MA 225-001 Final Exam

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1. For each of the following sentences, (i) translate into a symbolic sentence that uses the symbols $\wedge$, $\vee$, $\sim$, $\forall$ and $\exists$ wherever appropriate; (ii) write a clear denial as a symbolic sentence; and (iii) translate your denial into ordinary English. The universe for each statement is the set of all people.

   (a) There is a person who does not love anybody.

   (b) Everybody is happy or nobody is happy.

2. Definitions:

   (a) Let a and b be integers. We say a *divides* b if there is an integer k such that $ak = b$.

   (b) An integer a is *even* if there is an integer k such that $a = 2k$.

   (c) An integer a is *odd* if there is an integer k such that $a = 2k + 1$.

   (d) A real number x is *rational* if there exist integers a and b, with $b \neq 0$, such that $x = \frac{a}{b}$.

   (e) A real number x is *irrational* if it is not rational.

   Using these definitions, prove the following statements.

   (a) If a, b and c are integers such that a divides b and a^2 divides c, then a^2 divides $c - ab$.

   (b) If $2a^2 + 7a + 3$ is an even integer, then a is odd. (Give a contrapositive proof. You may assume that every integer is either even or odd.)

   (c) If x and y are real numbers such that x is rational and y is irrational, then $\frac{1}{2}(x+y)$ is irrational. (Give a proof by contradiction.)

3. Let A, B, C and D be sets. Prove that if $A \subseteq \tilde{B}$ and $C \subseteq D$, then $A - D \subseteq \tilde{B} \cap \tilde{C}$.

4. Let A, B and C be sets. Prove that if $A \subseteq C$ and $B \cap \tilde{A} \subseteq C$ then $B \subseteq C$. (Hint: any point is either in A or not in A.)

5. Prove by induction: For every natural number n,

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}.$$

6. Define a relation on $\mathbb{R}\times\mathbb{R}$ by $(x,y)\,R\,(a,b)$ if $x\le a$ and $y\ge b$.

   (a) Show that R is reflexive and transitive

   (b) Show by example that R is not symmetric.

7. Define $f:\mathbb{Z}_4\to\mathbb{Z}_6$ by $f(x/\equiv_4)=3x/\equiv_6$.

   (a) Show that f is a function.

   (b) Is f one-to-one? Justify your answer.

   (c) Is f onto? Justify your answer.

8. Define $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases}2x+1 & \text{if } x<0,\\ x^2+1 & \text{if } x\ge 0\ .\end{cases}$$

   (a) Prove that f is one-to-one.

   (b) Prove that f is onto.

   (c) Give the inverse function.

9. Let $S=\{x\in\mathbb{Z}: x\le 0\}=\{\ldots,-3,-2,-1,0\}$. Prove that S is denumerable in the following steps:

   (a) Define a function $f:\mathbb{N}\to S$ that is a bijection.

   (b) Prove that the function you have defined is a bijection, i.e., prove that it is one-to-one and onto.

10. Let $f:A\to B$ and $g:B\to C$ be functions. Assume that g is one-to-one and $g\circ f$ is onto. Prove that f is onto.