MA 225

S. Schecter

Test 2 Review Questions

Answers

2.2 7F Assume $A \subseteq B$. Let $x \in A \cap C$. Then $x \in A$ and $x \in C$. Since $x \in A$, $x \in B$. Thus $x \in B$ and $x \in C$, so $x \in B \cap C$.

2.2 19a $A \subseteq B \Leftrightarrow \forall x\,(x \in A \Rightarrow x \in B) \Leftrightarrow \forall x \sim (x \in A \wedge x \notin B)$
$\Leftrightarrow \sim \exists x\,(x \in A \wedge x \notin B) \Leftrightarrow \sim \exists x\,(x \in A - B) \Leftrightarrow A - B = \emptyset$.

19f Assume $A \subseteq C$ and $B \subseteq C$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$.

Case 1. $x \in A$. Since $A \subseteq C$, $x \in C$.

Case 2. $x \in B$. Since $B \subseteq C$, $x \in C$.

Thus $x \in C$ in either case, so $A \cup B \subseteq C$.

2.3 1c In text.

In $\bigcup \{D_n : n \in \mathbb{N}\} = (-\infty, 1)$ $\quad \bigcap \{D_n : n \in \mathbb{N}\} = (-1, 0]$

6a In text.

10a In text.

Induction problems

a $P(1)$ is $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$: True.

Assume $P(n)$.

Then $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (n+1)(n+2) =$

$$\frac{n(n+1)(n+2)}{3} + (n+1)(n+2) = \frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3} =$$

$$\frac{(n+3)(n+1)(n+2)}{3} = \frac{(n+1)(n+2)(n+3)}{3}.$$

Thus $P(n+1)$ is true.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$.

b $P(1)$ is $\frac{1}{1\cdot 2} = \frac{1}{1+1}$. It's true.

Assume $P(n)$.

Then $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$

$$= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

Thus $P(n+1)$ is true.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$.

c $P(1)$ is the statement 1^3-1 is divisible by 3. It's true.

Assume $P(n)$.

Then $(n+1)^3 - (n+1) = n^3+3n^2+3n+1 - (n+1) =$
$(n^3-n) + (3n^2+3n)$. Since n^3-n is divisible by 3 by assumption, and $3n^2+3n$ is clearly divisible by 3, we see that $P(n+1)$ is true.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$.

d $P(1)$ is the statement 1^3-1 is divisible by 6. It's true.

Assume $P(n)$.

Then as in part c, $(n+1)^3-(n+1) = (n^3-3) + 3(n^2+n)$.
n^3-3 is divisible by 6 by assumption. Since n^2+n is always even, it is divisible by 2, so $3(n^2+n)$ is also

divisible by 6. Thus $P(n+1)$ is true.
By induction, $P(n)$ is true $\forall\, n \in \mathbb{N}$.

e $P(1)$ is the statement $1! < (1+1)!$. It's true.
Assume $P(n)$.
Then $1! + 2! + \cdots + n! + (n+1)! < (n+1)! + (n+1)!$
$= 2(n+1)! < (n+2)(n+1)! = (n+2)!$
Thus $P(n+1)$ is true.
By induction, $P(n)$ is true $\forall n \in \mathbb{N}$.

f $P(1)$ is the statement $1^1 > 1!$ It's false.
$P(2)$ " " " $2^2 > 2!$ It's true.
Assume $P(n)$ is true for some $n \geq 2$.
Then $(n+1)^{(n+1)} = (n+1)(n+1)^n > (n+1)n^n > (n+1)n! = (n+1)!$
Thus $P(n+1)$ is true.
By induction, $P(n)$ is true $\forall n \geq 2$.