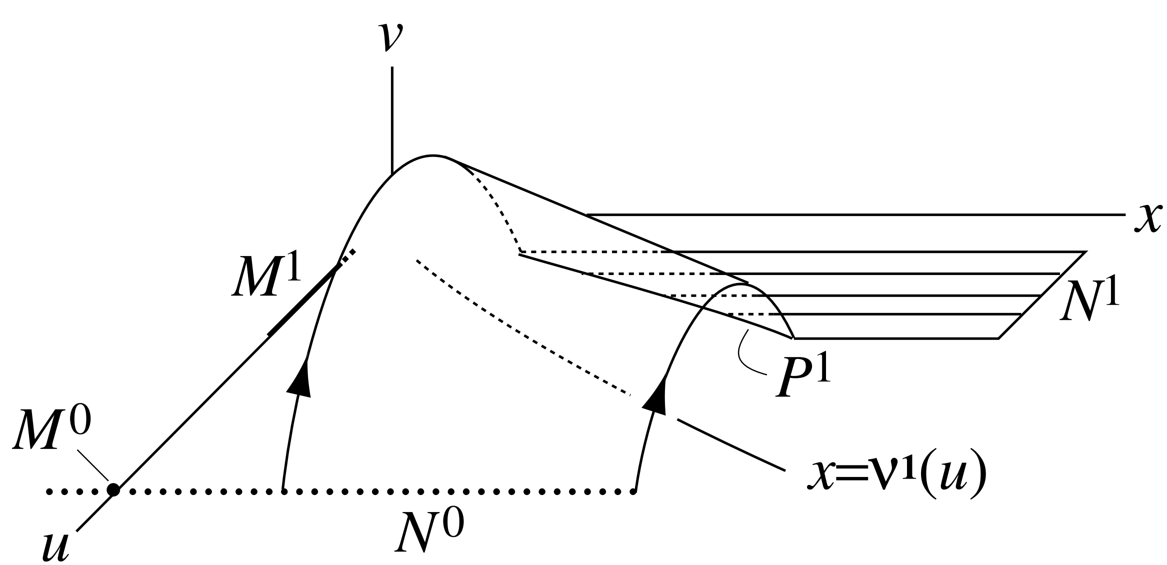
Idea of proof:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= (Df(u) - xI)v, \\ \dot{x} &= \epsilon. \end{aligned}$$

Let

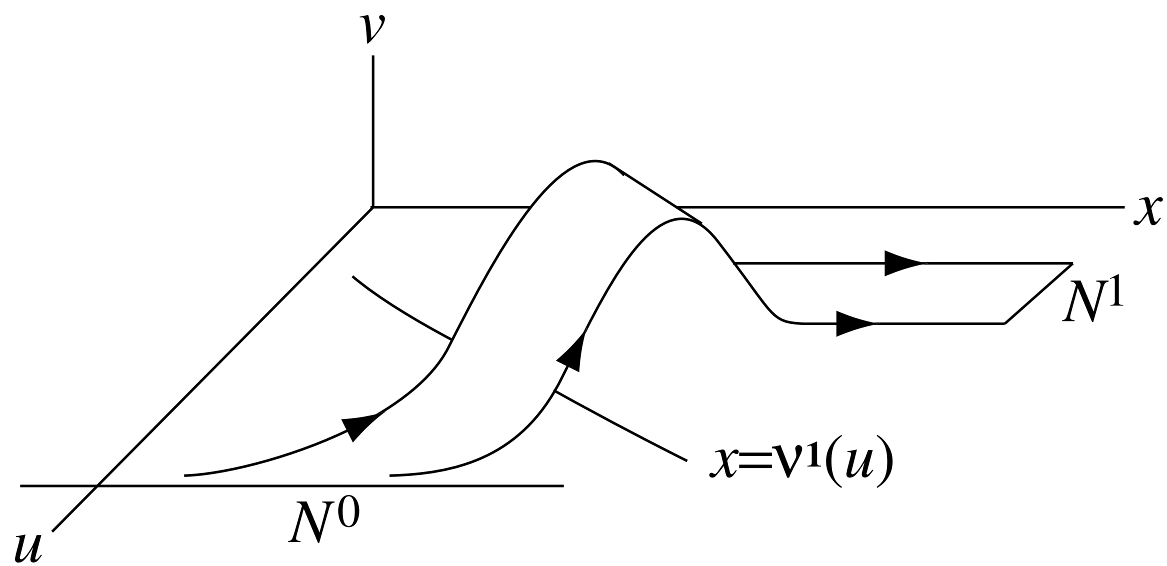
- $M^0 = \{\bar{u}^0\} = \{u^\ell\}$.
- $N^0 = I_{u^\ell}$.

$W_0^u(N^0)$ meets $W_0^s(S^1)$ transverally in a 1-parameter family of connecting orbits to a 1-dimensional manifold P^1.



Let M^1 = projection of P^1 onto u-space, $N^1 = \{(u, 0, x) \in S^1 : u \in M^1\}$.

For small $\epsilon > 0$ the Exchange Lemma implies that $W^u_\epsilon(N^0)$ is C^1-close to $W^u_0(N^1)$.



$W^u_0(N^1)$ meets $W^s_0(S^2)$ transverally in a 2-parameter family of connecting orbits to P^2.

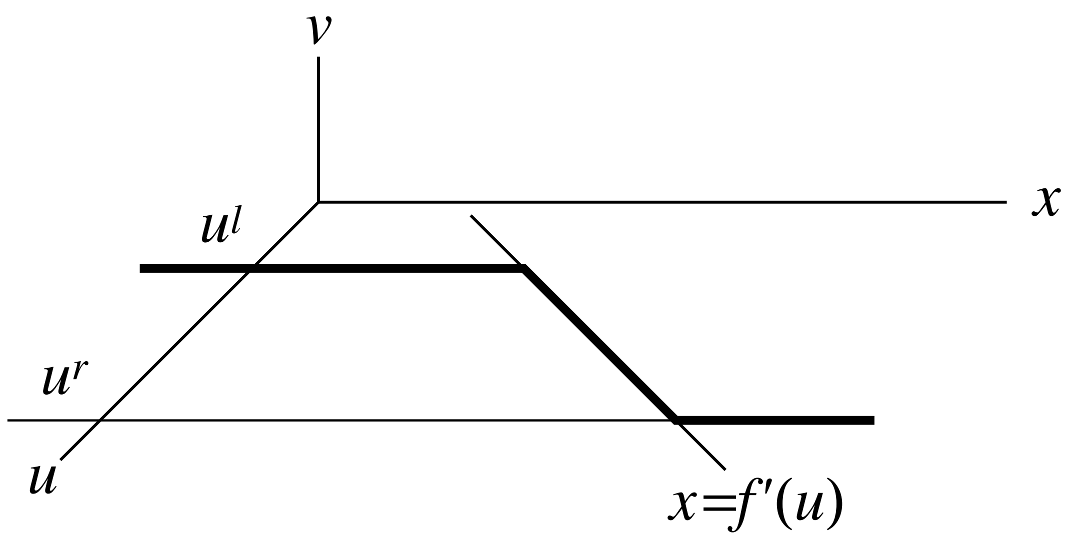
For small $\epsilon > 0$ the Exchange Lemma implies that $W^u_\epsilon(N^0)$ is C^1-close to $W^u_0(N^2)$.

By induction, $W^u_\epsilon(N^0)$ is C^1-close to $W^u_0(N^n) = N^n$, i.e., it is C^1-close to ux space when x is large.

There it meets $W^s_\epsilon(I_{u^r})$ transversally.

Riemann-Dafermos solutions near Riemann solutions with rarefactions

Suppose $n = 1$ and the Riemann solution is a single rarefaction. Singular solution of the Dafermos boundary value problem ($\epsilon = 0$):



Normal hyperbolicity of the plane of equilibria is lost along the curve $x = f'(u)$, $v = 0$.

Make the change of variables $x = f'(u) + \sigma$. Now normal hyperbolicity is lost along the u-axis.

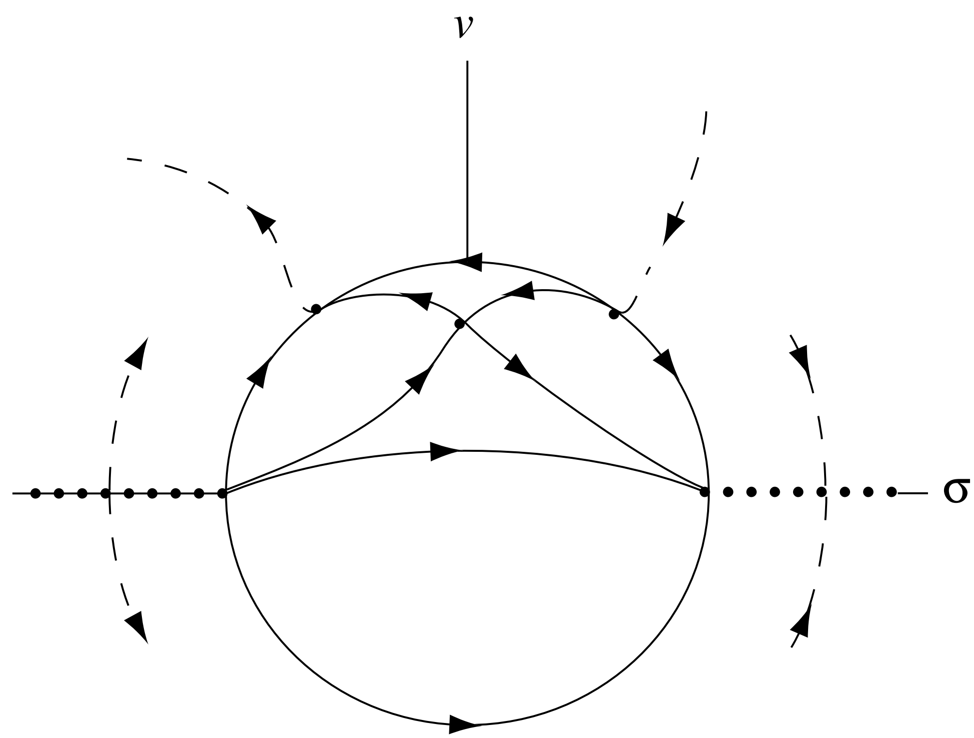
"Blow up" the u-axis in $uv\sigma\epsilon$-space:

$$\begin{aligned} u &= u, \\ v &= \bar{r}^2 \bar{v}, \\ \sigma &= \bar{r}\bar{\sigma}, \\ \epsilon &= \bar{r}^2 \bar{\epsilon}, \end{aligned}$$

$(\bar{v}, \bar{\sigma}, \bar{\epsilon}) \in S^2$ and $\bar{r} \geq 0$.

After this transformation the spherical cylinder $\bar{r} = 0$ consists entirely of equilibria. Divide by $\bar{r}$. Division by $\bar{r}$ desingularizes the system on $\bar{r} = 0$ but leaves it invariant.

Flow in blow-up space:



$u = u^\ell$ is fixed; in the figure we look straight down the ϵ-axis. We see the sphere $u = u^\ell$, $\bar{r} = 0$, and outside it the plane $u = u^\ell$, $\epsilon = 0$, in which the origin is blown up to a circle.

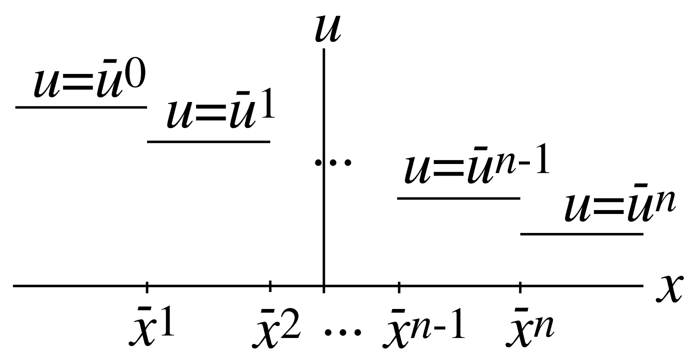
For $n > 1$ this sort of analysis must be supplemented by a **"General Exchange Lemma"** (S.) to deal with the hyperbolic directions.

4. Time-asymptotic Stability

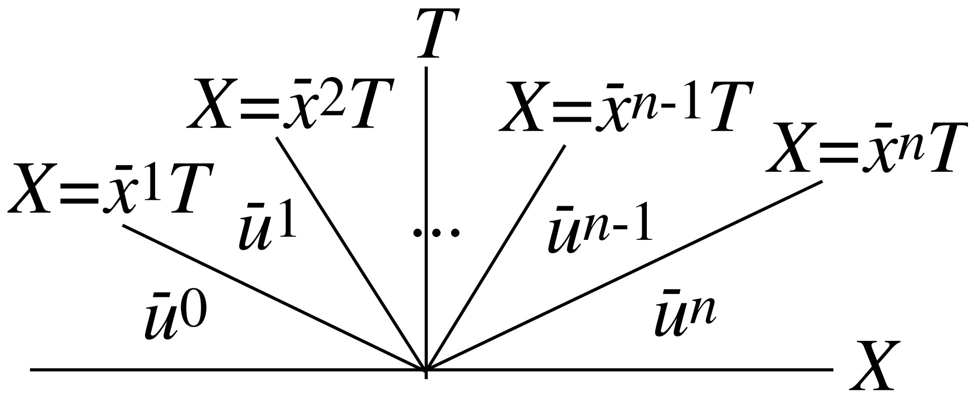
(a) Riemann solutions

Riemann solution of (CL) with n Lax shock waves:

$$u_0(x) = \begin{cases} \bar{u}^0 & \text{for } x < \bar{x}^1, \\ \bar{u}^i & \text{for } \bar{x}^i < x < \bar{x}^{i+1},\ i = 1, \ldots, n-1, \\ \bar{u}^n & \text{for } x > \bar{x}^n. \end{cases}$$



(RS) $$u_0(X,T) = \begin{cases} \bar{u}^0 & \text{for } X < \bar{x}^1 T, \\ \bar{u}^i & \text{for } \bar{x}^i T < x < \bar{x}^{i+1} T \\ & \qquad\qquad i = 1, \ldots, n-1, \\ \bar{u}^n & \text{for } x > \bar{x}^n T. \end{cases}$$



To study the stability of (RS) as a solution of (CL), one considers the linear system

$$\text{(LINEQ)} \qquad U_T + \begin{cases} Df(\bar{u}^0)U_X = 0 & \text{for } X < \bar{x}^1 T, \\ Df(\bar{u}^i)U_X = 0 & \text{for } \bar{x}^i T < X < \bar{x}^{i+1} T, \\ & \qquad\qquad i = 1, \dots, n-1, \\ Df(\bar{u}^n)U_X = 0 & \text{for } \bar{x}^n T < X, \end{cases}$$

$$\text{(JUMP)} \quad \begin{aligned} &(Df(\bar{u}^i) - \bar{x}^i I)U(\bar{x}^i T+, T) - (Df(\bar{u}^{i-1}) - \bar{x}^i I)U(\bar{x}^i T-, T) \\ &\qquad - S^i(T)(\bar{u}^i - \bar{u}^{i-1}) = 0, \quad i = 1, \dots, n, \end{aligned}$$

In each sector, $Df(\bar{u}^i)$ is constant, so solutions (which may include discontinuities) propagate along straight-line characteristics.

Along the lines $X = \bar{x}^i T$, data arrive from both sides along incoming characteristics, and one uses (JUMP) to solve for S^i and for the continuation of the solution along outgoing characteristics.

Majda's stability condition is just the condition that one can do this.

One can interpret (LINEQ)–(JUMP) as describing the scattering of incoming small shock waves by the large shock waves that comprise the original Riemann solution.

Several authors have found sufficient conditions that guarantee that, in a weighted BV or L^1 norm, the total norm of the scattered shocks is smaller than the total norm of the incoming shocks (Schochet, Bressan and Colombo, Bressan and Marson, Wang, Lewicka). This theory yields sufficient conditions for stability of the Riemann solution.

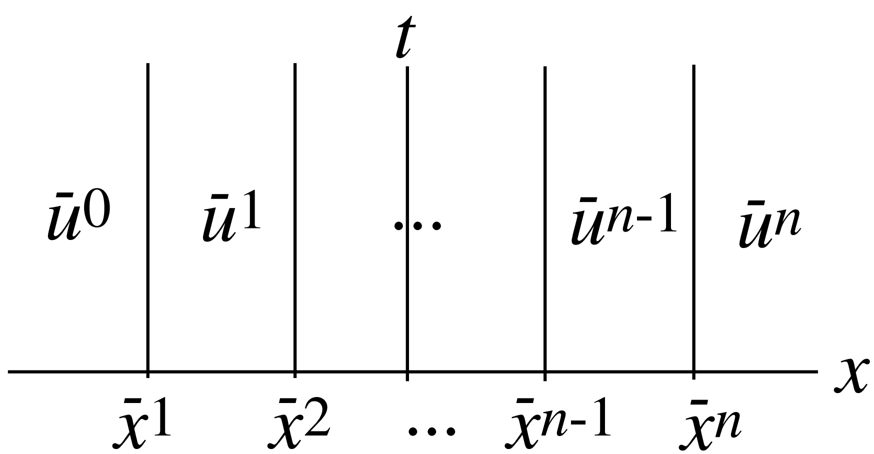
Since solutions of (LINEQ)–(JUMP) are typically discontinuous along the skew lines $X = \bar{s}^i T$, it is not reasonable to expect solutions of the form $e^{\lambda T} U(X)$, i.e., it is not reasonable to look for eigenvalues and eigenfunctions.

Alternate approach:

In (LINEQ) and (JUMP), let $x = \frac{X}{T}$, $t = \ln T$:

(LINEQ2)
$$U_t + \begin{cases} (Df(\bar{u}^0) - xI)U_x = 0 & \text{for } x < \bar{x}^1, \\ (Df(\bar{u}^i) - xI)U_x = 0 & \text{for } \bar{x}^i < x < \bar{x}^{i+1}, \\ & \qquad i = 1, \dots, n-1, \\ (Df(\bar{u}^n) - xI)U_x = 0 & \text{for } \bar{x}^n < x, \end{cases}$$

(JUMP2)
$$\begin{aligned} (Df(\bar{u}^i) - \bar{x}^i I)U(\bar{x}^i+, t) - (Df(\bar{u}^{i-1}) - \bar{x}^i I)U(\bar{x}^i-, t) \\ - S^i(t)(\bar{u}^i - \bar{u}^{i-1}) = 0, \quad i = 1, \dots, n, \end{aligned}$$



The characteristics are no longer straight lines, but the lines $X = \bar{x}^i T$ become $x = \bar{x}^i$, so it is reasonable to look for eigenvalues and eigenfunctions.

Theorem (Lin). (LINEQ2)–(JUMP2) generates a C_0 semigroup in the space L^2_η of L^2 functions that are $O((1+|x|)^{-\eta})$, $\eta > 0$. In L^2_η the linear operator has no essential spectrum in $\Re\lambda > -\eta$, but it can have both eigenvalues and resonance lines there. If γ is greater than the real parts of all eigenvalues and all numbers α such that $\Re\lambda = \alpha$ is a resonance line, then all solutions are $O(e^{\gamma t})$.

A resonance line is a line $\Re\lambda = \alpha$ on which the linear operator $A - \lambda I$ has a bounded inverse for each λ, but $\|(A-\lambda I)^{-1}\|$ is not uniformly bounded.

It follows that if the linear system has no eigenvalues or resonance lines in the half-plane $\Re\lambda \geq 0$, then all solutions decay exponentially.

On the other hand, if a resonance line lies in the right half-plane, solutions can grow even though the spectrum lies in the left half-plane.

Recent work of Lewicka and Zumbrun suggests that the BV stability criterion may be equivalent to Lin's criterion.

(b) Traveling waves of viscous conservation laws

(VCL) $$u_T + f(u)_X = u_{XX}$$

Consider a traveling wave solution $u(\xi)$, $\xi = X - sT$, with

(TWBC) $$u(-\infty) = u^-, \quad u(+\infty) = u^+.$$

In the variables (ξ, T), (VCL) becomes

(VCL*) $$u_T + (Df(u) - sI)u_\xi = u_{\xi\xi},$$

which has $u(\xi)$ as a stationary solution. Linearize at $u(\xi)$:

$$U_T + (Df(u) - sI)U_\xi + D^2 f(u) u_\xi U = U_{\xi\xi} \qquad \text{with } u = u(\xi).$$

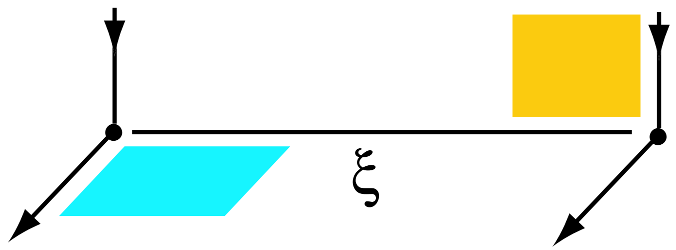
Solutions of the form $e^{\lambda T}U(\xi)$ ($\lambda =$ eigenvalue, $U(\xi) =$ eigenfunction) satisfy

$$\lambda U + (Df(u) - sI)U_\xi + D^2 f(u) u_\xi U = (B(u)U_\xi)_\xi \qquad \text{with } u = u(\xi).$$

Written as a system:

$$\begin{pmatrix} U_\xi \\ V_\xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ \lambda I + D^2 f(u) u_\xi & Df(u) - sI \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \qquad \text{with } u = u(\xi).$$

The space we are working in should be chosen so that for $\Re\lambda \geq -\delta$, the spaces of acceptable $U(\xi)$ for $\xi \to -\infty$ and for $\xi \to \infty$ have complementary dimension.



At $u(\pm\infty) = u^\pm$ the matrix is:

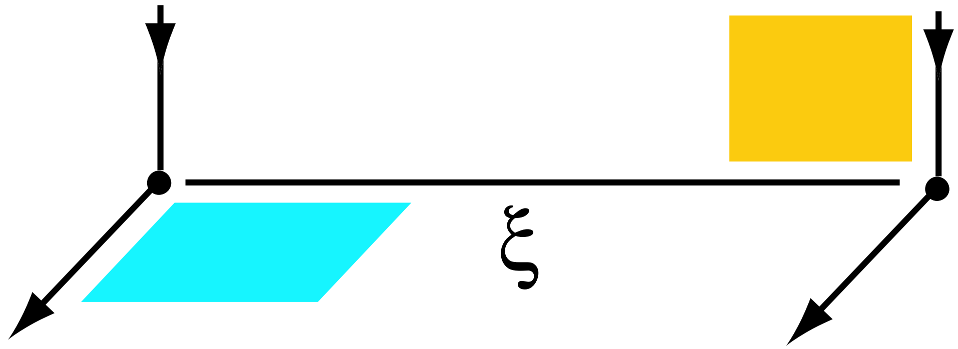
$$\begin{pmatrix} 0 & I \\ \lambda I & Df(u^\pm) - sI \end{pmatrix}$$

Its eigenvalues at $u(-\infty) = u^-$ are

$$\frac{1}{2}\Big(\nu_i(u^-) - s \pm \big((\nu_i(u^-) - s)^2 + 4\lambda\big)^{\frac{1}{2}}\Big), \quad i = 1, \dots, n.$$

Its eigenvalues at $u(\infty) = u^+$ are

$$\frac{1}{2}\Big(\nu_i(u^+) - s \pm \big((\nu_i(u^+) - s)^2 + 4\lambda\big)^{\frac{1}{2}}\Big), \quad i = 1, \dots, n.$$



For $n = 1$ and $\lambda = 0$:

$$u^- : +, 0, \quad u^+ : -, 0.$$

Use spaces of exponentially decreasing functions (Sattinger).
For $n = 2$, a repeller-to-saddle traveling wave (viscous Lax 1-shock), and $\lambda = 0$:

$$u^- : +, +, 0, 0, \quad u^+ : -, +, 0, 0.$$

There is no good way to choose the space. (For any choice, $\lambda = 0$ is in the closure of the essential spectrum.)

For $\Re\lambda > 0$ the spaces of bounded solutions have dimension n at each end. Can choose bases

$$X_1(\lambda,\xi),\ldots,X_n(\lambda,\xi) \text{ on } -\infty < \xi \le 0$$

$$X_{n+1}(\lambda,\xi),\ldots,X_{2n}(\lambda,\xi) \text{ on } 0 \le \xi < \infty.$$

Then for $\Re\lambda > 0$,

$$E(\lambda) = \det\big(X_1(\lambda,0) \;\; \cdots \;\; X_{2n}(\lambda,0)\big)$$

is 0 if and only if λ is an eigenvalue.

$E(\lambda)$ is the *Evans function.* It is analytic in λ.

The proof that spectral stability implies nonlinear stability is hard (Zumbrun and collaborators).

(c) Riemann-Dafermos solutions

(DR) $$u_t + (Df(u) - xI)u_x = \epsilon u_{xx}.$$

(BC) $$u(-\infty, t) = u^\ell, \quad u(+\infty, t) = u^r.$$

Let $u_\epsilon(x)$ be a Riemann-Dafermos solution.

Linearized Dafermos operator at $u_\epsilon(x)$:

(LD) $$U_t + (Df(u) - xI)U_x + D^2 f(u) u_x U = \epsilon U_{xx} \qquad \text{with } u = u_\epsilon(x).$$

Theorem (Lin). For small $\epsilon > 0$, the linearized Dafermos operator is sectorial in a space of functions that grow like $e^{-\frac{x^2}{\epsilon}}$. Its essential spectrum lies in $\Re\lambda \leq -\delta$.

Hence linear and nonlinear stability are determined by the eigenvalues.

Eigenvalues λ and corresponding eigenfunctions $U(x)$ of the linearized Dafermos operator at $u_\epsilon(x)$ satisfy

$$\lambda U + (Df(u) - xI)U_x + D^2 f(u) u_x U = \epsilon U_{xx} \qquad \text{with } u = u_\epsilon(x).$$
$$U(\pm\infty) = 0.$$

Written as a system with $x = \epsilon\xi$ and $\rho = \epsilon\lambda$ (**fast eigenvalues**):

$$\text{(LDE)} \qquad \begin{pmatrix} U_\xi \\ V_\xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ \rho I + D^2 f(u) u_\xi & Df(u) - xI \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}.$$

For each ρ, (LDE) has an n-dimensional space of solutions that go to 0 exponentially as $x \to -\infty$, and an n-dimensional space of solutions that go to 0 exponentially as $x \to \infty$.

Proposition (S.). Choosing bases yields an Evans function $E(\rho)$ that is analytic in ρ.

To find eigenvalues, combine (DODE) and (LDE) into one system. Add ρ as a state variable:

Combined system:

$$
\begin{aligned}
\dot{u} &= v, \\
\dot{v} &= (Df(u) - xI)v, \\
\dot{x} &= \epsilon, \\
\dot{U} &= V, \\
\dot{V} &= \rho U + (Df(u) - xI)V + D^2 f(u)vU, \\
\dot{\rho} &= 0.
\end{aligned}
$$

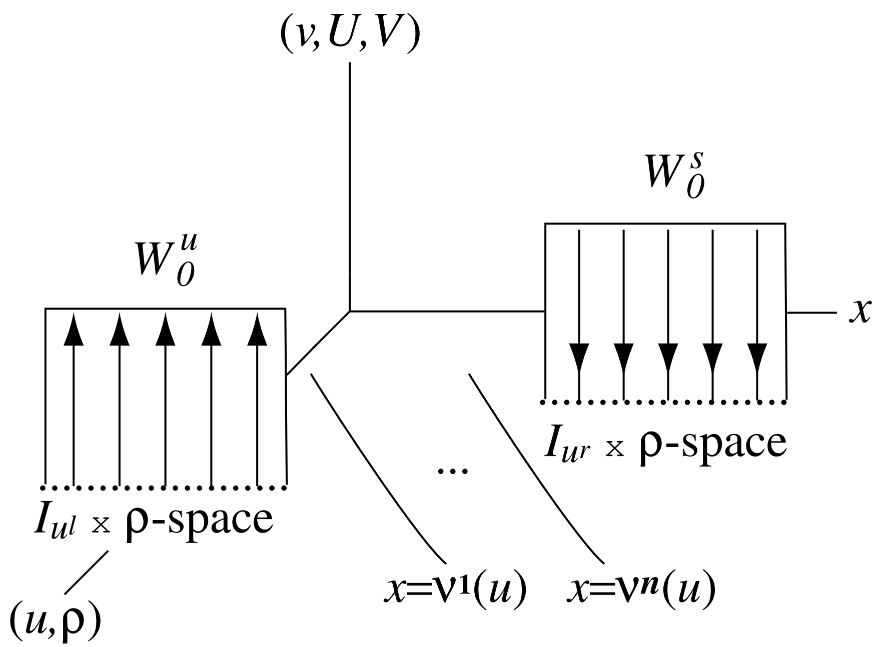
Let $\epsilon = 0$:

$$
\begin{aligned}
\dot{u} &= v, \\
\dot{v} &= (Df(u) - xI)v, \\
\dot{x} &= 0, \\
\dot{U} &= V, \\
\dot{V} &= \rho U + (Df(u) - xI)V + D^2 f(u)vU, \\
\dot{\rho} &= 0.
\end{aligned}
$$

Combined system with $\epsilon = 0$:

$$\begin{aligned}
\dot{u} &= v, \\
\dot{v} &= (Df(u) - xI)v, \\
\dot{x} &= 0, \\
\dot{U} &= V, \\
\dot{V} &= \rho U + (Df(u) - xI)V + D^2 f(u)vU, \\
\dot{\rho} &= 0.
\end{aligned}$$

$ux\rho$-space (the space $v = U = V = 0$) consists of equilibria.



For $\epsilon > 0$, intersections of W_0^u and W_0^s represent triples (Riemann-Dafermos solution, eigenfunction, "eigenvalue").

Let $u_0(x)$ be a structurally stable Riemann solution with n Lax shock waves and viscous profiles $q^i(\xi)$.

Let $E^i(\rho)$ be the Evans function for the ith viscous profile $q^i(\xi)$, considered as a traveling wave of (VCL).

Theorem (Lin–S.). For $\bar{\rho} > 0$:

(1) Suppose $E^i(\bar{\rho}) \neq 0$ for all $i = 1, \ldots, n$. Then for small $\epsilon > 0$, $u_\epsilon(x)$ does not have a number near $\frac{\bar{\rho}}{\epsilon}$ as an eigenvalue.

(2) Suppose there is a number ℓ such that $E^\ell(\bar{\rho}) = 0$, $\frac{d}{d\rho}E^\ell(\bar{\rho}) \neq 0$, and $E^i(\bar{\rho}) \neq 0$ for $i \neq \ell$. Then there is a curve $\rho(\epsilon) = \bar{\rho} + O(\epsilon)$ such that $\frac{\bar{\rho}}{\epsilon}$ is an eigenvalue of $u_\epsilon(x)$.

Proof of (2). Using the Exchange Lemma, follow W^u_ϵ forward and W^s_ϵ backward until they meet at the ℓth jump. Use the assumption to check transversality there.

The Exchange Lemma cannot be used at the ℓth jump, because W^u_0 meets the stable manifold of $(\bar{u}^\ell, 0, \bar{x}^\ell, 0, 0, \bar{\rho})$ in a **two-dimensional** manifold: viscous profile $\times$ multiples of eigenfunction.

$\rho = 0$ is special: $E^i(0) = 0$ for every i!

To study eigenvalues near $\rho = 0$, let $\rho = \epsilon\lambda$ (**slow eigenvalues**):

$$
\begin{aligned}
\dot{u} &= v, \\
\dot{v} &= (Df(u) - xI)v, \\
\dot{x} &= \epsilon, \\
\dot{U} &= V, \\
\dot{V} &= \epsilon\lambda U + (Df(u) - xI)V + D^2 f(u) vU, \\
\dot{\lambda} &= 0.
\end{aligned}
$$

Set $\epsilon = 0$:

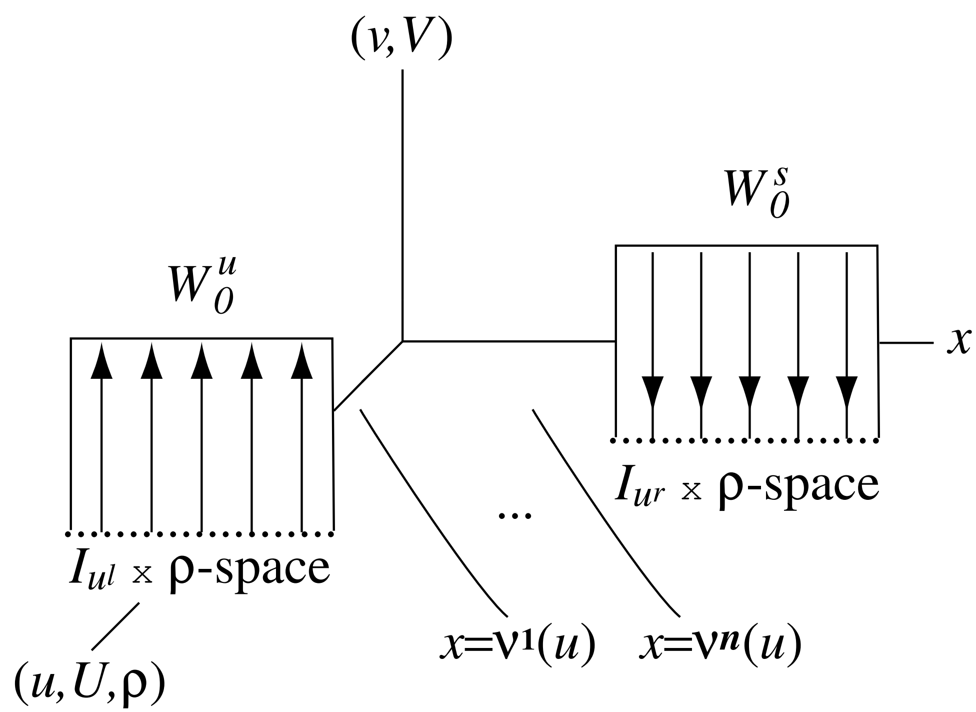
$$
\begin{aligned}
\dot{u} &= v, \\
\dot{v} &= (Df(u) - xI)v, \\
\dot{x} &= 0, \\
\dot{U} &= V, \\
\dot{V} &= (Df(u) - xI)V + D^2 f(u) vU, \\
\dot{\lambda} &= 0.
\end{aligned}
$$

Then

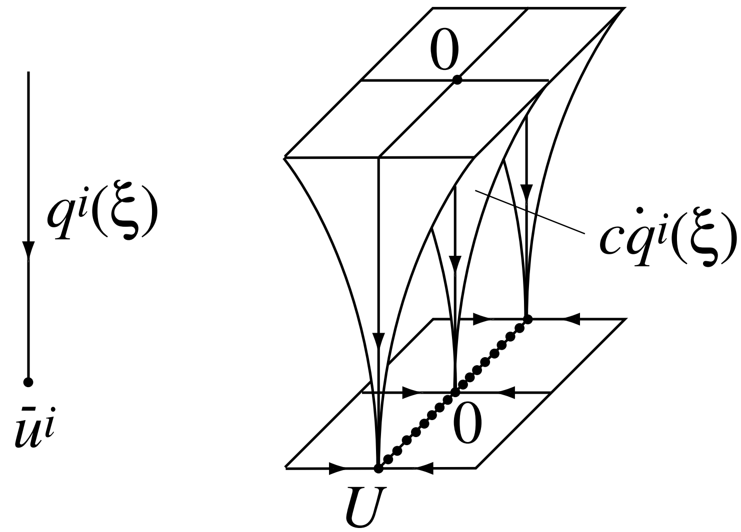
- $uxU\lambda$-space (the space $v = V = 0$) consists of equilibria.
- The UV-system is the linearization of the uv-system.

$\epsilon = 0$:

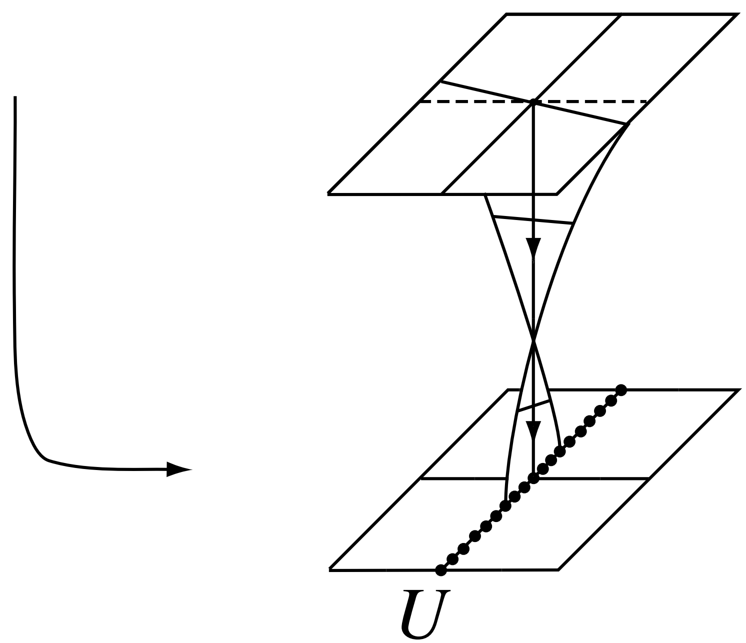
$$
\begin{aligned}
\dot{u} &= v, \\
\dot{v} &= (Df(u) - xI)v, \\
\dot{x} &= 0, \\
\dot{U} &= V, \\
\dot{V} &= (Df(u) - xI)V + D^2 f(u) vU, \\
\dot{\lambda} &= 0.
\end{aligned}
$$



$\epsilon = 0$:



$\epsilon > 0$:



The jump in U of order ϵ must be followed using the flow on perturbed $uxU\lambda$-space:

Flow on perturbed $uxU\lambda$-space:

$$
\begin{aligned}
\dot{u} &= 0, \\
\dot{x} &= \epsilon, \\
\dot{U} &= \epsilon\Big(-\lambda(Df(u) - xI)^{-1} + O(\epsilon)\Big)U, \\
\dot{\lambda} &= 0.
\end{aligned}
$$

Dividing the third equation by the second, we obtain

$$U_x = -\lambda(Df(u) - xI)^{-1}U,$$

or, equivalently,

$$\lambda U + (Df(u) - xI)U_x = 0.$$

To lowest order in ϵ, λ is an eigenvalue of the linearized Dafermos operator at $u_\epsilon(x)$ provided there exist $\mu^1, \dots, \mu^n$ not all 0 such that the following system has a solution with $U = 0$ for $x < \bar{x}^1$ and $\bar{x}^n < x$:

$$\text{(SLOWEIG)} \qquad \lambda U + \begin{cases} (Df(\bar{u}^0) - xI)U_x = 0 & \text{for } x < \bar{x}^1, \\ (Df(\bar{u}^i) - xI)U_x = 0 & \text{for } \bar{x}^i < x < \bar{x}^{i+1}, \\ & \qquad\qquad i = 1, \dots, n-1, \\ (Df(\bar{u}^n) - xI)U_x = 0 & \text{for } \bar{x}^n < x, \end{cases}$$

$$\text{(SLOWJUMP)} \quad (Df(\bar{u}^i) - \bar{x}^i I)U(\bar{x}^i+) - (Df(\bar{u}^{i-1}) - \bar{x}^i I)U(\bar{x}^i-) \\ - \mu^i(\bar{u}^i - \bar{u}^{i-1}) = 0, \quad i = 1, \dots, n.$$

The same system arises when one looks for eigenvalues of the linear system (LINEQ2)–(JUMP2), the linearization of (CL) at (RS).

Theorem (Lin-S.). Suppose (1) (SLOWEIG)–(SLOWJUMP) has a nontrivial solution for $\lambda = \bar{\lambda}$ and (2) a nondegeneracy condition is satisfied. Then for small $\epsilon > 0$, W^u_ϵ and W^s_ϵ meet transversally along a one-parameter family of orbits

$$(u_\epsilon(x), u'_\epsilon(x), x, aU_\epsilon(x), aV_\epsilon(x), \lambda_\epsilon),$$

with $\lambda_\epsilon = \bar{\lambda} + O(\epsilon)$.

5. Open problem: put the pieces together

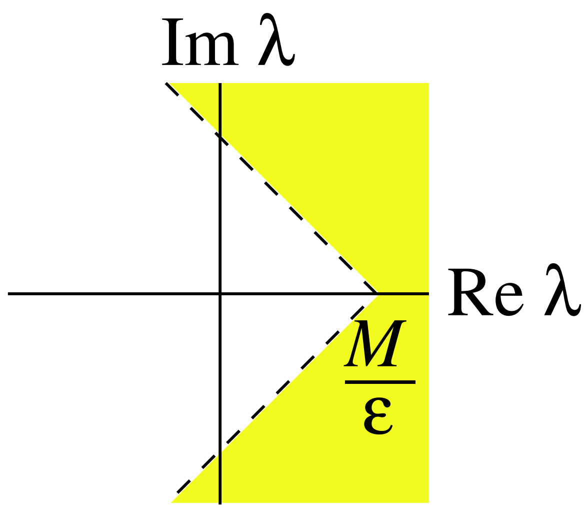
Linearized Dafermos operator at $u_\epsilon(x)$:

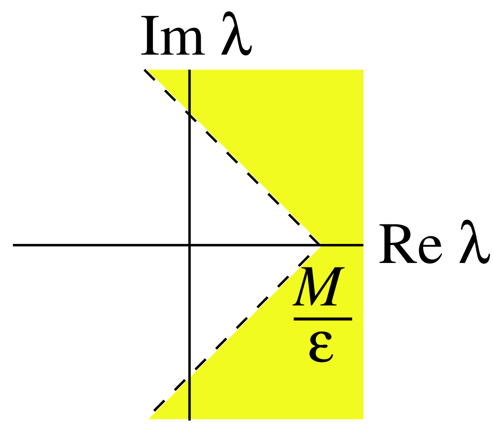
$$\text{(LD)} \qquad U_t + (Df(u) - xI)U_x + D^2 f(u) u_x U = \epsilon U_{xx} \qquad \text{with } u = u_\epsilon(x).$$

Write as $U_t = L_\epsilon U$ on some space.

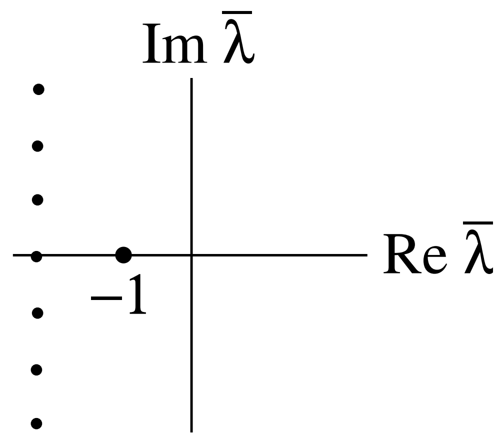
Theorem (Lin). L_ϵ is sectorial in a space of functions that grow like $e^{-\frac{x^2}{\epsilon}}$. Its essential spectrum lies in $\Re\lambda \leq -\delta$.

In fact, there are constants K and M such that $\|(L_\epsilon - \lambda I)^{-1}\| \leq \frac{K}{|\lambda|}$ on the set





On the other hand, in the examples we have calculated, (slow) eigenvalues $\lambda_\epsilon = \bar{\lambda} + O(\epsilon)$ of L_ϵ occur for $\bar{\lambda}$ **evenly spaced on a finite number of vertical lines**. There also appear to be n slow eigenvalues emanating from $\bar{\lambda} = -1$.



There is also a resolvent estimate for L_0: $\|(L_0 - \lambda I)^{-1}\| \le K$ to the right of a vertical line.

We have not been able to put the pieces together to get a good understanding of L_ϵ.

We have not extended the work on stability to more general Riemann solutions (including rarefactions, composite waves, undercompressive shock waves) or to more general viscosities.

A Ph.D. student, Monique Thomas, is working on extensions to scalar conservation laws with third and fourth order regularization.