MA 225 S. Schecter

Test 3 Review Answers

15

① $\forall x \in \mathbb{R}$ $\sin x = \sin x$ so xRx: reflexive.

$\forall x, y \in \mathbb{R}$, if xRy, then $\sin x = \sin y$, so $\sin y = \sin x$, so yRx: symmetric.

$\forall x, y, z \in \mathbb{R}$, if xRy and yRz, then $\sin x = \sin y$ and $\sin y = \sin z$, so $\sin x = \sin z$, so xRz: transitive.

15

② Let A be an open interval. Then A is not disjoint from itself, so it is not true that ARA. Thus R is not reflexive.

Let A and B be open intervals. If ARB, then A and B are disjoint, so B and A are disjoint, so BRA. Thus R is symmetric.

It can happen that ARB and BRC but it is not true that ARC. Example: $A = (0,1)$ $B = (3,4)$ $C = (0,2)$. Thus R is not transitive.

10

③ $\{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

10

④ $0/\equiv_2 \; = \; 2/\equiv_2 \; = \; \{\dots, -4, -2, 0, 2, 4, \dots\}$.

However, $f(0/\equiv_2) = 0/\equiv_4 = \{\dots, -8, -4, 0, 4, 8, \dots\}$

and $f(2/\equiv_2) = 2/\equiv_4 = \{\dots, -10, -6, -2, 2, 6, 10, \dots\}$

Since $0/\equiv_4$ and $2/\equiv_4$ are <u>different</u> equivalence classes, f <u>does</u> <u>not</u> define a function.

10 ⑤ $y = \frac{x+2}{x-2} = xy - 2y = x + 2 \iff xy - x = 2y + 2$

$\iff x(y-1) = 2y+2 \iff x = \frac{2y+2}{y-1}$

Thus $Rng(f) = \{y \in \mathbb{R} : y \neq 1\}$.

24 ⑥ f is one-to-one:

Let $x_1, x_2 \in \mathbb{R}$. Suppose $f(x_1) = f(x_2)$.

Case 1. $x_1 \geq 0$ and $x_2 \geq 0$. Then $x_1^2 = x_2^2$. Since $x_1 \geq 0$ and $x_2 \geq 0$, taking the square root yields $x_1 = x_2$.

Case 2. $x_1 < 0$ and $x_2 < 0$. Then $4x_1 = 4x_2$, so $x_1 = x_2$.

Case 3. $x_1 \geq 0$ and $x_2 < 0$. Then $x_1^2 = 4x_2$. Since $x_1^2 \geq 0$ and $4x_2 < 0$, this case cannot occur.

f is onto:

Let $y \in \mathbb{R}$.

Case 1. Let $y \geq 0$. Let $x = \sqrt{y} \geq 0$. Then $f(x) = x^2 = (\sqrt{y})^2 = y$.

Case 2. Let $y < 0$. Let $x = \frac{y}{4} < 0$. Then $f(x) = 4x = 4\left(\frac{y}{4}\right) = y$.

$$f^{-1}(y) = \begin{cases} \sqrt{y} & y \geq 0 \\ \frac{y}{4} & y < 0 \end{cases}$$

16 ⑦ a) and b) Domain $= \mathbb{N} \times \mathbb{N}$, Codomain $= \mathbb{N}$

c) $\pi_1 \circ f$ and $g \circ \pi_1$ have the same domain, namely $\mathbb{N} \times \mathbb{N}$. Let $(x,y) \in \mathbb{N} \times \mathbb{N}$. Then $\pi_1 \circ f(x,y) = \pi_1(f(x,y)) = \pi_1(x^2, y) = x^2$ and $g \circ \pi_1(x,y) = g(\pi_1(x,y)) = g(x) = x^2$.

(8) a) Let $z \in C$. Since f is onto, $\exists\, x \in A$ such that $f(x) = z$. Let y be any element of B. Then $(x,y) \in A \times B$, and $f \circ \pi_1 (x,y) = f(\pi_1(x,y)) = f(x) = z$. Therefore $f \circ \pi_1$ is onto.

b) $f \circ \pi_1$ is one-to-one if B has only one element.