MA 225-002 Answers to Final Exam of May 9, 2005

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1. (a) $\exists x \forall y \sim (x \text{ loves } y)$

   (b) $\forall x (x \text{ is happy}) \vee \sim \exists x (x \text{ is happy})$

2. (a) Let a, b and c be integers such that a divides b and a^2 divides c. Then there exists an integer k such that $b = ak$, and there exists an integer l such that $c = a^2 l$. Hence $c - ab = a^2 l - a \cdot ak = a^2(l - k)$. Since $l - k$ is an integer, a^2 divides $c - ab$.

   (b) For a contrapositive proof, assume a is even. We must show that $2a^2 + 7a + 3$ is odd. Since a is even, there exists an integer k such that $a = 2k$. Hence $2a^2 + 7a + 3 = 2(2k)^2 + 7(2k) + 3 = 2(4k^2 + 7k + 1) + 1$. Since $4k^2 + 7k + 1$ is an integer, $2a^2 + 7a + 3$ is odd.

   (c) For a proof by contradiction, assume x and y are real numbers such that x is rational, y is irrational, and $\frac{1}{2}(x + y)$ is rational. Since x and $\frac{1}{2}(x + y)$ are rational, there are integers k, l, m, and n, with $l \neq 0$ and $n \neq 0$, such that $x = \frac{k}{l}$ and $\frac{1}{2}(x + y) = \frac{m}{n}$. Therefore $y = 2 \cdot \frac{1}{2}(x + y) - x = 2 \cdot \frac{m}{n} - \frac{k}{l} = \frac{2ml - kn}{nl}$. Since $2ml - kn$ and nl are integers and $nl \neq 0$, y is rational. This contradicts the assumption that y is irrational.

3. Assume $A \subseteq \tilde{B}$ and $C \subseteq D$. Let $x \in A - D$. Then $x \in A$ and $x \notin D$. Since $x \in A$ and $A \subseteq \tilde{B}$, $x \in \tilde{B}$. Now $x \notin D$ implies $x \in \tilde{D}$, and $C \subseteq D$ implies $\tilde{D} \subseteq \tilde{C}$. Therefore $x \in \tilde{C}$. Since $x \in \tilde{B}$ and $x \in \tilde{C}$, $x \in \tilde{B} \cap \tilde{C}$. Therefore $A - D \subseteq \tilde{B} \cap \tilde{C}$.

4. Assume $A \subseteq C$ and $B \cap \tilde{A} \subseteq C$. Let $x \in B$. Then $x \in B \cap A$ or $x \in B \cap \tilde{A}$. If $x \in B \cap A$, then $x \in A$; since $A \subseteq C$, $x \in C$. If $x \in B \cap \tilde{A}$, then, since $B \cap \tilde{A} \subseteq C$, $x \in C$. Thus $x \in C$ in both cases; hence $B \subseteq C$.

5. Prove by induction that the following statement $P(n)$ is true for every natural number n:

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}.$$

   $P(1)$ is the statement $\frac{1}{1 \cdot 4} = \frac{1}{3 \cdot 1 + 1}$. It is true.

   Assume $P(n)$ is true. Then

$$\begin{aligned}
&\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{(3n-2)(3n+1)} + \frac{1}{(3(n+1)-2)(3(n+1)+1)} \\
&= \frac{n}{3n+1} + \frac{1}{(3(n+1)-2)(3(n+1)+1)} = \frac{n}{3n+1} + \frac{1}{(3n+1)(3n+4)} \\
&= \frac{n(3n+4)+1}{(3n+1)(3n+4)} = \frac{3n^2+4n+1}{(3n+1)(3n+4)} = \frac{(3n+1)(n+1)}{(3n+1)(3n+4)} = \frac{n+1}{3n+4} \\
&= \frac{n+1}{3(n+1)+1}.
\end{aligned}$$

   Thus $P(n+1)$ is true.

6. Define a relation on $\mathbb{R} \times \mathbb{R}$ by $(x, y)\, R\, (a, b)$ if $x \leq a$ and $y \geq b$.

   (a) Since $x \leq x$ and $y \geq y$, $(x, y)\, R\, (x, y)$. Therefore R is reflexive. To show R is transitive, assume $(x, y)\, R\, (a, b)$ and $(a, b)\, R\, (u, v)$. Then $x \leq a$ and $a \leq u$, and $y \geq b$ and $b \geq v$. Therefore $x \leq u$, and $y \geq v$. Hence $(x, y)\, R\, (u, v)$. Thus R is transitive.

   (b) Show by example that R is not symmetric: $(1,4)R(2,3)$ is true, but $(2,3)R(1,4)$ is not true.

7. Define $f : \mathbb{Z}_4 \to \mathbb{Z}_6$ by $f(x/\equiv_4) = 3x/\equiv_6$.

   (a) Show that f is a function: if $x_1 \equiv_4 x_2$, then there is an integer k such that $x_1 - x_2 = 4k$. Then $3x_1 - 3x_2 = 12k = 6 \cdot 2k$. Since $2k$ is an integer, $3x_1 \equiv_6 3x_2$, so f is a function.

(b) Is f one-to-one? No: $f(0/\equiv_4) = 0/\equiv_6$ and $f(2/\equiv_4) = 6/\equiv_6 = 0/\equiv_6$. However, $0/\equiv_4 \neq 2/\equiv_4$.

(c) Is f onto? No. $f(0/\equiv_4) = 0/\equiv_6$, $f(1/\equiv_4) = 3/\equiv_6$, $f(2/\equiv_4) = 6/\equiv_6 = 0/\equiv_6$, $f(3/\equiv_4) = 9/\equiv_6 = 3/\equiv_6$. Hence the range of f includes just two of the six elements of $\mathbb{Z}_6$. (Or: f is not onto because $\mathbb{Z}_4$ has just four elements, so the range of f includes at most four of the six elements of $\mathbb{Z}_6$.)

8. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 0, \\ x^2 + 1 & \text{if } x \geq 0\ . \end{cases}$$

(a) f is one-to-one: Let x_1 and x_2 belong to $\mathbb{R}$, and let $f(x_1) = f(x_2)$.
Case 1: $x_1 < 0$ and $x_2 < 0$. Then $2x_1 + 1 = 2x_2 + 1$. Algebra yields $x_1 = x_2$.
Case 2: $x_1 \geq 0$ and $x_2 \geq 0$. Then $x_1^2 + 1 = x_2^2 + 1$. Therefore $x_1^2 = x_2^2$. Since $x_1 \geq 0$ and $x_2 \geq 0$, $x_1 = x_2$.
Case 3: $x_1 < 0$ and $x_2 \geq 0$. Then $f(x_1) = 2x_1 + 1 < 1$ and $f(x_2) = x_2^2 + 1 \geq 1$. This case cannot occur.

(b) f is onto: Let $y \in \mathbb{R}$.
Case 1: $y < 1$. Let $x = \frac{1}{2}(y-1)$. Then $x < 0$. Therefore $f(x) = 2x + 1 = 2 \cdot \frac{1}{2}(y-1) + 1 = y$.
Case 2: Let $y \geq 1$. Let $x = \sqrt{y-1}$. This makes sense because $y - 1 \geq 0$. Then $x \geq 0$. Therefore $f(x) = x^2 + 1 = (\sqrt{y-1})^2 + 1 = y$.

(c) Inverse function: from part (b),

$$f^{-1}(y) = \begin{cases} \frac{1}{2}(y-1) & \text{if } y < 1, \\ \sqrt{y-1} & \text{if } y \geq 1. \end{cases}$$

9. Let $S = \{x \in \mathbb{Z} : x \leq 0\} = \{\ldots, -3, -2, -1, 0\}$. Define $f : \mathbb{N} \to S$ by $f(n) = 1 - n$. (Clearly f is a function from $\mathbb{N}$ into $\mathbb{Z}$. Since $n \in \mathbb{N} \Rightarrow n \geq 1 \Rightarrow -n \leq -1 \Rightarrow 1 - n \leq 0$, we see that an acceptable codomain for f is indeed S.)

f is one-to-one: $f(n_1) = f(n_2) \Rightarrow 1 - n_1 = 1 - n_2 \Rightarrow n_1 = n_2$.

f is onto: Let $x \in S$. Let $n = 1 - x$. Since $x \in \mathbb{Z}$, $n \in \mathbb{Z}$. Also, $x \leq 0 \Rightarrow -x \geq 0 \Rightarrow 1 - x \geq 1$. Therefore $n \in \mathbb{N}$. We have $f(n) = 1 - n = 1 - (1 - x) = x$. Therefore f is onto.

10. Let $f : A \to B$ and $g : B \to C$ be functions. Assume that g is one-to-one and $g \circ f$ is onto. Prove that f is onto.

    Let $y \in B$. Let $z = g(y) \in C$. Since $g \circ f$ is onto, there exists $x \in A$ such that $(g \circ f)(x) = g(f(x)) = z$. Thus $g(y) = z$ and $g(f(x)) = z$. Since g is one-to-one, $y = f(x)$. Therefore f is onto.