MA 225-001 S. Schecter

Test 3 Answers

① a) Let $(x,y) \in \mathbb{R}\times\mathbb{R}$. Then $xy = xy$. Therefore $(x,y) R (x,y)$

Let $(x,y), (u,v) \in \mathbb{R}\times\mathbb{R}$. Assume $(x,y) R (u,v)$. Then $xy = uv$, so $uv = xy$, so $(u,v) R (x,y)$.

Let $(x,y), (u,v), (w,z) \in \mathbb{R}\times\mathbb{R}$. Assume $(x,y) R (u,v)$ and $(u,v) R (w,z)$. Then $xy = uv$ and $uv = wz$, so $xy = wz$, so $(x,y) R (w,z)$.

b) Any ordered pair of real numbers (u,v) such that $uv = 4$ is in the equivalence class of $(1,4)$. Examples: $(2,2)$, $(4,1)$, $(\frac{1}{2}, 8)$, $(-1,-4)$.

② $0/\equiv_2 \;=\; 2/\equiv_2$ but

$$f(0/\equiv_2) = 2\cdot 0/\equiv_3 = 0/\equiv_3 = \{\ldots, -3, 0, 3, \ldots\}$$

$$f(2/\equiv_2) = 2\cdot 2/\equiv_3 = 4/\equiv_3 = \{\ldots, -2, 1, 4, \ldots\}$$

Since $0/\equiv_3 \neq 4/\equiv_3$, this is not a function.

③ a) Let x_1, x_2 be in the same equiv. class in $\mathbb{Z}_8$. Then $x_1 \equiv_8 x_2$, so $\exists k \in \mathbb{Z}$ s.t. $x_1 - x_2 = 8k = 2\cdot 4k$, so $x_1 \equiv_4 x_2$ So x_1 and x_2 are in the some equivalence class in $\mathbb{Z}_4$. This shows that f is a function: each equivalence class in $\mathbb{Z}_8$ goes to just one equivalence class in $\mathbb{Z}_4$.

b) f is not one-to-one. For example,

$$f(0/\equiv_8) = 0/\equiv_4$$

$$f(4/\equiv_8) = 4/\equiv_4 = 0/\equiv_4.$$

However, $0/\equiv_8 \neq 4/\equiv_8$.

Another argument due to T.J. Brannon: Since $\mathbb{Z}_8$ has more elements than $\mathbb{Z}_4$, no function from $\mathbb{Z}_8$ to $\mathbb{Z}_4$ can be one-to-one.

(4) One-to-one: Let $x_1, x_2 \in (0,\infty)$. Assume $f(x_1) = f(x_2)$.

Case 1. $0 < x_1 < 4$ and $0 < x_2 < 4$. Then $\frac{4}{x_1} = \frac{4}{x_2}$ so $x_1 = x_2$.

Case 2. $x_1 \geq 4$ and $x_2 \geq 4$. Then $5 - x_1 = 5 - x_2$ so $x_1 = x_2$.

Case 3. $0 < x_1 < 4$ and $x_2 \geq 4$. Then $f(x_1) = \frac{4}{x_1} > 1$ and $f(x_2) = 5 - x_2 \leq 1$, so $f(x_1) \neq f(x_2)$. This case cannot occur.

Thus if $f(x_1) = f(x_2)$ then $x_1 = x_2$, so f is one-to-one.

Onto. Let $y \in \mathbb{R}$.

Case 1. $y > 1$. Let $x = \frac{4}{y}$. Then $0 < x < 4$,

so $f(x) = \frac{4}{x} = \frac{4}{\frac{4}{y}} = y$.

Case 2. $y \leq 1$. Let $x = 5 - y$. Then $x \geq 4$,

so $f(x) = 5 - x = 5 - (5-y) = y$.

Thus f is onto.

⑤ Notice that $g \circ f : \mathbb{R} \to [0,\infty)$. Let $z \in [0,\infty)$. Then $\sqrt{z} \in [0,\infty)$. Since $[0,\infty) \subseteq Rng(f)$, $\exists x \in \mathbb{R}$ such that $f(x) = \sqrt{z}$. Then $(g \circ f)(x) = g(f(x)) = g(\sqrt{z}) = (\sqrt{z})^2 = z$. Thus $g \circ f$ is onto.

Points

① a) 20 b) 4 ② 14 ③ a) 14 b) 8

④ a) 14 b) 12 ⑤ 14