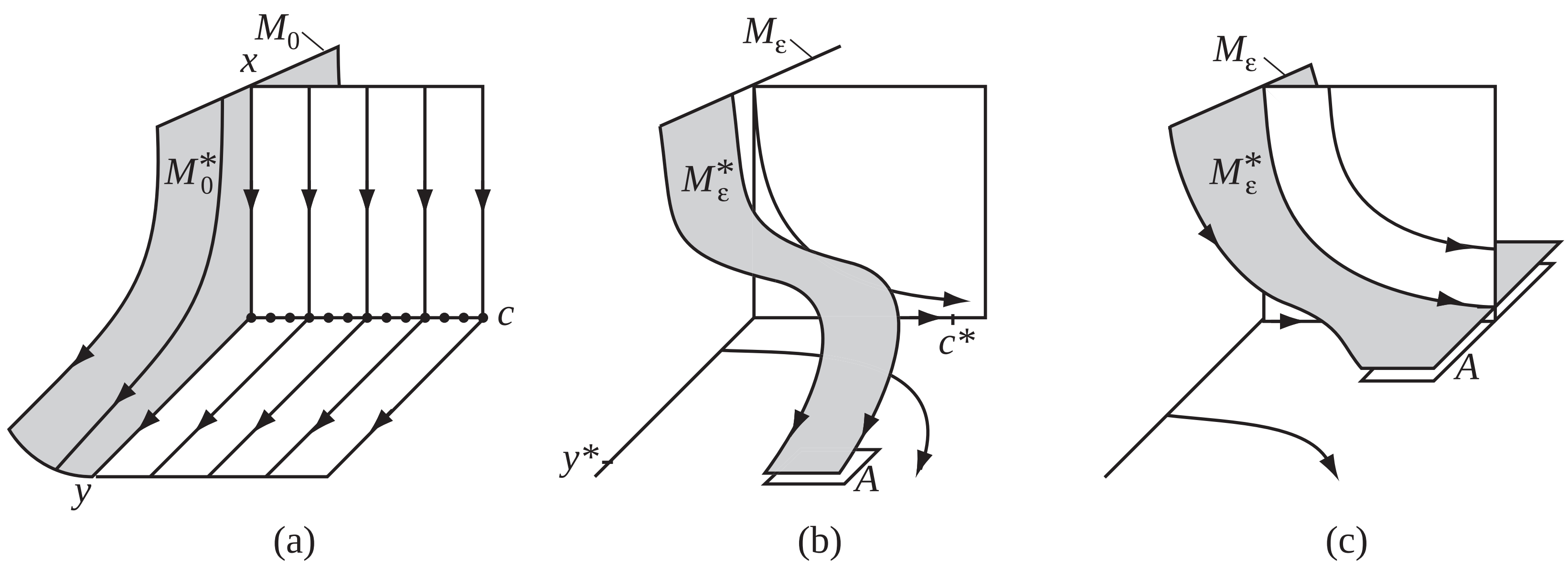
Exchange Lemmas



Steve Schecter

North Carolina State University

Plan

(1) Boundary value problems

(2) Exchange Lemma of Jones and Kopell

(3) General Exchange Lemma

(4) Exchange Lemma of Jones and Tin

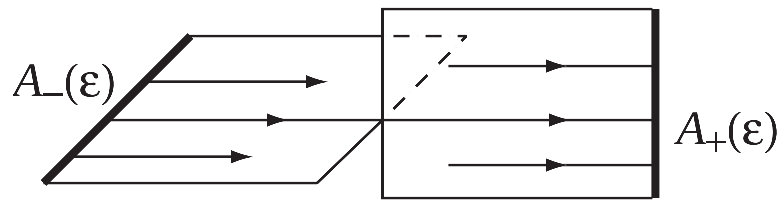
(5) Loss-of-stability turning points: Liu's Exchange Lemma

(6) Gain-of-stability turning points

(7) Basis of proof: Generalized Deng's Lemma

Boundary Value Problems

$$\dot{\xi} = F(\xi, \varepsilon), \quad \xi(t_-) \in A_-(\varepsilon), \quad \xi(t_+) \in A_+(\varepsilon),$$



To show existence of a solution: show that the manifold of solutions that start on $A_-(\varepsilon)$ and the manifold of solutions that end on $A_+(\varepsilon)$ meet transversally.

Remarks

- The problem with $\varepsilon = 0$ may be degenerate in some major way .
- Such problems are called *singularly perturbed*.
- The geometric approach to these problems, which focuses on *tracking manifolds of potential solutions* rather than on asymptotic expansions of solutions, is called *geometric singular perturbation theory* (Fenichel, Kopell, Jones, . . .).

Exchange Lemma of Jones and Kopell

Slow-Fast Systems

$$\dot{a} = f(a,b,\varepsilon), \quad \dot{b} = \varepsilon g(a,b,\varepsilon), \quad (a,b) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Set $\varepsilon = 0$:

$$\dot{a} = f(a,b,0), \quad \dot{b} = 0.$$

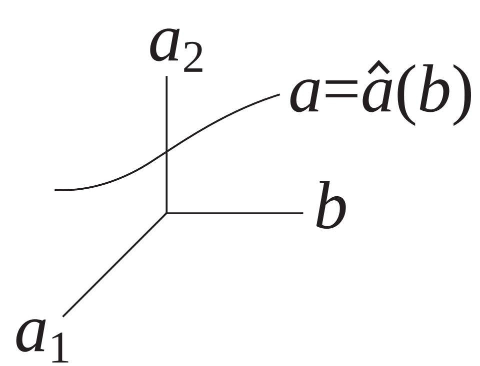
Assume:

(1) $f(\hat{a}(b),b,0) = 0$.

(2) $D_a f(\hat{a}(b),b,0)$ has

- k eigenvalues with negative real part.
- l eigenvalues with positive real part.
- $k+l=n$.

(3) $g(\hat{a}(b),b,0) \neq 0$.



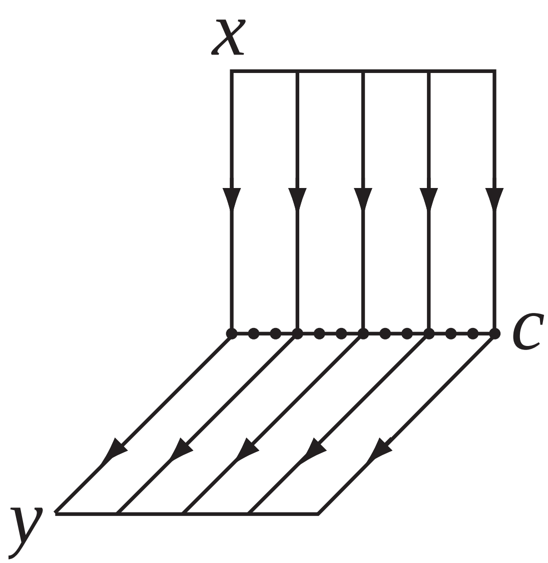
After a change of coordinates:

$$\dot{x} = A(x,y,c,\varepsilon)x,$$
$$\dot{y} = B(x,y,c,\varepsilon)y,$$
$$\dot{c} = \varepsilon((1,0,\ldots,0) + L(x,y,c,\varepsilon)xy),$$

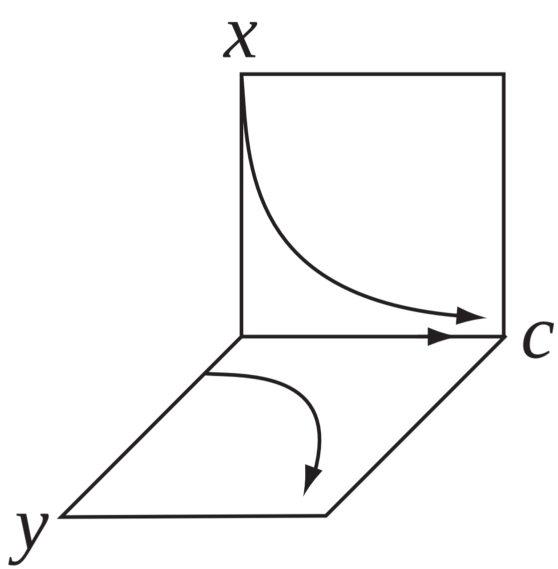
$(x,y,c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$,

$A(0,0,c,0)$ has eigenvalues with negative real part,

$B(0,0,c,0)$ has eigenvalues with positive real part.



Flow with $\varepsilon = 0$.



Flow with $\varepsilon > 0$.

Exchange Lemma of Jones and Kopell with $m = 1$

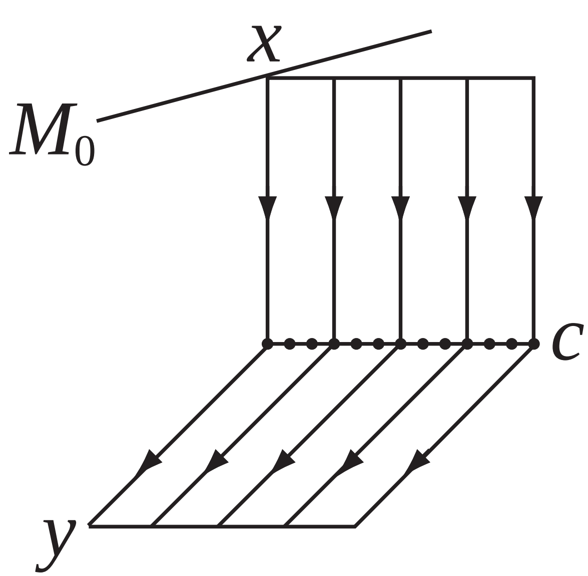
Assume:

(1) $m = 1$ (for simplicity).

(2) For each ε, M_ε is a submanifold of xyc-space of dimension l.

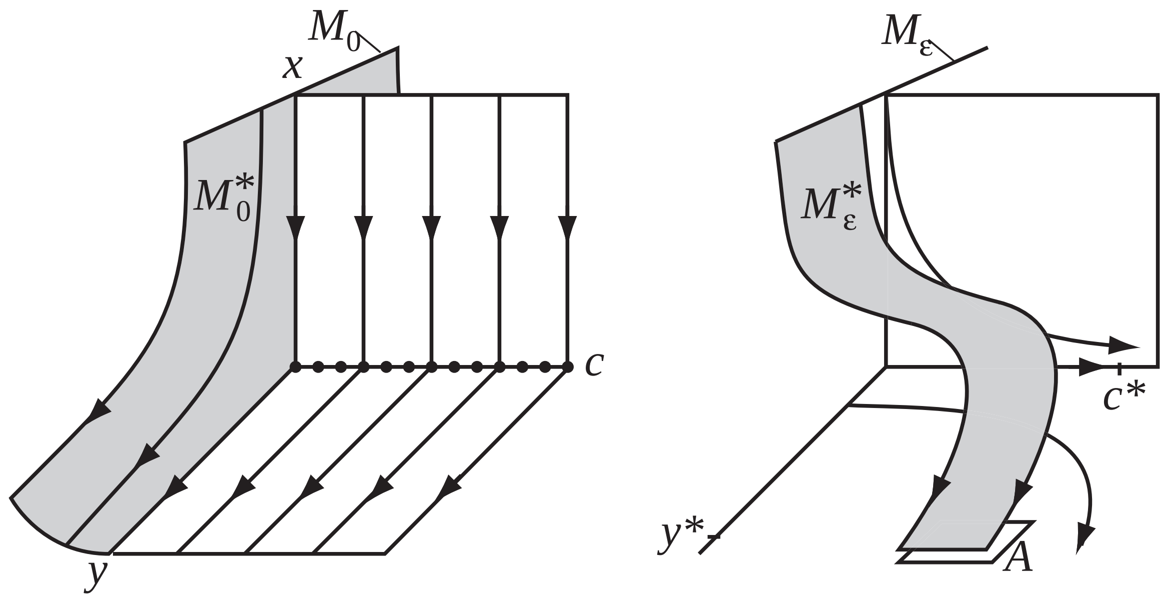
(3) $M = \{(x, y, c, \varepsilon) : (x, y, c) \in M_\varepsilon\}$ is itself a manifold.

(4) M_0 meets xc-space transversally at a point $(x_*, 0, 0)$.



$\varepsilon = 0$

Under the forward flow, each M_ε becomes a manifold M_ε^* of dimension $l + 1$.



Theorem 1 (Exchange Lemma of Jones and Kopell with $m = 1$, 1994). Consider a point $(0, y^*, c^*)$ with $y^* \neq 0$ and $0 < c^*$. Let A be a small neighborhood of (y^*, c^*) in yc-space. Then for small $\varepsilon_0 > 0$ there is a smooth function $\tilde{x} : A \times [0, \varepsilon_0) \to \mathbb{R}^k$ such that:

(1) $\tilde{x}(y, c, 0) = 0$.

(2) As $\varepsilon \to 0$, $\tilde{x} \to 0$ exponentially, along with its derivatives with respect to all variables.

(3) For $0 < \varepsilon < \varepsilon_0$, $\{(x, y, c) : (y, c) \in A \text{ and } x = \tilde{x}(y, c, \varepsilon)\}$ is contained in M_ε^*.

Transversality to xc-space is “exchanged” for closeness to yc-space.

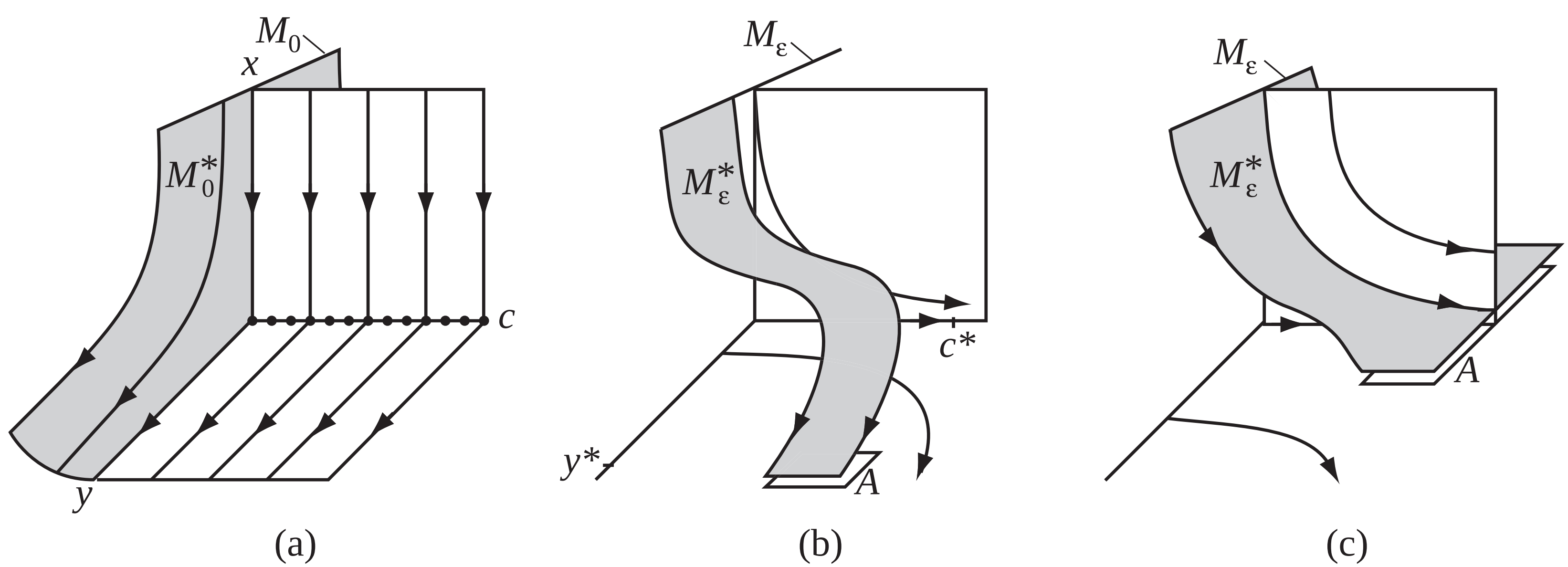
Brunovsky's Reformulation of Jones and Kopell's Exchange Lemma as an Inclination Lemma

Theorem 2 (1999). Let $0 < c^*$. Let A be a small neighborhood of $(0, c^*)$ in yc-space. Then for small $\varepsilon_0 > 0$ there is a smooth function $\tilde{x} : A \times [0, \varepsilon_0) \to \mathbb{R}^k$ such that:

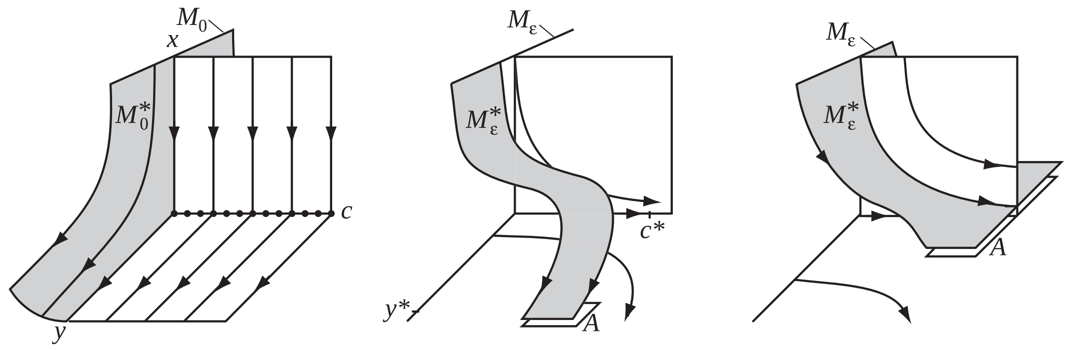
(1) $\tilde{x}(y, c, 0) = 0$.

(2) As $\varepsilon \to 0$, $\tilde{x} \to 0$ exponentially, along with its derivatives with respect to all variables.

(3) For $0 < \varepsilon < \varepsilon_0$, $\{(x, y, c) : (y, c) \in A$ and $x = \tilde{x}(y, c, \varepsilon)\}$ is contained in M^*_ε.



General Exchange Lemma



Important Features of the Exchange Lemma

(1) There is a normally hyperbolic invariant manifold (c-space) and a small parameter ε.

(2) There is a collection of submanifolds M_ε of xyc-space such that $M = \{(x,y,c,\varepsilon) : (x,y,c) \in M_\varepsilon\}$ is itself a manifold. M_0 meets xc-space transversally in a manifold N_0 (here a point).

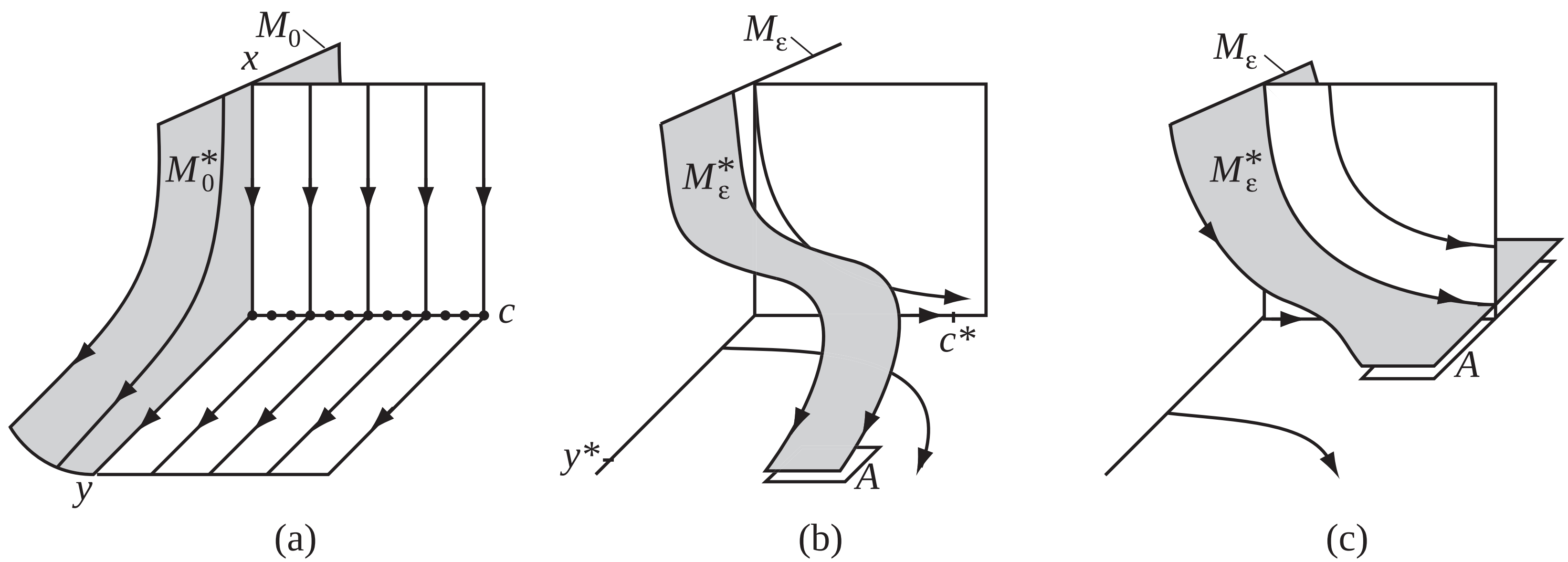
(3) N_0 projects along the stable fibration of xc space to a submanifold P_0 of c-space of the same dimension (here a point).

(4) For small $\varepsilon > 0$, the flow on c-space is followed for a long time.

(5) It takes P_ε to a set P_ε^* of dimension one greater. As $\varepsilon \to 0$, the limit of $P_\varepsilon^* \neq$ where the limiting DE takes P_0. Nevertheless, the limit of P_ε^* exists and has the same dimension. Call it P_0^*.

(6) As $\varepsilon \to 0$, $M_\varepsilon^* \to W^u(P_0^*)$.

General Exchange Lemma (S., 2007). (1)–(5) plus technical assumptions imply (6).



What's the point?

- To understand the flow on the normally hyperbolic invariant manifold may require rectification, blowing-up, etc.
- Once you've done this work, the General Exchange Lemma helps deal with the remaining dimensions.

Exchange Lemma of Jones and Tin

Consider again:

$$\begin{aligned}\dot{x} &= A(x,y,c,\varepsilon)x,\\ \dot{y} &= B(x,y,c,\varepsilon)y,\\ \dot{c} &= \varepsilon((1,0,\ldots,0)+L(x,y,c,\varepsilon)xy),\end{aligned}$$

$$(x,y,c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m,$$
$A(0,0,c,0)$ has eigenvalues with negative real part,
$B(0,0,c,0)$ has eigenvalues with positive real part.

Assume:

(1) $m \geq 1$.
(2) For each ε, M_ε is a submanifold of xyc-space of dimension $l+p$, $0 \leq p \leq m-1$.
(3) $M = \{(x,y,c,\varepsilon) : (x,y,c) \in M_\varepsilon\}$ is itself a manifold.
(4) M_0 meets xc-space transversally in a manifold N_0 of dimension p.
(5) N_0 projects smoothly to a submanifold P_0 of c-space of dimension p.
(6) The vector $(1,0,\ldots,0)$ is not tangent to P_0.

Then:

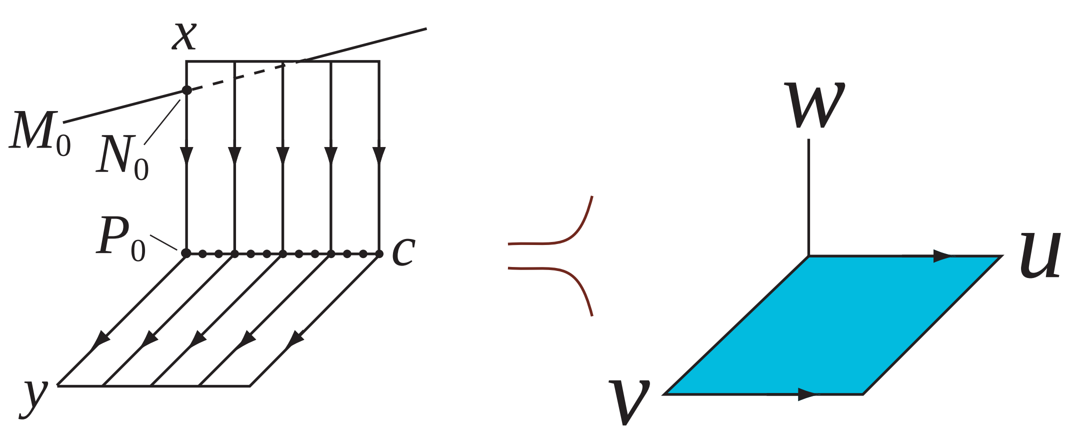
(1) Each M_ε meets xc-space transversally in a manifold N_ε of dimension p.

(2) N_ε projects smoothly to a submanifold P_ε of c-space of dimension p.

(3) The vector $(1,0,\ldots,0)$ is not tangent to P_ε.

After a change of coordinates $c = (u,v,w) \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^{m-1-p}$ that takes each P_ε to v-space, the system can be put in the form

$$\begin{aligned}
\dot{x} &= A(x,y,u,v,w,\varepsilon)x,\\
\dot{y} &= B(x,y,u,v,w,\varepsilon)y,\\
\dot{u} &= \varepsilon(1+e(x,y,u,v,w,\varepsilon)xy),\\
\dot{v} &= \varepsilon F(x,y,u,v,w,\varepsilon)xy,\\
\dot{w} &= \varepsilon G(x,y,u,v,w,\varepsilon)xy.
\end{aligned}$$



Under the forward flow, each M_ε becomes a manifold M_ε^* of dimension $l+p+1$. Each P_ε becomes a manifold P_ε^* of dimension $p+1$, which in our coordinates is just uv-space.

Theorem 3 (Exchange Lemma of Jones and Tin). Let $0 < u^*$. Let A be a small neighborhood of $(0,u^*,0)$ in yuv-space. Then for small $\varepsilon_0 > 0$ there are smooth function $\tilde{x} : A \times [0,\varepsilon_0) \to \mathbb{R}^k$ and $\tilde{w} : A \times [0,\varepsilon_0) \to \mathbb{R}^{m-p-1}$such that:

(1) $\tilde{x}(y,u,v,0) = 0$.
(2) $\tilde{w}(y,u,v,0) = \tilde{w}(0,u,v,\varepsilon) = 0$.
(3) As $\varepsilon \to 0$, $(\tilde{x},\tilde{w}) \to 0$ exponentially, along with its derivatives with respect to all variables.
(4) For $0 < \varepsilon < \varepsilon_0$, $\{(x,y,u,v,w) : (y,u,v) \in A \text{ and } (x,w) = (\tilde{x},\tilde{w})(y,u,v,\varepsilon)\}$ is contained in M_ε^*.

Remark

The theorem also applies to

$$\begin{aligned}
\dot{x} &= A(x,y,c,\varepsilon)x,\\
\dot{y} &= B(x,y,c,\varepsilon)y,\\
\dot{c} &= \varepsilon(1,0,\ldots,0) + L(x,y,c,\varepsilon)xy,
\end{aligned}$$

It is really about perturbations of systems with a family of normally hyperbolic equilibria, not about slow-fast systems.

Loss-of-Stability Turning Points: Liu's Exchange Lemma

Liu considers a slow-fast system

$$\dot{a} = f(a,b,\varepsilon),$$
$$\dot{b} = \varepsilon g(a,b,\varepsilon),$$

with $a \in \mathbb{R}^{k+l+1}$ and $b \in \mathbb{R}^{m-1}$, $m \geq 2$. Assume:

(1) $f(0,b,\varepsilon) = 0$. (Hence for each ε, b-space is invariant, and for $\varepsilon = 0$ it consists of equilibria.)

(2) $D_a f(0,b,0)$ has

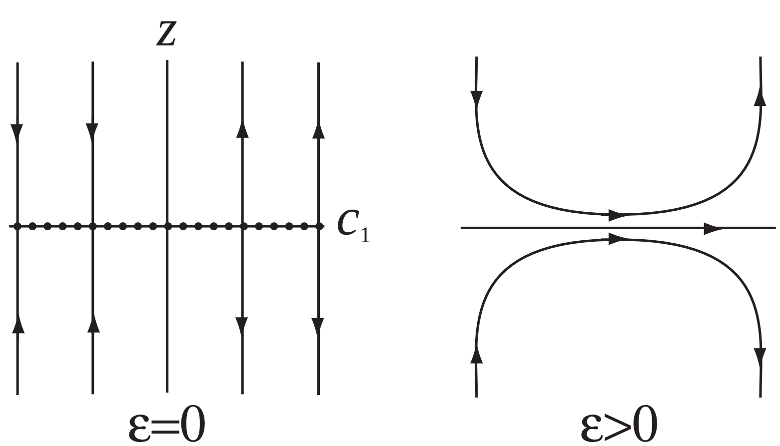
- k eigenvalues with real part less than $\lambda_0 < 0$;
- l eigenvalues with greater than $\mu_0 > 0$;
- a last eigenvalue $\nu(b)$ such that $\nu(0) = 0$.

(3) $D\nu(0)g(0,0,0) > 0$.

After a change of coordinates:

$$\begin{aligned}
\dot{x} &= A(x,y,z,c,\varepsilon)x,\\
\dot{y} &= B(x,y,z,c,\varepsilon)y,\\
\dot{z} &= h(z,c,\varepsilon)z + k(x,y,z,c,\varepsilon)xy,\\
\dot{c} &= \varepsilon((1,0,\dots,0) + l(z,c,\varepsilon)z + L(x,y,z,c,\varepsilon)xy),
\end{aligned}$$

$$\begin{aligned}
(x,y,z,c) &\in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}^{m-1},\\
A(0,0,0,c,0) &\text{ has eigenvalues with negative real part,}\\
B(0,0,0,c,0) &\text{ has eigenvalues with positive real part,}\\
h(0,(0,c_2,\dots,c_{m-1}),0) &= 0,\\
\frac{\partial h}{\partial c_1} &> 0.
\end{aligned}$$



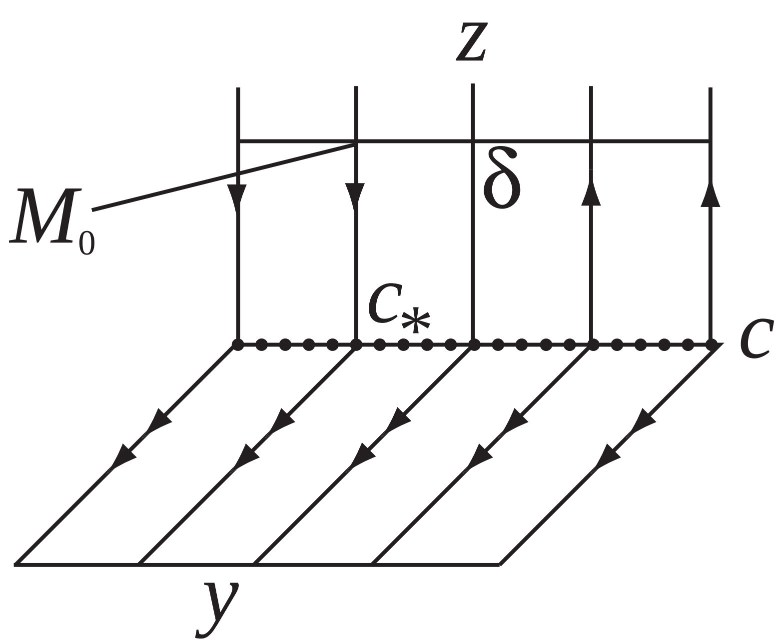
Assume:

(1) $m = 2$ (for simplicity).

(2) For each ε, M_ε is a submanifold of $xyzc$-space of dimension l.

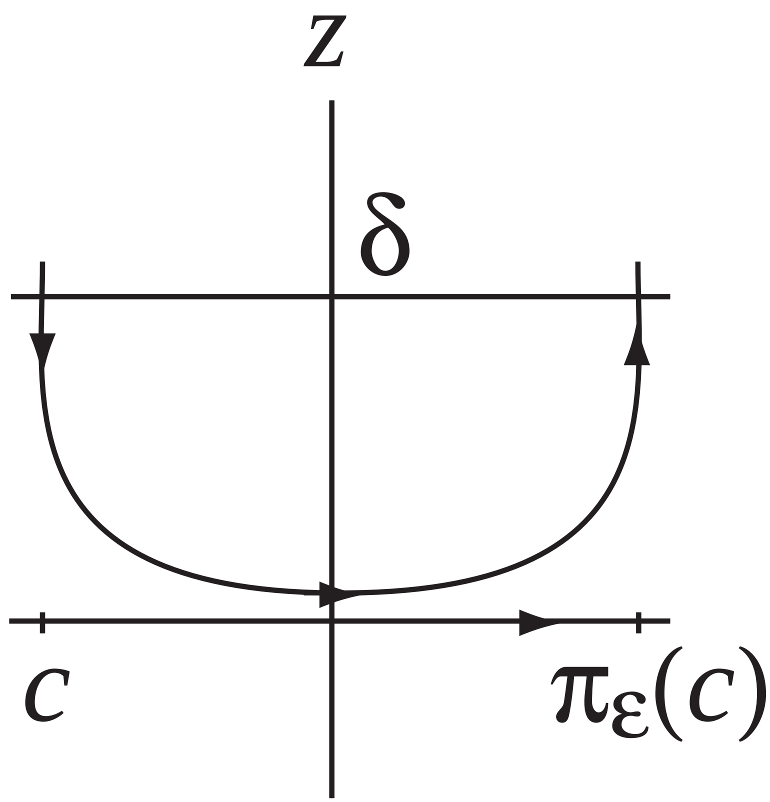
(3) $M = \{(x,y,z,c,\varepsilon) : (x,y,z,c) \in M_\varepsilon\}$ is itself a manifold.

(4) M_0 meets xzc-space transversally at a point $(x_*,0,\delta,c_*)$ with $\delta \neq 0$ and $c_* < 0$. We may assume that $M \subset \{(x,y,z,c,\varepsilon) : z = \delta\}$.



Each M_ε meets xzc-space transversally at $(x,y,z,c) = (x(\varepsilon),0,c(\varepsilon),\delta)$ with $(x(0),c(0)) = (x_*,c_*)$.

Define Poincare maps on $z=\delta$ by $c \rightarrow \pi_\varepsilon(c)$.



Define π_0 implicitly by

$$\int_c^{\pi_0(c)} h(0,u,0)\,du = 0.$$

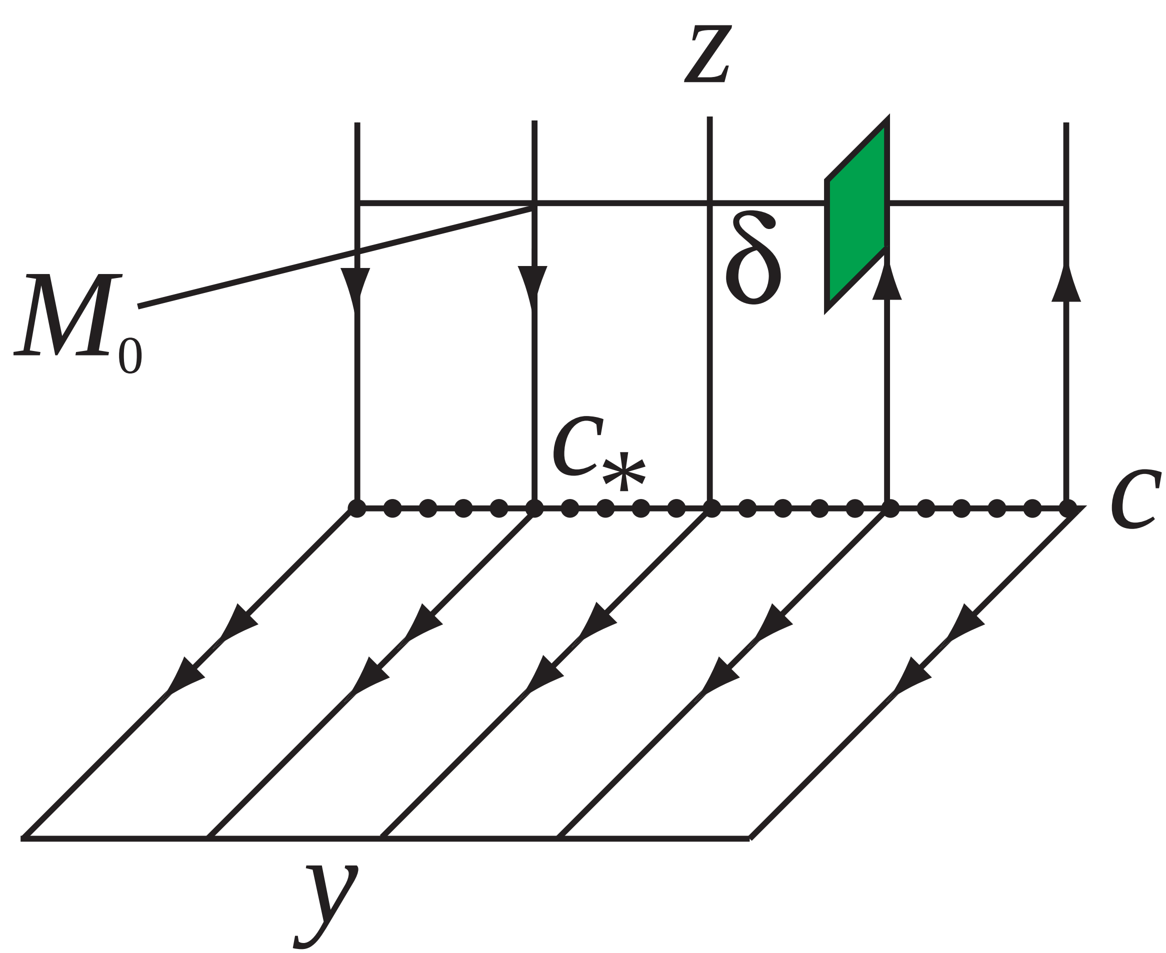
$\pi_\varepsilon \rightarrow \pi_0$, along with its derivatives, as $\varepsilon \rightarrow 0$ (De Maesschalck, 2008).

Under the forward flow, each M_ε becomes a manifold M_ε^* of dimension $l+1$.

Theorem 4 (Liu's Exchange Lemma, 2000). In zc-space, consider a short integral curve C_ε through $(z,c) = (\delta, \pi_\varepsilon(c(\varepsilon)))$. Let

$$A_\varepsilon = \{(x,y,z,c) : x = 0, \|y\| \text{ is small, } (z,c) \in C_\varepsilon\}.$$

Then M_ε^* is close to A_ε. As $\varepsilon \to 0$ the distance goes to 0 exponentially.



Gain-of-Stability Turning Points

(Rarefactions in the Dafermos Regularization)

Consider the system

$$\dot{u} = v,$$
$$\dot{v} = (A(u) - xI)v,$$
$$\dot{x} = \varepsilon,$$

with $(u, v, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $A(u)$ an $n \times n$ matrix.

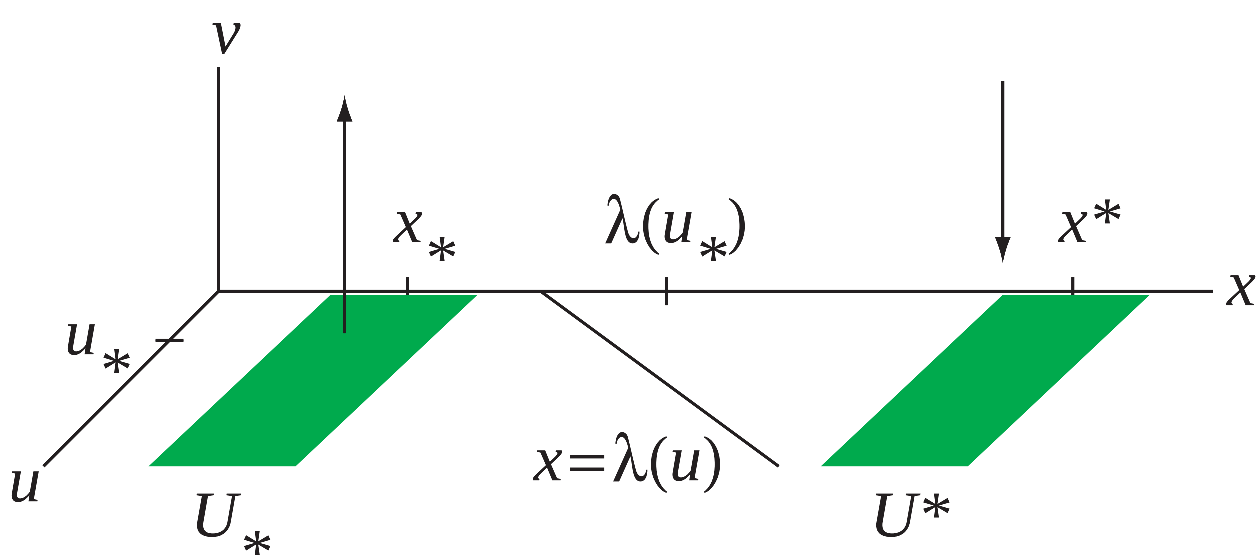
Let $n = k + l + 1$. Assume that on an open set U in $\mathbb{R}^n$:

- There are numbers $\lambda_1 < \lambda_2$ such that $A(u)$ has
  - k eigenvalues with real part less than λ_1,
  - l eigenvalues with real part greater than λ_2,
  - a simple real eigenvalue $\lambda(u)$ with $\lambda_1 < \lambda(u) < \lambda_2$.
- $A(u)$ has an eigenvector $r(u)$ for the eigenvalue $\lambda(u)$ such that $D\lambda(u)r(u) = 1$.

Notice ux-space is invariant for every ε. For $\varepsilon = 0$ it consists of equilibria, but loses normal hyperbolicity along the surface $x = \lambda(u)$.

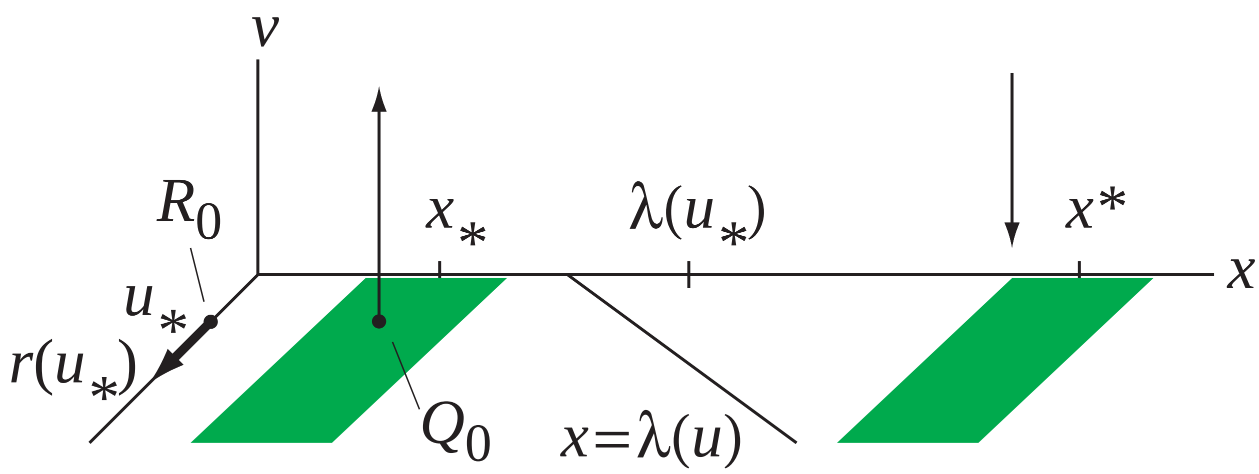
Choose $u_* \in U$, x_*, x^* such that $\lambda_1 < x_* < \lambda(u_*) < x^* < \lambda_2$. Let

$$U_* = \{(u,v,x) : u \in U, v = 0, |x - x_*| < \delta\},$$
$$U^* = \{(u,v,x) : u \in U, v = 0, |x - x^*| < \delta\}.$$



For $\varepsilon = 0$, U_* and U^* are normally hyperbolic manifolds of equilibria of dimension $n+1$. For U_*, the stable and unstable manifolds of each point have dimensions k and $l+1$ respectively; for U^*, the stable and unstable manifolds of each point have dimensions $k+1$ and l respectively.

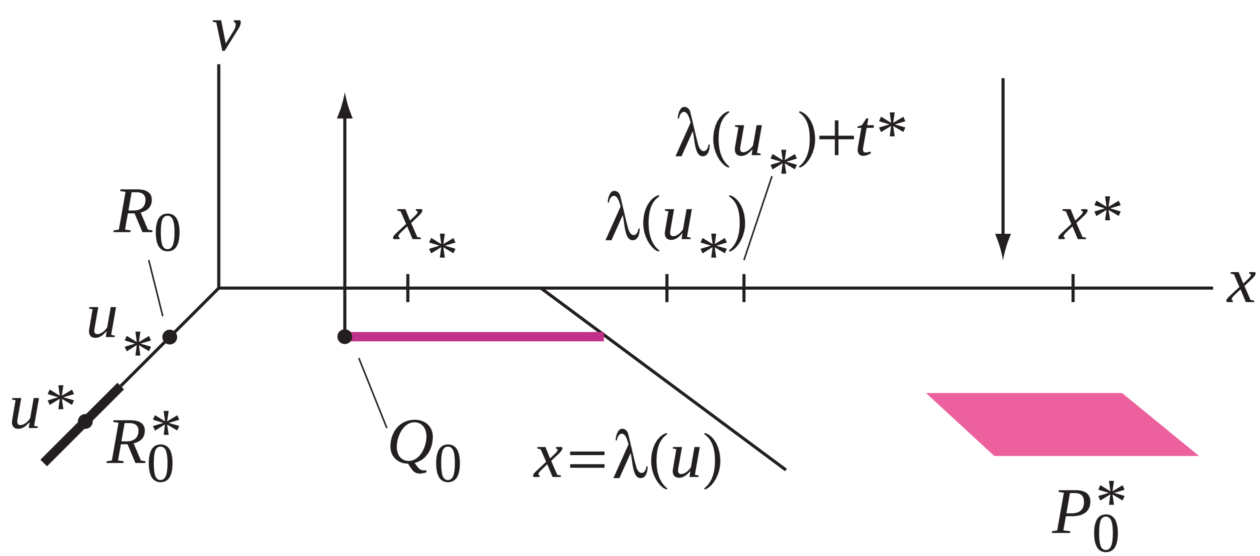
For $\varepsilon > 0$, U_* and U^* are normally hyperbolic invariant manifolds on which the system reduces to $\dot{u} = 0$, $\dot{x} = \varepsilon$.



For each $\varepsilon \geq 0$, let M_ε be a submanifold of uvx-space of dimension $l+1+p$, $0 \leq p \leq n-1$. Assume:

- $M = \{(u,v,x,\varepsilon) : (u,v,x) \in M_\varepsilon\}$ is itself a manifold.
- M_0 is transverse to $W_0^s(U_*)$ at a point in the stable fiber of $(u_*,0,x_*)$. The intersection of M_0 and $W_0^s(U_*)$ is a smooth manifold S_0 of dimension p.
- S_0 projects smoothly to a submanifold Q_0 of ux-space of dimension p.
- The vector $(\dot{u},\dot{x}) = (0,1)$ is not tangent to Q_0. Therefore Q_0 projects smoothly to a submanifold R_0 of u-space of dimension p.
- $r(u_*)$ is not tangent to R_0.

Under the flow, each M_ε becomes a manifold M_ε^* of dimension $l+2+p$.



Let $\phi(t,u)$ be the flow of $\dot{u} = r(u)$. Choose $t^* > 0$ such that $\lambda(u_*) + t^* < x^*$. Let

$$R_0^* = \cup_{|t-t^*|<\delta}\phi(t,R_0), \quad P_0^* = \{(u,v,x) : u \in R_0^*, v = 0, |x-x^*| < \delta\}.$$

R_0^* and P_0^* have dimensions $p+1$ and $p+2$ respectively.

Let $u^* = \phi(t^*, u_*)$.

Theorem 7. Near $(u^*, 0, x^*)$, M_ε^* is close to $W_0^u(P_0^*)$.

How the flow on the normally hyperbolic invariant manifold is analyzed

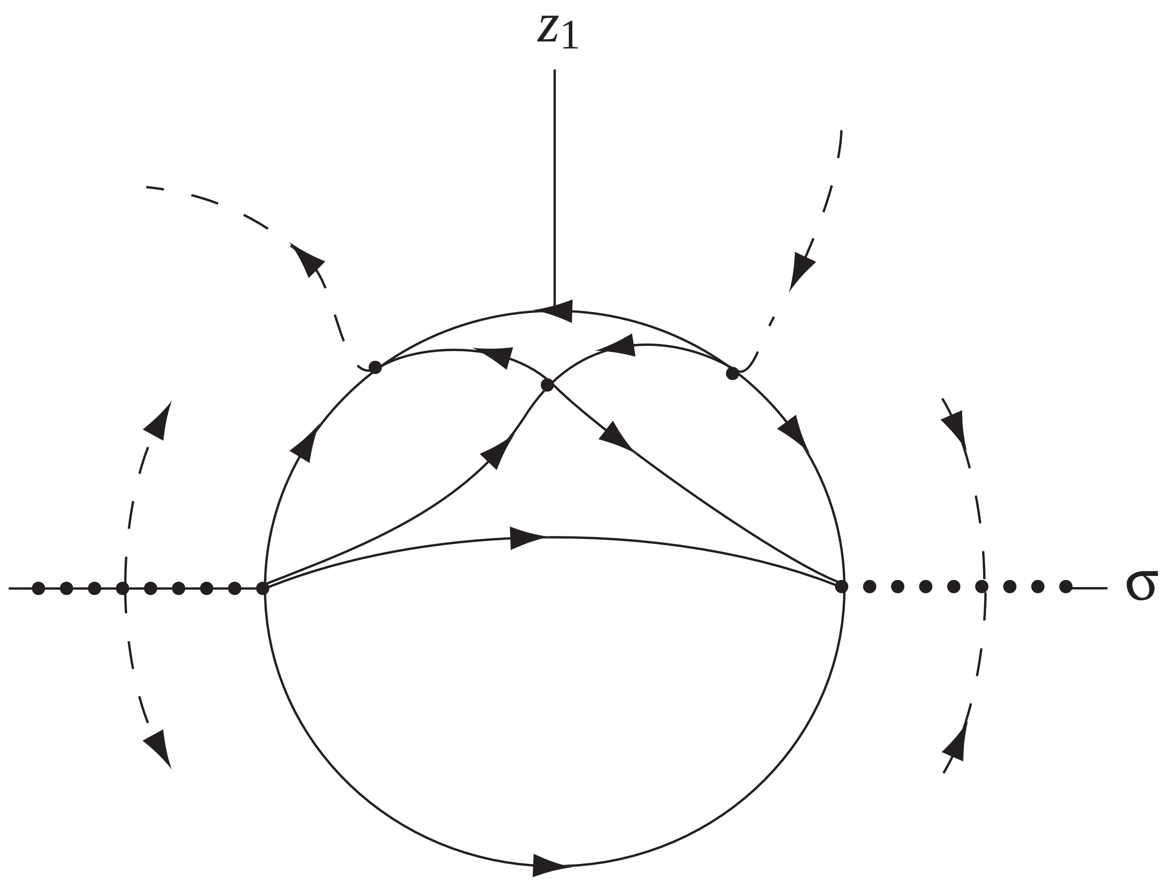
There is a normally hyperbolic invariant manifold with coordinates (u, z_1, x, ε) with z_1 a coordinate along $r(u)$ in v-space.

The equilibria $z_1 = \varepsilon = 0$ lose normal hyperbolicity when $x = \lambda(u)$. We therefore make the change of variables $x = \lambda(u) + \sigma$ and blow up the set $z_1 = \sigma = \varepsilon = 0$:

$$
\begin{aligned}
u &= u, \\
z_1 &= \bar{r}^2 \bar{z}_1, \\
\sigma &= \bar{r}\bar{\sigma}, \\
\varepsilon &= \bar{r}^2 \bar{\varepsilon},
\end{aligned}
$$

with $\bar{z}_1^2 + \bar{\sigma}^2 + \bar{\varepsilon}^2 = 1$.

For the new system, the spherical cylinder $\bar{r} = 0$ consists entirely of equilibria. Divide by $\bar{r}$ to desingularize.



Blown-up flow for fixed u. The ε-axis points toward you.

Generalized Deng's Lemma

In the literature, there are three ways to prove exchange lemmas:

- Jones and Kopell's approach, which is to follow the tangent space to M_ε forward using the extension of the linearized differential equation to differential forms.
- Brunovsky's approach, which is to locate M_ε^* by solving a boundary value problem in Silnikov variables.
- Krupa–Sandstede–Szmolyan approach (1997), using Lin's method.

We follow Brunovsky's approach, which is based on work of Bo Deng (1990). Brunovsky generalized a lemma of Deng that gives estimates on solutions of boundary value problems in Silnikov variables. Our proof of the Generalized Exchange Lemma is based on a further generalization of Deng's Lemma.

Let $(x,y,c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$. Let V be an open subset of $\mathbb{R}^m$. On a neighborhood of $\{0\} \times \{0\} \times V$, consider the C^{r+1} differential equation

$$\begin{aligned} \dot{x} &= A(x,y,c)x, \\ \dot{y} &= B(x,y,c)y, \\ \dot{c} &= C(c) + E(x,y,c)xy. \end{aligned}$$

Let $\phi(t,c)$ be the flow of $\dot{c} = C(c)$. For each $c \in V$ there is a maximal interval I_c containing 0 such that $\phi(t,c) \in V$ for all $t \in I_c$. Let the linearized solution operator of the system, with $\varepsilon = 0$, along the solution $(0,0,\phi(t,c^0))$ be

$$\begin{pmatrix} \overline{x}(t) \\ \overline{y}(t) \\ \overline{c}(t) \end{pmatrix} = \begin{pmatrix} \Phi^s(t,s,c^0) & 0 & 0 \\ 0 & \Phi^u(t,s,c^0) & 0 \\ 0 & 0 & \Phi^c(t,s,c^0) \end{pmatrix} \begin{pmatrix} \overline{x}(s) \\ \overline{y}(s) \\ \overline{c}(s) \end{pmatrix}$$

Assume:

(E1) There are numbers $\lambda_0 < 0 < \mu_0$, $\beta > 0$, and $M > 0$ such that for all $c^0 \in N$ and $s, t \in I_{c^0}$,

$$\begin{aligned} \|\Phi^s(t,s,c^0)\| &\leq Me^{\lambda_0(t-s)} && \text{if } t \geq s, \\ \|\Phi^u(t,s,c^0)\| &\leq Me^{\mu_0(t-s)} && \text{if } t \leq s, \\ \|\Phi^c(t,s,c^0)\| &\leq Me^{\beta|t-s|} && \text{for all } t, s. \end{aligned}$$

(E2) $\lambda_0 + r\beta < 0 < \lambda_0 + \mu_0 - r\beta$.

We wish to study solutions of Silnikov's boundary value problem on an interval $0 \le t \le \tau$:

$$x(0) = x^0, \quad y(\tau) = y^1, \quad c(0) = c^0.$$

We denote the solution by $(x, y, c)(t, \tau, x^0, y^1, c^0)$.

Theorem 9 (Generalized Deng's Lemma, S. 2008). Let V_0 and V_1 be compact subsets of V such that $V_0 \subset \text{Int } V_1)$. For each $c^0 \in N_0$ let J_{c^0} be the maximal interval such that $\phi(t, c^0) \in \text{Int } (V_1)$ for all $t \in J_{c^0}$. Then for λ and μ a little closer to 0 than λ_0 and μ_0, there is a number $\delta_0 > 0$ such that if $\|x^0\| \le \delta_0$, $\|y^1\| \le \delta_0$, $c^0 \in N_0$, and $\tau > 0$ is in J_{c^0}, then Silnikov's boundary value problem has a solution $(x, y, c)(t, \tau, x^0, y^1, c^0)$ on the interval $0 \le t \le \tau$. Moreover, there is a number $K > 0$ such that for all (t, τ, x^0, y^1, c^0) as above,

$$\|x(t, \tau, x^0, y^1, c^0)\| \le K e^{\lambda t},$$

$$\|y(t, \tau, x^0, y^1, c^0)\| \le K e^{\mu(t-\tau)},$$

$$\|c(t, \tau, x^0, y^1, c^0) - \phi(t, c^0)\| \le K e^{\lambda t + \mu(t-\tau)}.$$

In addition, if $\mathbf{i}$ is any $|\mathbf{i}|$-tuple of integers between 1 and $2+k+l+m$, with $1 \le |\mathbf{i}| \le r$, then

$$\|D_{\mathbf{i}} x(t, \tau, x^0, y^1, c^0)\| \le K e^{(\lambda + |\mathbf{i}|\beta)t},$$

$$\|D_{\mathbf{i}} y(t, \tau, x^0, y^1, c^0)\| \le K e^{(\mu - |\mathbf{i}|\beta)(t-\tau)},$$

$$\|D_{\mathbf{i}} c(t, \tau, x^0, y^1, c^0) - D_{\mathbf{i}} \phi(t, c^0)\| \le K e^{(\lambda + |\mathbf{i}|\beta)t + (\mu - |\mathbf{i}|\beta)(t-\tau)}.$$