MA 225-002 Test 3 Definitions

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$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

A *relation* from A to B is a subset R of $A \times B$. We say that *a is R-related to b*, and write $a\,R\,b$, if $(a, b) \in R$.

A relation from A to A is usually just called a relation *on* A.

Let A be a set and let R be a relation on A.

1. R is *reflexive* if for all $x \in A$, $x\,R\,x$.
2. R is *symmetric* if for all x and y in A, if $x\,R\,y$, then $y\,R\,x$.
3. R is *transitive* if for all x, y and z in A, if $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$.

A relation R on a set A is an *equivalence relation* if R is reflexive, symmetric and transitive.

Let R be an equivalence relation on A and let $x \in A$. The *equivalence class of* x is

$$[x] = \{y \in A : x\,R\,y\}.$$

$[x]$ is also written x/R.

Let m be a positive integer. The relation $\equiv_m$ on $\mathbb{Z}$ is defined by

$$x \equiv_m y \text{ if } m \text{ divides } x - y.$$

The set of equivalence classes for $\equiv_m$ is denoted $\mathbb{Z}_m$, which is pronounced "$\mathbb{Z}$ mod m." It has m elements: $[0]$, $[1]$, ..., $[m-1]$.

A *function* f *from* A *to* B is a relation from A to B that satisfies

1. For each $x \in A$ there exists $y \in B$ such that $x\,f\,y$.
2. If $x\,f\,y$ and $x\,f\,z$ then $y = z$.

A function f from A to B is usually denoted $f : A \to B$. We always write $f(x) = y$ to mean $x\,f\,y$. A is called the *domain* of f, and B is called the *codomain* of f. Condition (2) says that functions are well-defined: There is no ambiguity in which element of B is $f(x)$. Condition (1) says that $f(x)$ is defined for every $x \in A$.

The *range of* f is $\{y \in B : \text{ there exists } x \in A \text{ such that } f(x) = y\}$.

Common functions:

1. The *identity function on* A: $I_A : A \to A$, $I_A(x) = x$.
2. For $A \subseteq B$, the *inclusion map from* A *to* B: $i : A \to B$, $i(x) = x$.
3. The *projections* of $A \times B$ onto A and B: $\Pi_1 : A \times B \to A$, $\Pi_1(x, y) = x$, and $\Pi_2 : A \times B \to B$, $\Pi_2(x, y) = y$.

Two functions f and g are *equal* if

1. They have the same domain A.
2. For all $x \in A$, $f(x) = g(x)$.

Our text does not require that the codomains be the same.

If $f : A \to B$ and $D \subseteq A$, the *restriction of* f *to* D is denoted $f|_D$ and is defined as follows: $f|_D : D \to B$, $f|_D(x) = f(x)$.

Let $f : A \to B$ and $g : B \to C$ be functions. $g \circ f : A \to C$ is defined by $(g \circ f)(x) = g(f(x))$.

A function $f : A \to B$ is called *surjective* or *onto* if the range of f is B. In other words, f is onto if for each $y \in B$ there exists $x \in A$ such that $f(x) = y$.

A function $f : A \to B$ is called *injective* or *one-to-one* if for all x_1, $x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. Equivalently, f is one-to-one if for all x_1, $x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

If $f : A \to B$ is called a *bijection* if it is one-to-one and onto. In this case one can define its inverse function $f^{-1} : B \to A$ as follows: $f^{-1}(y) = x$ if and only if $y = f(x)$.