20. Show that coexistence occurs in the competing species model governed by system (7)–(8) if and only if there is an asymptotically stable critical point in the first quadrant.

21. When one of the populations in a competing species model is being harvested, the phase plane diagram is completely changed. For example, consider the competition model

$$\frac{dx}{dt} = x(1 - 4x - y) - h ,$$

$$\frac{dy}{dt} = y(1 - 2y - 5x) ,$$

where h is the constant harvest rate. For $h = 0, 1/32$, and $5/32$, determine the critical points, discuss the type and stability of the critical points, and sketch the phase plane diagrams. In each case discuss the implications of the competition model.

12.4 ENERGY METHODS

Many differential equations arise from problems in mechanics and for these equations it is natural to study the effect that energy in the system has on the solutions to the system. By analyzing the energy, we can often determine the stability of a critical point (even in the more elusive case when it is a center of the corresponding linear equation). As we will see in the next section, these ideas can be generalized to systems that are unrelated to problems in mechanics.

Conservative Systems

In Chapter 4 we considered some mechanical systems that were governed by Newton's second law

$$F = ma = m\frac{d^2x}{dt^2}$$

(*force equals mass times acceleration*). When the force $F = F(t, x, x')$ depends only on x, we can introduce its antiderivative $-U(x)$,

$$F = F(x) = -\frac{dU(x)}{dx} , \quad \text{or} \quad U(x) = -\int F(x)dx + K ,$$

and express the second law as

$$m\frac{d^2x}{dt^2} + \frac{dU(x)}{dx} = 0 . \tag{1}$$

Under this condition, the quantity

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x) \tag{2}$$

is constant during the motion, because

$$\frac{d}{dt}\left\{\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x)\right\} = \frac{2}{2}m\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dU}{dx}\frac{dx}{dt} = \frac{dx}{dt}\left\{m\frac{d^2x}{dt^2} - F\right\} = 0 .$$

Since $(m/2)(dx/dt)^2$ is the *kinetic energy* of the system, it is customary to refer to $U(x)$ as the *potential energy* and $(m/2)(dx/dt)^2 + U(x)$ as the *total energy* of the system. We have thus shown that the total energy is constant. This "energy conservation" principle appeared in a different context in Section 4.8, where it was called the "energy integral lemma."

By dividing the equations throughout by the bothersome constant m, we obtain the *standard form of the differential equation for a conservative system*

(3) $$\frac{d^2x}{dt^2} + g(x) = 0 ,$$

where $g(x) := U'(x)/m$; the equivalent *phase plane system*

(4) $$\frac{dx}{dt} = v ,$$

(5) $$\frac{dv}{dt} = -g(x) ;$$

the *potential function*

$$G(x) := \int g(x)dx + C ;$$

and the *energy function*

$$E(x, v) := v^2/2 + G(x) .$$

The fact that the total energy is constant means that the *level curves*

(6) $$E(x, v) = k , \quad k \text{ a constant,}$$

contain the phase plane trajectories of the system (4)–(5); that is, the curves (6) are *integral curves.*

Example 1 Find the energy function $E(x, v)$ for the following equations. Select E so that the energy at the critical point $(0, 0)$ is zero, that is, $E(0, 0) = 0$.

(a) $\dfrac{d^2x}{dt^2} + x - x^3 = 0$

(b) $\dfrac{d^2x}{dt^2} + \sin x = 0$ (pendulum equation)

(c) $\dfrac{d^2x}{dt^2} + x - x^4 = 0$

Solution (a) Here $g(x) = x - x^3$. By integrating g, we obtain the potential function $G(x) = (1/2)x^2 - (1/4)x^4 + C$. Thus, $E(x, v) = (1/2)v^2 + (1/2)x^2 - (1/4)x^4 + C$. Now $E(0, 0) = 0$ implies that $C = 0$. Hence, the energy function we seek is $E(x, v) = (1/2)v^2 + (1/2)x^2 - (1/4)x^4$.

(b) Since $g(x) = \sin x$, the potential function is $G(x) = -\cos x + C$, which gives $E(x, v) = (1/2)v^2 - \cos x + C$. Setting $E(0, 0) = 0$, we find $C = 1$. Thus, the energy function is $E(x, v) = (1/2)v^2 - \cos x + 1$.

(c) Here $g(x) = x - x^4$ and the potential function is $G(x) = (1/2)x^2 - (1/5)x^5 + C$. Setting $E(0, 0) = 0$, we get $C = 0$, so the energy function is $E(x, v) = (1/2)v^2 + (1/2)x^2 - (1/5)x^5$. ◆

Notice that the pendulum equation in part (b) also has a critical point at $(\pi, 0)$. Setting the energy to be zero at $(\pi, 0)$—that is, $E(\pi, 0) = 0$—we arrive at the energy function $E(x, v) = (1/2)v^2 - \cos x - 1$.

For conservative systems, much useful information about the phase plane can be gained directly from the potential function $G(x)$. A convenient way of doing this is to place the graph of the potential function $z = G(x)$ (the potential plane) directly above the phase plane diagram with the z-axis aligned with the v-axis (see Figure 12.21). This places the relative extrema for $G(x)$ directly over the critical points for the system, since $G'(x_0) = g(x_0) = 0$ implies that $(x_0, 0)$ is a critical point for (4)–(5).

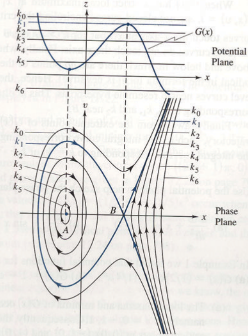

Figure 12.21 Relationship between potential plane (above) and phase plane (below)

To understand the relationship between the potential plane and the phase plane, let's look at the level curves for the energy function. In general, we have

$$\frac{v^2}{2} + G(x) = k ,$$

where $G(x)$ is the potential function with $G'(x) = g(x)$. Solving for v gives

$$v = \pm\sqrt{2[k - G(x)]} .$$

Thus, the velocity v exists (is real-valued) only when $k - G(x) \geq 0$. This generally results in three types of behavior.

When $G(x)$ has a strict local minimum at x_0 (above the point A) as illustrated in Figure 12.21, the integral curves $E(x, v) = k$, where k is slightly greater than $G(x_0)$, are closed curves encircling the critical point $A = (x_0, 0)$. To see this, observe that there is an interval (a, b) containing x_0 with $G(a) = G(b) = k_5$ and $G(x) < k_5$ for x in (a, b). Since $v = \sqrt{2[k_5 - G(x)]}$ is defined on (a, b) with $v = 0$ at $x = a$ and b, the two curves $v = \pm\sqrt{2[k_5 - G(x)]}$ join up at a and b to produce the closed curve. As this occurs for any k satisfying $G(x_0) < k \leq k_5$, the critical point A is encircled by closed trajectories corresponding to periodic solutions. Thus, A is a center.

When $G(x)$ has a strict local maximum at x_1 (above the point B), the integral curves $E(x, v) = k$ are *not* closed curves encircling the critical point $B = (x_1, 0)$. For $k = G(x_1)$, the curves touch the x-axis at x_1, but for $k > G(x_1)$, $v = \sqrt{2[k - G(x)]}$ is strictly positive and the level curves do not touch the x-axis. Finally, when $k < G(x_1)$, there is an interval about x_1 above and below in which there are *no* points of the curve $E(x, v) = k$ (the quantity under the radical in the formula for v is negative). Hence, the critical point B is a saddle point, and the level curves near B resemble hyperbolas. This is illustrated in Figure 12.21 by the level curves corresponding to k_0, k_1, and k_2 near B.

Finally, away from the extreme points of $G(x)$, the level curves may be part of a closed trajectory, such as the integral curve corresponding to $k = k_2$, or may be unbounded, such as the integral curves corresponding to $k = k_1$ or k_6.

Example 2 Use the potential plane to help sketch the phase plane diagram for the equations:

(a) $\dfrac{d^2x}{dt^2} + x - x^3 = 0$. **(b)** $\dfrac{d^2x}{dt^2} + \sin x = 0$. **(c)** $\dfrac{d^2x}{dt^2} + x - x^4 = 0$.

Solution In Example 1 we found that potential functions for these equations are **(a)** $G(x) = (1/2)x^2 - (1/4)x^4$; **(b)** $G(x) = 1 - \cos x$; **(c)** $G(x) = (1/2)x^2 - (1/5)x^5$.

(a) The local maxima and minima of $G(x)$ occur when $G'(x) = g(x) = x - x^3 = 0$; namely, at $x = 0, -1, 1$. Consequently, the phase plane diagram for equation (a) has critical points at $(0, 0)$, $(-1, 0)$, and $(1, 0)$. Since $G(x)$ has a strict local minimum at $x = 0$, the critical point $(0, 0)$ is a center. Furthermore, $G(x)$ has local maxima at

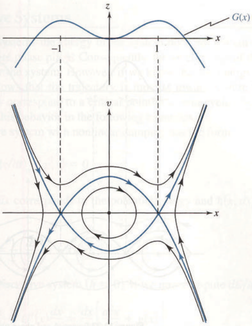

Figure 12.22 Potential and phase plane for $d^2x/dt^2 + x - x^3 = 0$

$x = \pm 1$, so $(-1, 0)$ and $(1, 0)$ are saddle points. The potential plane and the phase plane are displayed in Figure 12.22.

(b) The local maxima and minima of $G(x) = 1 - \cos x$ occur when $G'(x) = g(x) = \sin x = 0$. Here $G(x)$ has local minima for $x = 0, \pm 2\pi, \pm 4\pi, \ldots$ and local maxima for $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$. Consequently, the critical points $(2n\pi, 0)$, where n is an integer, are centers, while the critical points $((2n + 1)\pi, 0)$ are saddle points. The potential and phase planes are given in Figure 12.23 on page 798.

(c) The extreme values for $G(x) = (1/2)x^2 - (1/5)x^5$ occur when $G'(x) = g(x) = x - x^4 = 0$; that is, for $x = 0$ and 1. Here G has a minimum at $x = 0$ and a maximum at $x = 1$. Hence, the critical point $(0, 0)$ is a center, and the critical point $(1, 0)$ is a saddle point (see Figure 12.24 on page 798). ◆

You may have already observed that the three conservative systems in Example 2 have the *same* corresponding linear equation $d^2x/dt^2 + x = 0$. As we know, the origin is a stable center for this linear equation. Thus, Theorem 2 of the preceding section gives us no information about the stability of the origin for the three almost linear equations. However, using the energy method approach, we were able to determine that for these three conservative systems the origin is indeed a stable center.

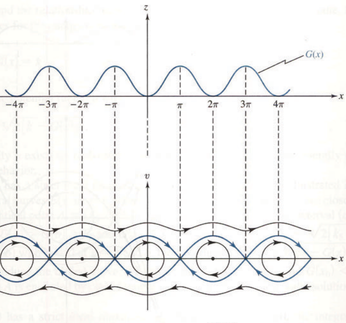

Figure 12.23 Potential and phase plane for $d^2x/dt^2 + \sin x = 0$

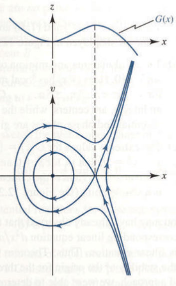

Figure 12.24 Potential and phase plane for $d^2x/dt^2 + x - x^4 = 0$

Nonconservative Systems

For nonconservative systems the energy of the system does *not* remain constant along its trajectories in the Poincaré phase plane. Consequently, the level curves of the energy function are *not* integral curves for the system. However, if we know that the energy is decreasing along a trajectory, then it follows that the trajectory is moving toward a state that has a lower total energy. This state may correspond to a critical point or a limit cycle (defined in Section 12.6). We will demonstrate this behavior in the following examples.

A nonconservative system with nonlinear damping has the form

$$\frac{d^2x}{dt^2} + h(x, dx/dt) + g(x) = 0 ,$$

where $G(x) := \int g(x)dx$ corresponds to the potential energy and $h(x, dx/dt)$ represents damping. Let

$$E(x, dx/dt) := \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + G(x) ,$$

as in the case of a conservative system ($h \equiv 0$). If we now compute dE/dt, we find

$$\frac{dE}{dt} = \frac{dx}{dt}\frac{d^2x}{dt^2} + G'(x)\frac{dx}{dt} = \frac{dx}{dt}\left[\frac{d^2x}{dt^2} + g(x)\right]$$

$$= -\frac{dx}{dt}h(x, dx/dt) ,$$

where we have used $d^2x/dt^2 + g(x) = -h(x, dx/dt)$. Consequently, the *energy E* is decreasing when $vh(x, v) > 0$ and is increasing when $vh(x, v) < 0$.

Example 3 Sketch the phase plane for the damped pendulum equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \sin x = 0 ,$$

where b is a positive constant.

Solution We first observe that $vh(x, v) = v(bv) = bv^2 > 0$ for $v \neq 0$. Hence, the energy is continually decreasing along a trajectory. The level curves for the energy function $E(x, v) = (1/2)v^2 + 1 - \cos x$ are just the integral curves for the (undamped) simple pendulum equation and are sketched in Figure 12.23. (Recall that the integral curves $E(x, v) = k$ are symmetric with respect to the x-axis and shrink to points on the x-axis as k decreases.) Hence, a trajectory must move toward the x-axis. Note that the critical points for the damped pendulum are the same as for the simple pendulum. Moreover, they are of the same type. The resulting phase plane diagram is sketched in Figure 12.25 on page 800. In this case the separatrices (colored curves) divide the plane into strips. A typical trajectory in a strip decays and spirals into an asymptotically stable spiral point (colored dots) located at multiples of 2π along the x-axis. ◆

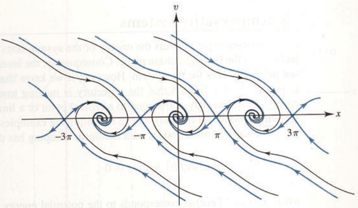

Figure 12.25 Phase plane diagram for damped pendulum

In the next example, the trajectories approach a limit cycle.

Example 4 Sketch the phase plane diagram for the equation

$$\frac{d^2x}{dt^2} + \left[x^2 + \left(\frac{dx}{dt}\right)^2 - 1\right]\frac{dx}{dt} + x = 0 .$$

Solution Here $h(x, v) = (x^2 + v^2 - 1)v$. Hence, $vh(x, v) = v^2(x^2 + v^2 - 1)$, which is zero on the unit circle $x^2 + v^2 = 1$, positive outside the unit circle, and negative inside. Consequently, trajectories beginning inside the unit circle spiral out and trajectories beginning outside the circle spiral in (see Figure 12.26). The unit circle (in color) is the trajectory for the solution $x(t) = \cos t$ and is called a **limit cycle.** ◆

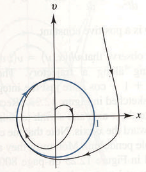

Figure 12.26 Phase plane diagram for $d^2x/dt^2 + [x^2 + (dx/dt)^2 - 1]dx/dt + x = 0$

EXERCISES 12.4

In Problems 1–6, find the potential energy function $G(x)$ and the energy function $E(x, v)$ for the given equations. Select E so that $E(0, 0) = 0$.

1. $\dfrac{d^2x}{dt^2} + x^2 - 3x + 1 = 0$

2. $\dfrac{d^2x}{dt^2} + \cos x = 0$

3. $\dfrac{d^2x}{dt^2} + \dfrac{x^2}{x-1} = 0$

4. $\dfrac{d^2x}{dt^2} + x - \dfrac{x^3}{6} + \dfrac{x^5}{120} = 0$

5. $\dfrac{d^2x}{dt^2} + x^3 + x^2 - x = 0$

6. $\dfrac{d^2x}{dt^2} + e^x - 1 = 0$

In Problems 7–12, use the potential plane to help sketch the phase plane diagrams for the given equations.

7. $\dfrac{d^2x}{dt^2} + x^3 - x = 0$

8. $\dfrac{d^2x}{dt^2} + 9x = 0$

9. $\dfrac{d^2x}{dt^2} + 2x^2 + x - 1 = 0$

10. $\dfrac{d^2x}{dt^2} - \sin x = 0$

11. $\dfrac{d^2x}{dt^2} + \dfrac{x}{x-2} = 0$

12. $\dfrac{d^2x}{dt^2} + (x-1)^3 = 0$

In Problems 13–16, use the energy function to assist in sketching the phase plane diagrams for the given nonconservative systems.

13. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x - x^3 = 0$

14. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x - x^4 = 0$

15. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + \dfrac{x}{x-2} = 0$

16. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + 2x^2 + x - 1 = 0$

17. **Nonlinear Spring.** The general nonlinear spring equation

$$\frac{d^2x}{dt^2} + \alpha x + \beta x^3 = 0 ,$$

where $\alpha > 0$ and β are parameters, is used to model a variety of physical phenomena. Holding $\alpha > 0$ fixed, sketch the potential function and the phase plane diagram for $\beta > 0$ and also for $\beta < 0$. Describe how the behavior of solutions to the equation differs in these cases.

18. **Cusps.** We have observed that where the potential energy function $G(x)$ for a conservative system has a strict maximum (minimum), the corresponding critical point is a saddle point (center). At a critical point $(x_0, 0)$ for which $G(x)$ also has a point of inflection (e.g., $G'(x_0) = g(x_0) = 0$, $G''(x_0) = g'(x_0) = 0$, and $G'''(x_0) = g''(x_0) \neq 0$), the curve in the phase plane has a **cusp.** Demonstrate this by sketching the potential plane and the phase plane diagrams for the equation

$$\frac{d^2x}{dt^2} + x^2 = 0 .$$

19. **General Relativity.** In studying the relativistic motion of a particle moving in the gravitational field of a larger body, one encounters the equation

$$\frac{d^2u}{d\theta^2} + u - \lambda u^2 - 1 = 0 ,$$

where u is inversely proportional to the distance of the particle from the body, θ is an angle in the plane of motion, and λ is a parameter with $0 < \lambda < 1$.

(a) Sketch the phase plane diagram for $0 < \lambda < 1/4$.

(b) Sketch the phase plane diagram for $1/4 < \lambda < 1$.

(c) What observations can be made about the motion of the particle in these two cases?