The entry-exit function and geometric singular perturbation theory

Steve Schecter[1]

[1]Department of Mathematics
North Carolina State University
Raleigh, North Carolina, USA

Miami University September 24 2016

Collaborators: Peter De Maesschalck, Anna Ghazaryan, Vahagn Manukian

Thanks to NSF

The model

$$
\begin{aligned}
u_t &= D_u u_{xx} + u(1-u) - \frac{u}{\alpha + u} v, \\
v_t &= D_v v_{xx} + \delta v \left(1 - \frac{\beta}{u} v\right).
\end{aligned}
$$

- $u=$ prey, $v=$ predator.
- Both populations are subject to overcrowding.
- Predator carrying capacity is proportional to prey population.
- For fixed predator population, prey is consumed at a rate that stabilizes as its population increases.

Traveling waves

Ghazaryan, Manukian, S., Proc. Roy. Soc. London Ser. A 471 (2015).

- Look for traveling waves with velocity $c > 0$, set $z = x - ct$.
- Rescale space so $c = 1$.
- Set $\epsilon = \frac{D_u}{c^2}$, $\mu = \frac{D_v}{D_u}$. (Small $\epsilon > 0$ means small diffusion.)

Traveling waves $(u, v)(z)$ satisfy:

$$
\begin{aligned}
0 &= \epsilon u_{zz} + u_z + u(1-u) - \frac{u}{\alpha + u} v, \\
0 &= \epsilon \mu \, v_{zz} + v_z + \delta v \left(1 - \frac{\beta}{u} v\right).
\end{aligned}
$$

Rewrite as a first-order system.

A slow-fast system in slow form

$$
\begin{aligned}
u_z &= U, \\
\epsilon U_z &= -U - u(1-u) + \frac{u}{\alpha + u} v, \\
v_z &= V, \\
\epsilon \mu V_z &= -V - \delta v \left(1 - \frac{\beta}{u} v\right).
\end{aligned}
$$

Normally attracting critical manifold (set $\epsilon = 0$):

$$
U = -u(1-u) + \frac{u}{\alpha + u} v, \quad V = -\delta v \left(1 - \frac{\beta}{u} v\right).
$$

Slow system:

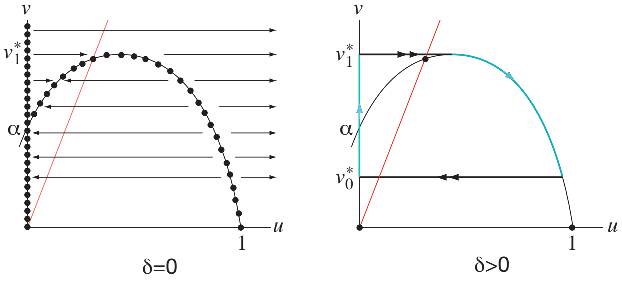
$$
u_z = \frac{u}{\alpha + u} v - u(1-u), \quad v_z = \delta v \left(\frac{\beta}{u} v - 1\right).
$$

Undefined for $u = 0$. Multiply by $(\alpha + u)u$:

Another slow-fast system, in fast form

$$\begin{aligned}\dot{u} &= u^2\big(v-(1-u)(\alpha+u)\big),\\ \dot{v} &= \delta v(\alpha+u)\,(\beta v-u)\,.\end{aligned}$$

Small $\delta > 0$ means slowly changing predator population. $u = 0$ is now invariant. **Note the factor u^2.**



For small $\delta > 0$, numerical simulation shows a closed orbit near a "singular orbit" with a certain value of v_0^*.

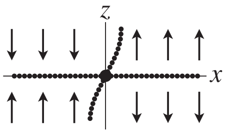
Classical analog

$$\dot{x} = \epsilon f(x,z),$$
$$\dot{z} = g(x,z)z,$$

with $x \in \mathbb{R}$, $z \in \mathbb{R}$,

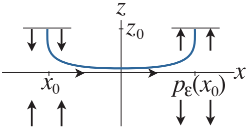
$$f(x,0) > 0, \quad g(x,0) \text{ has the sign of } x.$$

- **Note the factor z.**
- For $\epsilon = 0$, the x-axis consists of equilibria.
- Normally attracting for $x < 0$, normally repelling for $x > 0$. **Loss of normal hyperbolicity at $z = 0$.**



- For $\epsilon > 0$, x-axis remains invariant, flow is to the right.

Entry-exit function: attraction and repulsion balance



For small $\epsilon > 0$, a solution that starts at (x_0, z_0), with x_0 negative and $z_0 > 0$ small, reintersects the line $z = z_0$ at $(p_\epsilon(x_0), z_0)$.

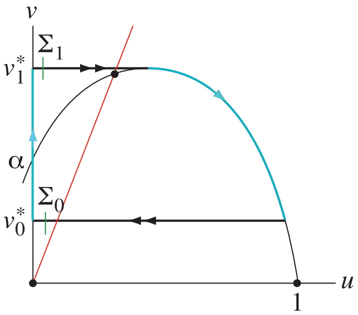
Theorem

As $\epsilon \to 0$, $p_\epsilon(x_0) \to p_0(x_0)$ given implicitly by

$$\int_{x_0}^{p_0(x_0)} \frac{g(x,0)}{f(x,0)}\, dx = 0.$$

The solution leaves the x-axis when repulsion has built up to balance the attraction that occurred before $x = 0$.

If the theorem holds when z is replaced by z^2, how to prove existence of the closed orbit



- Define v_0^* by $\int_{v_0^*}^{v_1^*} \frac{v-\alpha}{\alpha\beta v^2}\, dv = 0$.
- Follow the flow backwards for small $\delta > 0$.
- $p_\delta : \Sigma_1 \to \Sigma_0$ (entry-exit function) would be smooth.
- $q_\delta : \Sigma_0 \to \Sigma_1$ is an exponential contraction.
- $q_\delta \circ p_\delta : \Sigma_1 \to \Sigma_1$ has a fixed point.

Reformulation

$$\dot{x} = \epsilon f(x, z),$$
$$\dot{z} = g(x, z)z.$$

$$f(x, 0) > 0, \quad g(x, 0) \text{ has the sign of } x.$$

Divide by $f(x, z) > 0$, let $h = g/f$:

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x, z)z.$$

$$h(x, 0) \text{ has the sign of } x.$$

Theorem

As $\epsilon \to 0$, $p_\epsilon(x_0) \to p_0(x_0)$ given implicitly by

$$\int_{x_0}^{p_0(x_0)} h(x, 0)\, dx = 0.$$

"Standard" proofs that $p_\epsilon(x_0) \to p_0(x_0)$

- Asymptotic expansions: Haberman, SIAM J. Appl. Math. 37 (1979), 69–106; Mishchenko, Kolesov, Kolesov, and Rozov, *Asymptotic methods in singularly perturbed systems*, 1994.
- Comparison to solutions constructed by separation of variables: S., J. Diff. Eq. 60 (1985), 131–141.
- Direct estimation of the solution and its derivatives using the variational equation: De Maesschalck, J. Diff. Eq. 244 (2008), 1448–1466.

De Maesschalck: Let $p(x_0, \epsilon) = p_\epsilon(x_0)$, $\epsilon \geq 0$. If f and g are C^r, $r \geq 1$, then p is C^r.

How to prove $p_\epsilon(x_0) \to p_0(x_0)$ in the C^0 sense

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x,z)z.$$

$h(x,0)$ has the sign of x.

Replace by

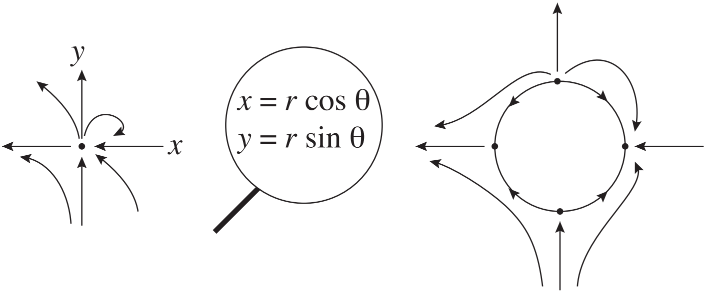
$$\dot{x} = \epsilon,$$
$$\dot{z} = (h(x,0) \pm \alpha)z.$$

$$\frac{dz}{dx} = \frac{(h(x,0) \pm \alpha)z}{\epsilon} \quad \Rightarrow \quad \frac{\epsilon}{z}dz = (h(x,0) \pm \alpha)dx.$$

If a solution that starts on the line $z = z_0$ at $x = x_0$ reintersects it when $x = x_1$, then

$$0 = \int_{x_0}^{x_1} (h(x,0) \pm \alpha)\, dx = 0.$$

- De Maesschalck's proof doesn't seem to work for z replaced by z^2.
- Proofs don't use the blow-up method of Geometric Singular Perturbation Theory, today the usual approach to loss of normal hyperbolicity in slow-fast systems.



De Maesschalck and S., J. Diff. Eq. 260 (2016), 6697–6715

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x, z)z^2.$$

$h(x, 0)$ has the sign of x.

Using blow-up, we prove:

Theorem

If h is C^∞, then:

1. *There is a C^∞ function $\tilde{p}$ of three variables such that $p(x_0, \epsilon) = \tilde{p}(x_0, \epsilon, \epsilon \log \epsilon)$.*
2. *If $h(x, z) - h(x, 0)$ is C^∞ flat in z, then p is a C^∞ function of (x_0, ϵ).*

A change of variables and the classical situation

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x,z)z.$$

$$z = \kappa(w) = \begin{cases} e^{-\frac{1}{w}} & \text{if } w > 0, \\ 0 & \text{if } w = 0. \end{cases}$$

$$\dot{x} = \epsilon,$$
$$\dot{w} = h(x,\kappa(w))w^2,$$

because

$$\dot{z} = \kappa'(w)\dot{w} = e^{-\frac{1}{w}}\frac{1}{w^2}\dot{w} \Rightarrow$$
$$\dot{w} = e^{\frac{1}{w}}\dot{z}w^2 = e^{\frac{1}{w}}h(x,\kappa(w))e^{-\frac{1}{w}}w^2.$$

Notice $h(x,\kappa(w)) - h(x,0)$ is C^∞ flat in w.

Classical result recovered

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x,z)z.$$

$h(x,0)$ has the sign of x.

Theorem
If h is C^∞, then $p(x_0,\epsilon)$ is C^∞.

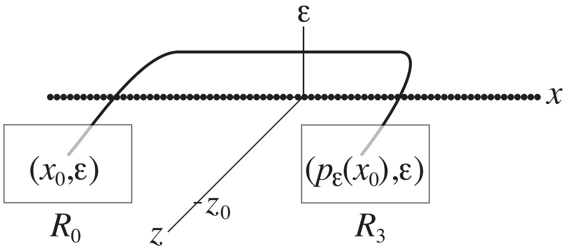
This result is not new, but it follows from part 2 of the Main Theorem.

Thus a linear result is a consequence of a quadratic result.

Proof of Main Theorem: Extension of the system

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x, z)z^2,$$
$$\dot{\epsilon} = 0.$$

$h(x, 0)$ has the sign of x.



Define $P : R_0 \to R_3$ by $P(x_0, \epsilon) = (p_\epsilon(x_0), \epsilon)$, with p_0 defined implicitly by $\int_{x_0}^{p_0(x_0)} h(x, 0)\, dx = 0$. Study P.

Blow-up transformation

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x, z)z^2,$$
$$\dot{\epsilon} = 0.$$

Let $(x, (\bar{z}, \bar{\epsilon}), r)$ be a point of $\mathbb{R} \times S^1 \times \mathbb{R}_+$; $\bar{z}^2 + \bar{\epsilon}^2 = 1$. Blow-up transformation:

$$x = x, \quad z = r\bar{z}, \quad \epsilon = r\bar{\epsilon}.$$

Our system pulls back to one on $\mathbb{R} \times S^1 \times \mathbb{R}_+$.

Division by r desingularizes the new system on the cylinder $r = 0$ but leaves it invariant.

Cylindrical coordinates

The blow-up can be visualized most completely in cylindrical coordinates.

For $(x, (\bar{z}, \bar{\epsilon}), r) \in \mathbb{R} \times S^1 \times \mathbb{R}_+$, let $\bar{z} = \cos\theta$ and $\bar{\epsilon} = \sin\theta$.

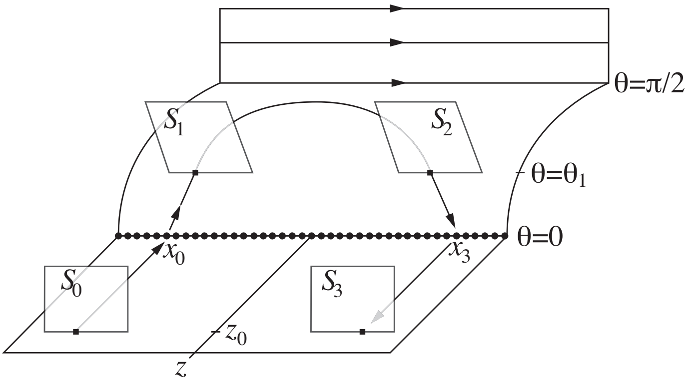
$$x = x, \quad z = r\cos\theta, \quad \epsilon = r\sin\theta.$$

After making the coordinate change and dividing by r, the system becomes

$$\begin{aligned}
\dot{x} &= \sin\theta, \\
\dot{r} &= r\cos^3\theta\, h(x, r\cos\theta), \\
\dot{\theta} &= -\cos^2\theta\sin\theta\, h(x, r\cos\theta).
\end{aligned}$$

Flow in cylindrical coordinates

$$\dot{x} = \sin\theta,$$
$$\dot{r} = r\cos^3\theta\, h(x, r\cos\theta),$$
$$\dot{\theta} = -\cos^2\theta\sin\theta\, h(x, r\cos\theta).$$



Affine coordinates

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x,z)z^2,$$
$$\dot{\epsilon} = 0.$$

New coordinates that blow up the x-axis to a plane:

$$x = x, \quad z = z, \quad \epsilon = zE.$$

The plane $z = 0$ in xzE-space corresponds to the line $z = \epsilon = 0$ in $xz\epsilon$-space

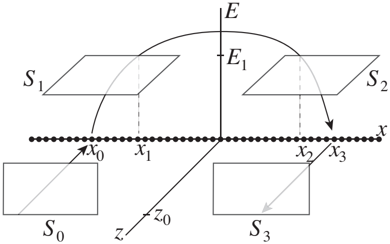
Change variables, divide by z (otherwise the plane $z = 0$ is all equilibria):

$$\dot{x} = E,$$
$$\dot{z} = h(x,z)z,$$
$$\dot{E} = -h(x,z)E$$

Flow in affine coordinates

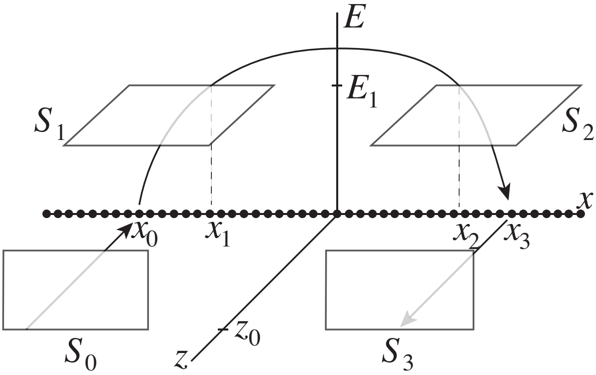
$$\begin{aligned}
\dot{x} &= E, \\
\dot{z} &= h(x,z)z, \\
\dot{E} &= -h(x,z)E
\end{aligned}$$

Notice $\epsilon = zE$ is a first integral.



$$0 = \int_{x_0}^{x_3} \frac{dE}{dx}\, dx = \int_{x_0}^{x_3} -h(x,0)\, dx \Rightarrow x_3 = p_0(x_0).$$

Return map as a composition



$$P = P_3 \circ P_2 \circ P_1.$$

P_2 is clearly C^∞.

It remains to study the smoothness of P_1 and P_3.

Simplification

$$\dot{x} = E,$$
$$\dot{z} = h(x, z)z,$$
$$\dot{E} = -h(x, z)E$$

At the left, divide by $-h(x, z) > 0$, let $k = -\frac{1}{h} > 0$:

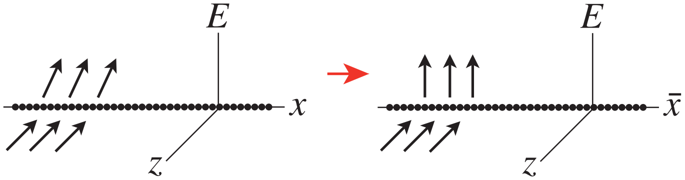
$$\dot{x} = k(x, z)E,$$
$$\dot{z} = -z,$$
$$\dot{E} = E.$$

Note that $zE = \epsilon$ is constant on solutions.

Normal form step 1

$$\dot{x} = k(x,z)E,$$
$$\dot{z} = -z,$$
$$\dot{E} = E.$$

Straighten flow on $z = 0$:



Then $\dot{\bar{x}} = \tilde{k}(\bar{x}, z, E)zE = \epsilon\tilde{k}(\bar{x}, z, E)$.

Normal form step 2

Proposition

Let $N \geq 1$. Then after a C^∞ coordinate change $\bar{x} = \eta(x, z, E)$,

$$\begin{aligned}\dot{\bar{x}} &= \epsilon a(\bar{x}, \epsilon) + \epsilon^N b(\bar{x}, z, E),\\ \dot{z} &= -z,\\ \dot{E} &= E,\end{aligned}$$

with a and b of class C^∞.

Proof: The case $N = 1$ was step 1 (with $a = 0$). If the proposition is true for some N, let

$$\hat{x} = \bar{x} + \epsilon^N \big(\beta(\bar{x}, z) + \gamma(\bar{x}, E)\big),$$

and choose β and γ to eliminate terms of order ϵ^N.

Integration change of variables

$$
\begin{aligned}
\dot{\bar{x}} &= \epsilon a(\bar{x}, \epsilon) + \epsilon^{N+2} b(\bar{x}, z, E), \\
\dot{z} &= -z, \\
\dot{E} &= E,
\end{aligned}
$$

Integrate from $(\bar{x}_0, 1, \epsilon)$ to $(\bar{x}_1, \frac{\epsilon}{E_1}, E_1)$. Change of variables:

$$
\bar{z} = zE \log z = \epsilon \log z, \quad \bar{E} = zE \log E = \epsilon \log E, \quad \tau = \frac{t}{\epsilon}.
$$

Use $\dot{\bar{z}} = \frac{\epsilon}{z} \dot{z} = -\epsilon$, etc.:

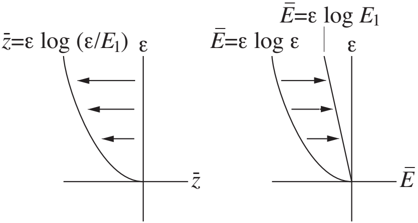
$$
\begin{aligned}
\bar{x}' &= a(\bar{x}, \epsilon) + \epsilon^{N+1} b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon}), \\
\bar{z}' &= -1, \\
\bar{E}' &= 1.
\end{aligned}
$$

Integrate from $(\bar{x}_0, 0, \epsilon \log \epsilon)$ to $(\bar{x}_1, \epsilon \log \frac{\epsilon}{E_1}, \epsilon \log E_1)$.

Region of integration

$$\bar{x}' = a(\bar{x}, \epsilon) + \epsilon^{N+1} b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon}),$$
$$\bar{z}' = -1,$$
$$\bar{E}' = 1.$$

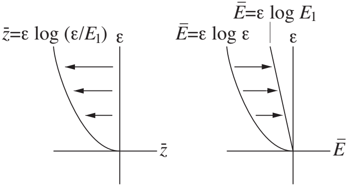
Regard ϵ as a parameter. Integrate from $(\bar{x}_0, 0, \epsilon \log \epsilon)$ to $(\bar{x}_1, \epsilon \log \frac{\epsilon}{E_1}, \epsilon \log E_1)$.



Within the region of integration $\mathcal{D}$, the system is C^N.

Smoothness

$$\bar{x}' = a(\bar{x}, \epsilon) + \epsilon^{N+1} b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon})$$



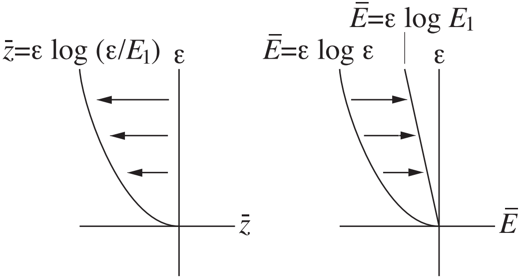
$$\frac{\partial^N}{\partial \epsilon^N} \epsilon^{N+1} b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon})$$

$$= \epsilon^{N+1} D_3 b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon}) \left(-\frac{\bar{E}}{\epsilon^2} \right)^N + \dots$$

$$= (-1)^N \frac{\bar{E}^N}{\epsilon^{N-1}} D_3 b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon}) + \dots .$$

Within $\mathcal{D}$, $\frac{\bar{E}^N}{\epsilon^{N-1}} \to 0$ as $(\bar{z}, \bar{E}, \epsilon) \to (0, 0, 0)$.

Output depends on $(\bar{x}_0, \epsilon, \epsilon \log \epsilon)$



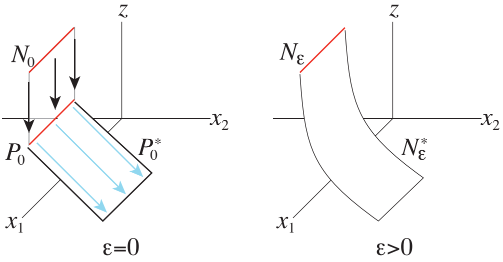
The solution with initial condition $(\bar{x}, \bar{z}, \bar{E}) = (\bar{x}_0, 0, \bar{E}_0)$ at $\tau = 0$ has $\bar{x}$-coordinate $\bar{x} = \phi(\bar{x}_0, \bar{E}_0, \epsilon, \tau)$, where ϕ is C^N as long as the solution remains in $\mathcal{D}$. Thus

$$\bar{x}_1 = \phi(\bar{x}_0, \bar{E}_0, \epsilon, \tau) = \phi(\bar{x}_0, \epsilon \log \epsilon, \epsilon, \epsilon \log E_1 - \epsilon \log \epsilon).$$

More compactly, $\bar{x}_1$ is a C^N function of $(\bar{x}_0, \epsilon, \epsilon \log \epsilon)$.

Exchange Lemma

$$\dot{x} = \epsilon f(x, z, \epsilon),$$
$$\dot{z} = g(x, z, \epsilon)z,$$
$$(x, z) \in \mathbb{R}^n \times \mathbb{R}, \quad f(x, 0, 0) \neq 0, \quad g(x, 0, 0) < 0.$$



Theorem

If N_0 and the N_ϵ ($\epsilon > 0$) fit together to form a smooth manifold of $xz\epsilon$-space, then, away from P_0, P_0^* and the N_ϵ^* ($\epsilon > 0$) do too.

Ting-Hao Hsu, preprint, 2016

Slow-fast system #1

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x,z)z,$$

$h(x,0)$ has the sign of x.

Dot indicates derivative with respect to fast time t.
Introduce the slow time $\tau = \epsilon t$ as a new dependent variable.

Slow-fast system #2

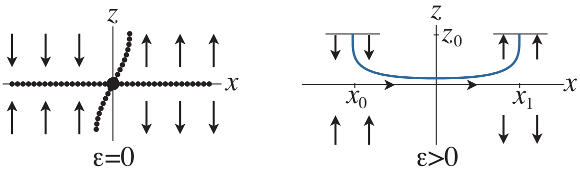
$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x,z)z,$$
$$\dot{\tau} = \epsilon.$$

Slow-fast system #2

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x, z)z,$$
$$\dot{\tau} = \epsilon.$$

For $x_0 < 0$:

- Define x_1 by $\int_{x_0}^{x_1} h(x, 0)\, dx = 0$.
- Define $\tau_1 = x_1 - x_0$.



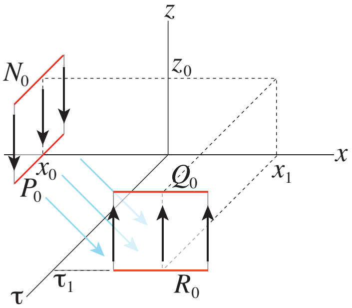
Slow-fast system #2

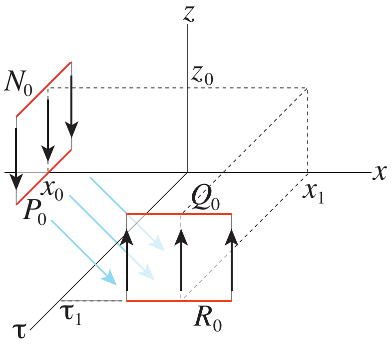
$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x,z)z,$$
$$\dot{\tau} = \epsilon.$$

Start and end 1-dimensional manifolds:

- Define N_ϵ: $(x,z,\tau) = (x_0, z_0, \sigma) : |\sigma| < \delta$.
- Define Q_ϵ: $(x,z,\tau) = (x_1+\sigma, z_0, \tau_1) : |\sigma| < \delta$.

Idea: For $\epsilon > 0$, start at $(x,z) = (x_0,z_0)$, return to $z = z_0$: will have $x \simeq x_1$ and $\Delta\tau \simeq \tau_1$.





- Project N_0 and Q_0 along the fast flow to $z = 0$. Get P_0 and R_0.
- Follow P_0 and R_0 using the slow system on $z = 0$:.
- Obtain P_0^* and R_0^*, both open subsets of $x\tau$-space.

This does not help us study the intersection of N_ϵ^* and Q_ϵ^*, which are near P_0^* and R_0^*.

Introduce an "extra variable" using $z = e^{-\frac{\zeta}{\epsilon}}$ or:

$$\zeta = -\epsilon \ln z, \quad \text{so} \quad \dot{\zeta} = -\epsilon \frac{\dot{z}}{z} = -\epsilon h(x, z).$$

Slow-fast system #3 (equivalent to original system on the invariant manifold $\zeta = -\epsilon \ln z$)

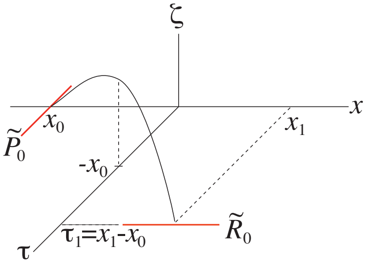
$$\begin{aligned}
\dot{x} &= \epsilon, \\
\dot{z} &= h(x, z)z, \\
\dot{\zeta} &= -\epsilon h(x, z), \\
\dot{\tau} &= \epsilon.
\end{aligned}$$

New start and end 1-dimensional manifolds:

- N_ϵ: $(x, z, \tau) = (x_0, z_0, \sigma) : |\sigma| < \delta \longrightarrow$
  $\tilde{N}_\epsilon$: $(x, z, \zeta, \tau) = (x_0, z_0, -\epsilon \ln z_0, \sigma) : |\sigma| < \delta$
- Q_ϵ: $(x, z, \tau) = (x_1 + \sigma, z_0, \tau_1) : |\sigma| < \delta \longrightarrow$
  $\tilde{Q}_\epsilon$: $(x, z, \zeta, \tau) = (x_1 + \sigma, z_0, -\epsilon \ln z_0, \tau_1) : |\sigma| < \delta$

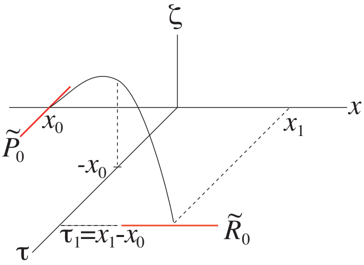
Project $\tilde{N}_0$ and $\tilde{Q}_0$ along the fast flow to $z = 0$, i.e., to $x\zeta\tau$-space:

- $\tilde{P}_0$: $(x, \zeta, \tau) = (x_0, 0, \sigma) : |\sigma| < \delta$.
- $\tilde{R}_0$: $(x, \zeta, \tau) = (x_1 + \sigma, 0, \tau_1) : |\sigma| < \delta$.



Follow using the slow system on $z = 0$:

$$x' = 1,$$
$$\zeta' = -h(x, 0),$$
$$\tau' = 1.$$

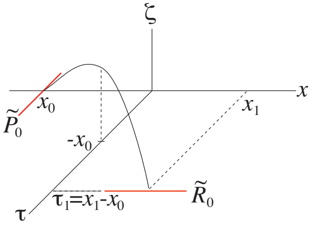


We easily obtain: $\tilde{P}_0^*$ and $\tilde{R}_0^*$ meet transversally at

$$
\begin{aligned}
(x, \zeta, \tau) &= (0, \int_{x_0}^{0} -h(\xi, 0)\, d\xi, -x_0) \\
&= (0, \int_{x_1}^{0} -h(\xi, 0)\, d\xi, -x_0).
\end{aligned}
$$

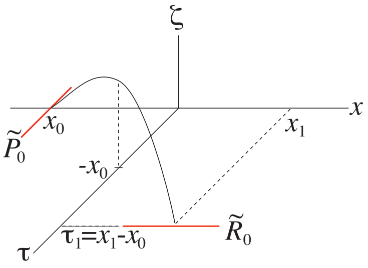
Equality follows from $\int_{x_0}^{x_1} h(\xi, 0)\, d\xi = 0$.

Summary for Slow-Fast System #3 in $xz\zeta\tau$-space



- Start and end 1-dimensional manifolds $\tilde{N}_\epsilon$ and $\tilde{Q}_\epsilon$.
- Project $\tilde{N}_0$ and $\tilde{Q}_0$ along the fast flow to $\tilde{P}_0$ and $\tilde{R}_0$ in $z = 0$, i.e., in $x\zeta\tau$-space.
- $\tilde{P}_0^*$ and $\tilde{R}_0^*$ meet transversally (2-dimensional manifolds in $\mathbb{R}^3$).
- By the Exchange Lemma, away from $\tilde{P}_0$ and $\tilde{R}_0$, $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ are close to $\tilde{P}_0^*$ and $\tilde{R}_0^*$ respectively.
- **But we cannot conclude that $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ meet transversally.**
    1. Two-dimensional manifolds in $\mathbb{R}^4$.
    2. Exchange Lemma can't follow $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ to $x = 0$.

Objection 1: $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ are 2-dimensional manifolds in $\mathbb{R}^4$.



- $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ are close to $\tilde{P}_0^*$ and $\tilde{R}_0^*$, which meet transversally in $x\zeta\tau$-space.
- Project $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ to $x\zeta\tau$-space (ignore small z-coordinate). The projections meet transversally there.
- But then $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ intersect, because on these manifolds, $\zeta = -\epsilon \ln z$.

Objection 2: Exchange Lemma can't follow $\tilde{N}^*_\epsilon$ and $\tilde{Q}^*_\epsilon$ to $x = 0$.

Slow-fast system #3

$$\dot{x} = \epsilon,$$
$$\dot{z} = h(x, z)z,$$
$$\dot{\zeta} = -\epsilon h(x, z),$$
$$\dot{\tau} = \epsilon.$$

Within the manifold $\zeta = -\epsilon \ln z$, in which both $\tilde{N}^*_\epsilon$ and $\tilde{Q}^*_\epsilon$ lie,

$$\dot{x} = \epsilon,$$
$$\dot{\zeta} = -\epsilon h(x, e^{-\frac{\zeta}{\epsilon}}),$$
$$\dot{\tau} = \epsilon.$$

In the slow time $\tau = \epsilon t$:

Within the manifold $\zeta = -\epsilon \ln z$, in which both $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ lie,

$$x' = 1,$$
$$\zeta' = -h(x, e^{-\frac{\zeta}{\epsilon}}),$$
$$\tau' = 1$$

Use this to follow $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ to $x = 0$.

Recent work that uses an "extra variable" to study exponential loss of normal hyperbolicity in geometric singular perturbation problems

K. U. Kristiansen, "Blowup for flat slow manifolds with applications to regularization of piecewise smooth systems using tanh and a model of aircraft ground dynamics," preprint, 2016.

$$\dot{x} = \epsilon e^{-z^{-1}},$$
$$\dot{z} = z^2\left(x - e^{-z^{-1}}\right).$$

$$q = e^{-z^{-1}} \Rightarrow \dot{q} = e^{-z^{-1}} z^{-2} \dot{z} = q(x-q).$$

$$\dot{x} = \epsilon q, \quad \dot{z} = z^2(x-q), \quad \dot{q} = q(x-q).$$