Completion of proof of C^1 Contraction Mapping Theorem

(3) Proof that $g(y)$ is differentiable.

We want to show that $Dg(y)$ is the linear map $A(y)$ that satisfies the equation

$$D_1T(g(y),y)\,A(y) + D_2T(g(y),y) = A(y).$$

So we write:

$$g(y+k) - g(y) - A(y)k = T(g(y+k), y+k) - T(g(y),y)$$
$$- [D_1T(g(y),y)A(y) + D_2T(g(y),y)]\,k =$$

$$T(g(y+k), y+k) - T(g(y), y+k) - D_1T(g(y),y)(g(y+k)-g(y))$$
$$+ D_1T(g(y),y)(g(y+k) - g(y) - A(y)k)$$
$$+ T(g(y), y+k) - T(g(y),y) - D_2T(g(y),y)\,k =$$

$$(*) \quad \int_0^1 D_1T(g(y) + t(g(y+k)-g(y)), y+k) - D_1T(g(y),y)\,dt\,(g(y+k)-g(y))$$

$$+ D_1T(g(y),y)(g(y+k)-g(y)-A(y)k)$$

$$(**) \quad + \int_0^1 D_2T(g(y), y+tk) - D_2T(g(y),y)\,dt\,k.$$

Therefore

$$(I - D_1T(g(y),y))(g(y+k)-g(y)-A(y)k) = (*) + (**)$$

So

$$g(y+k) - g(y) - A(y)k = (I - D_1T(g(y),y))^{-1}[(*) + (**)]$$

Now $\|D_1T(g(y), y)\| \le \lambda < 1$.

We have seen that if $\|B\| < 1$, then $(I-B)^{-1} = I + B + B^2 + \cdots$,

so $\|(I-B)^{-1}\| \le 1 + \|B\| + \|B\|^2 + \cdots$,

Therefore $\|(I - D_1T(g(y), y))^{-1}\| \le 1 + \lambda + \lambda^2 + \cdots = \frac{1}{1-\lambda}$.

By step (2) in the proof, there are numbers $\delta_1 > 0$ and $K \ge 1$ such that if $\|k\| < \delta_1$ then $\|g(y+k) - g(y)\| \le K\|k\|$.

Let $\varepsilon > 0$.

Since D_1T and D_2T are continuous functions of (x, y), $\exists \delta_2 > 0$ such that if $\|k\| < \delta_2$ then the integrands in $(*)$ and $(**)$ have norm $< \frac{1-\lambda}{2K}\varepsilon$.

Let $\delta = \min(\delta_1, \delta_2)$. If $\|k\| < \delta$, then

$\|g(y+k) - g(y) - A(y)k\| \le \frac{1}{1-\lambda}\left(\|(*)\| + \|(**)\|\right) \le$

$\frac{1}{1-\lambda}\left[\frac{1-\lambda}{2K}\varepsilon\|g(y+k) - g(y)\| + \frac{1-\lambda}{2K}\varepsilon\|k\|\right] \le \frac{1}{2K}\varepsilon \cdot K\|k\| + \frac{1}{2}\varepsilon\|k\|$

$= \varepsilon\|k\|$.

This proves that $Dg(y) = A(y)$. //