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Undercompressive shock waves and the Dafermos regularization

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Abstract
For a system of conservation laws in one space dimension, we identify all structurally stable Riemann solutions that include only shock waves. Shock waves are required to satisfy the viscous profile criterion for a given viscosity $(B(u)u_x)_x$. Undercompressive shock waves are allowed. We also show that all such Riemann solutions have nearby smooth solutions of the Dafermos regularization with the given viscosity.

Mathematics Subject Classification: 35L65, 35L67, 34C37, 34E15

1. Introduction

A *system of conservation laws* in one space dimension is a partial differential equation of the form

$$u_t + f(u)_x = 0, \tag{1.1}$$

with $t \geqslant 0$, $x \in \mathbb{R}$, $u(x,t) \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ a smooth map.

The simplest discontinuous solutions of (1.1) are the centred, piecewise constant *shock waves* defined by

$$u(x,t) = \begin{cases} u_- & \text{for } x < st, \\ u_+ & \text{for } x > st. \end{cases} \tag{1.2}$$

The question arises which such discontinuous functions should be admitted as solutions of (1.1). An easy necessary condition is that the triple (u_-, s, u_+) should satisfy the *Rankine–Hugoniot condition*

$$R(u_-, s, u_+) := f(u_+) - f(u_-) - s(u_+ - u_-) = 0. \tag{1.3}$$

This condition follows from the requirement that (1.2) be a weak solution of (1.1) [20].

To obtain a stronger condition, Courant and Friedrichs [5] and Gelfand [8] proposed that (1.1) be regularized by adding a small parabolic term. The differential equation becomes

$$u_t + f(u)_x = \epsilon(B(u)u_x)_x, \tag{1.4}$$

where for all $u \in \mathbb{R}^n$, all eigenvalues of the matrix $B(u)$ have positive real part. Ideally $B(u)$ should represent physically realistic diffusive terms that are ignored in (1.1). The shock wave (1.2) is to be admitted as a solution of (1.1) provided (1.4) has travelling wave solutions $u^\epsilon(x - st)$ that satisfy the boundary conditions

$$u(-\infty) = u_-, \qquad u'(-\infty) = 0, \tag{1.5}$$

$$u(+\infty) = u_+, \qquad u'(+\infty) = 0, \tag{1.6}$$

and that converge to (1.2) in the L^1 sense as $\epsilon \to 0$.

Now the scaling $x \to x/\epsilon$, $t \to t/\epsilon$ removes ϵ from (1.4). Thus, if

$$u_t + f(u)_x = (B(u)u_x)_x \tag{1.7}$$

has a travelling wave solution $u(x - st)$ that satisfies the boundary conditions (1.5)–(1.6), then we can set $u^\epsilon(x - st) = u((x - st)/\epsilon)$.

A travelling wave solution $u(x - st)$ of (1.7) that satisfies (1.5)–(1.6) exists if and only if the ordinary differential equation (ODE)

$$\dot{u} = B(u)^{-1}(f(u) - f(u_-) - s(u - u_-)) \tag{1.8}$$

has an equilibrium at u_+ (it automatically has one at u_-) and a connecting orbit from u_- to u_+. The condition that (1.8) has an equilibrium at u_+ is just the Rankine–Hugoniot condition (1.3). A shock wave (1.2) that has a corresponding connecting orbit for (1.8) is said to satisfy the *viscous profile criterion for* $B(u)$.

The question of whether (1.8) has an equilibrium at u_+ is independent of $B(u)$. Suppose $Df(u_-)$ is *strictly hyperbolic* (eigenvalues real and distinct), $B(u_-)$ is strictly stable with respect to $Df(u_-)$ (see below), the genuine nonlinearity condition [20] is satisfied at u_-, s is close to an eigenvalue of u_-, and the triple (u_-, s, u_+) satisfies the Rankine–Hugoniot condition. Then the dimensions of the stable and unstable manifolds of $u_\pm$ are also independent of the choice of $B(u)$ [16]. Moreover, Majda and Pego [16] show that there is a connection of (1.8) from u_- to u_+ if and only if the dimensions of $W^u(u_-)$ and $W^s(u_+)$ sum to $n + 1$. Such shock waves are termed *compressive*. Thus, roughly speaking, the existence of the connection is independent of the choice of $B(u)$. However, if we consider u_+ far from u_-, and especially if we consider *undercompressive* shock waves (the dimensions of $W^u(u_-)$ and $W^s(u_+)$ sum to at most n), then the existence of a connection depends strongly on the choice of $B(u)$.

The most important initial value problem for (1.1) is the *Riemann problem*, for which the initial condition is piecewise constant with a jump at $x = 0$:

$$u(x, 0) = \begin{cases} u_L & \text{for } x < 0, \\ u_R & \text{for } x > 0. \end{cases} \tag{1.9}$$

One seeks piecewise continuous weak solutions of Riemann problems in the scale-invariant form $u(x, t) = \hat{u}(\xi)$, $\xi = x/t$. Usually one requires that the solution consists of a finite number of constant parts, continuously changing parts (rarefaction waves) and jump discontinuities (shock waves). Shock waves occur when

$$\lim_{\xi \to s-} \hat{u}(\xi) = u_- \neq u_+ = \lim_{\xi \to s+} \hat{u}(\xi).$$

The triple (u_-, s, u_+) is required to satisfy the viscous profile admissibility criterion for a given $B(u)$.

Riemann problems are solved by piecing together shock waves and rarefaction waves. A more wholistic approach to Riemann problems, based on an artificial regularization of (1.1), was proposed by Dafermos [6].

Dafermos's regularization of (1.1) is

$$u_t + f(u)_x = \epsilon t u_{xx}. \tag{1.10}$$

Like the Riemann problem, but unlike (1.7), (1.10) has many scale-invariant solutions $u(x,t) = \hat{u}(\xi)$, $\xi = x/t$. They satisfy the non-autonomous ODE

$$(Df(u) - \xi I)\frac{\mathrm{d}u}{\mathrm{d}\xi} = \epsilon \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2}, \tag{1.11}$$

where we have written u instead of $\hat{u}$. Corresponding to the initial condition (1.9), Dafermos uses the boundary conditions

$$u(-\infty) = u_L, \qquad u'(-\infty) = 0, \tag{1.12}$$

$$u(+\infty) = u_R, \qquad u'(+\infty) = 0, \tag{1.13}$$

where a prime represents differentiation with respect to ξ. Dafermos conjectured that solutions of the boundary value problem (1.11)–(1.13) should converge to Riemann solutions in the L^1 sense as $\epsilon \to 0$. (Shock waves are to satisfy the viscous profile criterion for $B(u) = I$.) This has been proved for u_R close to u_L by Tzavaras [23].

Recently Szmolyan [21] has taken the opposite point of view. He regards (1.11)–(1.13) as a singular perturbation problem that has a given Riemann solution $\hat{u}(x/t)$ of (1.1), (1.9) as a singular solution when $\epsilon = 0$. Shock waves are assumed to satisfy the viscous profile criterion for $B(u) = I$. If $\hat{u}(x/t)$ is a Riemann solution that consists of n waves with different speeds, each a compressive shock wave or a rarefaction, Szmolyan shows using geometric singular perturbation theory [10] that for small $\epsilon > 0$, (1.11)–(1.13) has a solution near $\hat{u}(\xi)$. A novel aspect of the singular perturbation problem is that normal hyperbolicity is lost along rarefactions. Szmolyan deals with this difficulty by a blowing-up construction.

In fact, the Dafermos regularization can be used with a more general viscosity. In place of (1.10), one uses

$$u_t + f(u)_x = \epsilon t (B(u)u_x)_x. \tag{1.14}$$

A scale-invariant solution $u(x,t) = \hat{u}(\xi)$, $\xi = x/t$, satisfies the non-autonomous ODE

$$(Df(u) - \xi I)u' = \epsilon (B(u)u')'. \tag{1.15}$$

We use the boundary conditions (1.12)–(1.13). If $\hat{u}(x/t)$ is a Riemann solution of (1.1), (1.9) that consists of n waves with different speeds, each a compressive shock wave or a rarefaction, and whose shock waves satisfy the viscous profile criterion for $B(u)$, then Szmolyan's argument shows that for small $\epsilon > 0$, (1.15), (1.12)–(1.13) has a solution near $\hat{u}(\xi)$.

A disturbing fact about Riemann problems is that they sometimes have several solutions [1]. This, of course, does not make physical sense for an initial value problem. However, Riemann solutions play a second role in this subject as asymptotic states of (1.7), a context in which it does make sense for a Riemann problem to have several solutions.

More precisely, let $u(x,t)$ be a solution of (1.7) together with the boundary conditions

$$u(-\infty, t) = u_L, \qquad u(+\infty, t) = u_R, \tag{1.16}$$

and some initial condition $u(x,0) = u_0(x)$. Make the spatial change of coordinates $\xi = x/t$. Then $u(x,t)$ is transformed into

$$\tilde{u}(\xi, t) = u(\xi t, t). \tag{1.17}$$

In numerical computations of solutions $u(x,t)$ of (1.7), it is often observed that as $t \to \infty$, the rescaled solution $\tilde{u}(\xi,t)$ approaches a solution of the Riemann problem (1.1), (1.9), with shock waves that satisfy the viscous profile criterion for $B(u)$ [3].

In particular, multiple solutions of the Riemann problem should correspond to multiple asymptotic states of (1.7), (1.16), which should be approached for different initial conditions $u_0(x)$. This phenomenon has been shown to occur in careful numerical simulations [2].

The rigorous study of Riemann solutions as asymptotic states of (1.7) is not easy, since in general a Riemann solution of (1.1) does not correspond in a natural way to an exact solution of (1.7). However, if the Riemann solution is a single shock wave, then it corresponds to a travelling wave solution of (1.7), so its asymptotic stability can be studied by linearizing (1.7) at this travelling wave. This has been done for both compressive and undercompressive shock waves [14, 15, 24]. Alternatively, energy methods have been used to study the asymptotic stability of Riemann solutions consisting of a single compressive shock wave, a single rarefaction or a combination of weak compressive shock waves [18, 12, 9, 22, 13].

Regarding Riemann solutions as asymptotic states of (1.7), rather than as solutions of initial value problems, sheds a different light on the problem of computing them numerically. Let us consider the somewhat analogous problem of computing equilibrium solutions of the ODE with parameters $\dot{x} = f(x,\lambda)$, with $\lambda \in \mathbb{R}$ for simplicity. One way to do this is to solve an initial value problem $\dot{x} = f(x,\lambda_1)$, $x(0) = x_0$. If the solution tends to an equilibrium x_1 as $t \to \infty$, then one has found a solution (x_1,λ_1) of $f(x,\lambda) = 0$. One can then continue this solution to a curve of solutions by varying λ and repeatedly using Newton's method. It is not necessary to solve more initial value problems; the asymptotic states are computed directly. An advantage of continuation methods is that they easily follow a curve of solutions of $f(x,\lambda) = 0$ around a limit point, thus finding equilibria of $\dot{x} = f(x,\lambda)$ that are not asymptotically stable.

Solving (1.7) numerically, rescaling using (1.17) and observing the limit is analogous to finding an equilibrium of $\dot{x} = f(x,\lambda_1)$ by solving an initial value problem and observing the limit. Unfortunately, there does not seem to be a practical numerical method for accessing the asymptotic states of (1.7) (Riemann solutions) more directly. Numerical methods for (1.1) can be used to solve to (1.1) with Riemann initial data, but they do not accurately locate large or undercompressive shock waves. The reason is that the location and speed of such a wave can depend strongly on the viscous term in (1.7) [4]. However, a numerical method for (1.1) substitutes a numerical viscosity for this term. Another possibility is to construct Riemann solutions geometrically, using wave curves. This is the subject of a large literature, and is implemented for $n = 2$ in the interactive Riemann Problem Package of Isaacson *et al* (available at www.ams.sunysb.edu/˜plohr).

The correspondence between solutions of the boundary value problem (1.15), (1.12)–(1.13), and Riemann solutions of (1.1), (1.9) whose shock waves satisfy the viscous profile criterion for $B(u)$, suggests another approach: compute Riemann solutions by numerically solving the boundary value problem (1.15), (1.12)–(1.13) for a small $\epsilon > 0$. Numerical experiments using this idea are reported in [17]. In order to justify such an approach to interesting Riemann problems, one must show in greater generality that Riemann solutions of (1.1), (1.9), are close to solutions of (1.15), (1.12)–(1.13).

In [19], Schecter *et al* studied *structurally stable* Riemann solutions. These are Riemann solutions that are stable to perturbation of u_L, u_R, and f, in the sense that the nearby Riemann problem has a solution with the same number of waves, of the same types. Although this work was done for $n = 2$ and $B(u) = I$, the notions extend to more general n and $B(u)$. The question of whether a Riemann solution is structurally stable is separate from the question of whether it is asymptotically stable. Again an ODE analogy may be helpful: an equilibrium of $\dot{x} = f(x,\lambda_1)$ for which all eigenvalues of the linearization have non-zero real part is stable to perturbation of λ, but is not asymptotically stable unless all eigenvalues have negative real part.

Peter Szmolyan and I conjecture that for any structurally stable Riemann solution $\hat{u}(x/t)$, the Dafermos regularization has a solution near $\hat{u}(\xi)$ for small $\epsilon > 0$.

In this paper we take a step toward verifying this conjecture. For arbitrary n, we consider Riemann solutions with no rarefactions, consisting of a finite number of constant states and discontinuities. The discontinuities are required to satisfy the viscous profile criterion for a given $B(u)$, but they are not assumed to be compressive. Thus undercompressive shock waves are explicitly allowed. We first show which such solutions are structurally stable. We then show, using the exchange lemma of geometric singular perturbation theory, that all the structurally stable Riemann solutions have solutions of the Dafermos regularization nearby.

Throughout the paper we consider (1.1) and a fixed parabolic regularization (1.4), where all eigenvalues of $B(u)$ have positive real part. Whenever we consider a shock wave (1.2), we assume that for both $u_0 = u_-$ and $u_0 = u_+$, (i) $Df(u_0)$ is strictly hyperbolic, (ii) s is not an eigenvalue of $Df(u_0)$, and (iii) $B(u_0)$ is *strictly stable with respect to* $Df(u_0)$. Strict stability is defined as follows. Let $Df(u_0)$ have right eigenvectors $r_1, \dots, r_n$, and corresponding left eigenvectors $l_1, \dots, l_n$. Then $B(u_0)$ is strictly stable with respect to $Df(u_0)$ provided

(a) $l_k B(u_0) r_k > 0$ for $k = 1, \dots, n$; and
(b) the symbol $P(\zeta) = -\zeta^2 B(u_0) - \mathrm{i}\zeta Df(u_0)$ has no pure imaginary eigenvalues for $\zeta \neq 0$.

The rest of the paper is organized as follows. Sections 2, 3, and 4 give definitions and lemmas. In section 5 we characterize structurally stable Riemann solutions that contain only shock waves. In section 6 we show that these Riemann solutions have solutions of the Dafermos regularization nearby.

2. Regular shock waves

In this paper we will consider only *regular shock waves*. This means that that the connecting orbits of (1.8) are required to connect equilibria at which all eigenvalues have non-zero real part, and the unstable and stable manifolds of these equilibria are required to intersect in a regular manner.

In order to define regular shock waves more precisely, we first define an equilibrium u_0 of a differential equation $\dot{u} = g(u)$ on $\mathbb{R}^n$ to have *type k* if $Dg(u_0)$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. (We will not need to consider equilibria at which some eigenvalue has zero real part.)

Consider (1.8), a family of ODEs on $\mathbb{R}^n$ with parameters (u_-, s). The following result is proved in [16].

Proposition 2.1. *Let u_0 be an equilibrium of (1.8) with (u_-, s) fixed. Assume that $Df(u_0)$ is strictly hyperbolic, with eigenvalues $\lambda_1 < \cdots < \lambda_n$, and assume that $B(u_0)$ is strictly stable with respect to $Df(u_0)$. Then*

(a) The equilibrium u_0 has type 0 if and only if $s < \lambda_1$.
(b) For $k = 1, \dots, n-1$, u_0 has type k if and only if $\lambda_k < s < \lambda_{k+1}$.
(c) The equilibrium u_0 has type n if and only if $\lambda_n < s$.

Let $w = (u_-, s, u_+, \Gamma)$ with $u_\pm \in \mathbb{R}^n$, $s \in \mathbb{R}$, and $\Gamma \subset \mathbb{R}^n$. Assume that (u_-, s, u_+) satisfies the Rankine–Hugoniot condition (1.3), that $Df(u_\pm)$ is strictly hyperbolic, and that $B(u_\pm)$ is strictly stable with respect to $Df(u_\pm)$. Then w is a *shock wave of type* (k_-, k_+) if the ODE (1.8) has equilibria of type $k_\pm$ at $u_\pm$, and Γ is a connecting orbit from u_- to u_+.

Let $w = (u_-, s, u_+, \Gamma)$ be a shock wave of type (k_-, k_+). Let $W^u(u_-, (u_-, s))$ denote the unstable manifold of u_- for (1.8), which has dimension $n - k_-$, and let $W^s(u_+, (u_-, s))$ denote the stable manifold of u_+ for (1.8), which has dimension k_+. $W^u(u_-, (u_-, s))$ and $W^s(u_+, (u_-, s))$ have *regular intersection along* Γ if at any point of Γ, the dimension of the intersection of the tangent spaces with $W^u(u_-, (u_-, s))$ and $W^s(u_+, (u_-, s))$ is $\max(1, k_+ - k_-)$. Equivalently, the sum of these tangent spaces has dimension $\min(n + k_+ - k_- - 1, n)$.

A *regular shock wave* is a shock wave $w = (u_-, s, u_+, \Gamma)$ of one of the types (k_-, k_+) defined above, such that $W^u(u_-, (u_-, s))$ and $W^s(u_+, (u_-, s))$ have regular intersection along Γ.

We distinguish three kinds of regular shock waves.

(a) Overcompressive: $k_+ > 1 + k_-$. $W^u(u_-, (u_-, s))$ and $W^s(u_+, (u_-, s))$ intersect transversally along Γ in a manifold of connecting orbits of dimension $k_+ - k_- > 1$. This manifold of connecting orbits persists when (u_-, s) varies. Example: $n = 2$, $k_- = 0$ (repeller), $k_+ = 2$ (attractor).
(b) Compressive: $k_+ = 1 + k_-$. $W^u(u_-, (u_-, s))$ and $W^s(u_+, (u_-, s))$ intersect transversally along Γ in a manifold of dimension 1, namely Γ. The connecting orbit persists when (u_-, s) varies. Examples: $n = 2$, $k_- = 0$ (repeller), $k_+ = 1$ (saddle)—a Lax one-shock; $n = 2$, $k_- = 1$ (saddle), $k_+ = 2$ (attractor)—a Lax two-shock.
(c) Undercompressive: $k_+ < 1 + k_-$. Existence of the connecting orbit is a phenomenon of codimension $1 + k_- - k_+ > 0$. Example: $n = 2$, $k_- = 1$ (saddle), $k_+ = 1$ (saddle). Existence of the connection is a codimension-one phenomenon.

Let $w^* = (u_-^*, s^*, u_+^*, \Gamma^*)$ be a regular shock wave of type $T = (k_-, k_+)$. A point (u_-, s, u_+) near (u_-^*, s^*, u_+^*) also represents a regular shock wave of type T, with connecting orbit Γ near Γ^*, provided a system of e_T equations in the variables (u_-, s, u_+) is satisfied, where

$$
\begin{aligned}
e_T &:= n + \max(0, 1 + k_- - k_+) \\
&= \begin{cases} n & \text{if } T \text{ is an overcompressive or compressive type,} \\ n + 1 + k_- - k_+ & \text{if } T \text{ is an undercompressive type.} \end{cases}
\end{aligned}
$$

The equations are the Rankine–Hugoniot condition (1.3), $R(u_-, s, u_+) = 0$ (all cases), plus $1 + k_- - k_+$ additional equations in the undercompressive case. We denote these equations $S_1(u_-, s) = 0, \ldots, S_l(u_-, s) = 0$, $l = 1 + k_- - k_+$, and we define the *separation function* $S(u_-, s) := (S_1(u_-, s), \ldots, S_l(u_-, s))$. (The separation function is discussed in more detail in the next section.) We define $G_T(u_-, s, u_+)$ to be $R(u_-, s, u_+)$ if T is an overcompressive or compressive type, and to be $(R(u_-, s, u_+), S(u_-, s))$ if T is an undercompressive type. Then (u_-, s, u_+) near (u_-^*, s^*, u_+^*) also represents a shock wave of type T, with connecting orbit Γ near Γ^*, provided the system $G_T(u_-, s, u_+) = 0$ is satisfied.

(Γ^* and Γ are considered close provided there are corresponding solutions of $u^*(t)$ of $\dot{u} = H(u, \lambda^*)$ and $u(t)$ of $\dot{u} = H(u, \lambda)$ that are close in the sup norm on $C^0(\mathbb{R}, \mathbb{R}^n)$.)

In the remainder of the paper, *all shock waves are assumed to be regular*.

3. Separation function

Let us explain the separation function S, and its relation to transversality of unstable and stable manifolds, in more detail. Let $(u_-^*, s^*, u_+^*, \Gamma^*)$ be an undercompressive shock wave of type $T = (k_-, k_+)$. We shall write (1.8) as

$$\dot{u} = H(u, \lambda), \tag{3.1}$$

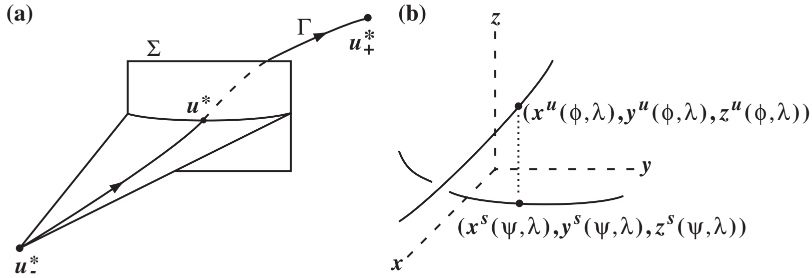

Figure 1. Geometry of the separation function. In (*a*), *u*-space is shown for $\lambda = \lambda^*$, with $n = 3$ and $(k_-, k_+) = (1, 2)$. The two-dimensional unstable manifold of u_-^* meets Σ in a curve, and the one-dimensional stable manifold of u_+^* meets Σ in a point. In (*b*), Σ is shown for a λ near λ^*, with $n = 4$ and $(k_-, k_+) = (2, 2)$. The unstable manifold of $u_-(\lambda)$ and the stable manifold of $u_+(\lambda)$ meet Σ in curves. In the *xyz*-coordinates used in the proof of proposition 3.1, the former is near the *x*-axis, the latter near the *y*-axis. The signed length of the dotted line is $S(\lambda)$.

where λ lies in a *p*-dimensional submanifold Λ of u_-s-space that contains $\lambda^* = (u_-^*, s^*)$. (This generality will be needed later.) Near u_-^* and u_+^* are equilibria $u_-(\lambda)$ and $u_+(\lambda)$ of (3.1). Let Σ be an $(n-1)$-dimensional submanifold of *u*-space that is transverse to Γ at a point u^*. The unstable manifold of $u_-(\lambda)$ for (3.1) meets Σ in a surface of dimension $n - k_- - 1$ that depends on λ. We parametrize this family of surfaces as $u^u(\phi, \lambda)$, where $\phi \in \mathbb{R}^{n-k_--1}$ and $u^u(0, \lambda^*) = u^*$. Similarly, the stable manifold of $u_+(\lambda)$ for (3.1) meets Σ in a surface of dimension $k_+ - 1$ that depends on λ. We parametrize this family of surfaces as $u^s(\psi, \lambda)$, where $\psi \in \mathbb{R}^{k_+-1}$ and $u^s(0, \lambda^*) = u^*$ (see figure 1).

Without loss of generality we may assume that $D_\phi u^u(0, \lambda^*)$ and $D_\psi u^s(0, \lambda^*)$ are injective. Then, since we are considering an undercompressive shock wave, the regular intersection assumption is equivalent to

$$\text{Range } D_\phi u^u(0, \lambda^*) \cap \text{Range } D_\psi u^s(0, \lambda^*) = \{0\}.$$

Proposition 3.1. *There is a map S from* Λ *to* $\mathbb{R}^{1+k_--k_+}$ *such that (3.1) has a connecting orbit from* $u_-(\lambda)$ *to* $u_+(\lambda)$ *near* Γ^* *if and only if* $S(\lambda) = 0$*.*

Proof. There is such a connection if and only if there is a triple (ϕ, ψ, λ) such that

$$L(\phi, \psi, \lambda) := u^u(\phi, \lambda) - u^s(\psi, \lambda) = 0. \tag{3.2}$$

We may choose coordinates (x, y, z) on Σ, $x \in \mathbb{R}^{n-k_--1}$, $y \in \mathbb{R}^{k_+-1}$, $z \in \mathbb{R}^{1+k_--k_+}$, such that $W^u(u_-(\lambda^*), \lambda^*)$ is the set $y = z = 0$, and $W^s(u_+(\lambda^*), \lambda^*)$ is the set $x = z = 0$. In terms of these coordinates,

$$L(\phi, \psi, \lambda) = (x^u(\phi, \lambda) - x^s(\psi, \lambda), y^u(\phi, \lambda) - y^s(\psi, \lambda), z^u(\phi, \lambda) - z^s(\psi, \lambda)),$$

with $y^u(\phi, \lambda^*) = 0$, $x^s(\psi, \lambda^*) = 0$, and $z^u(\phi, \lambda^*) = z^s(\psi, \lambda^*) = 0$. Then

$$DL(0, 0, \lambda^*) = \begin{pmatrix} D_\phi x^u(0, \lambda^*) & 0 & D_\lambda x^u(0, \lambda^*) - D_\lambda x^s(0, \lambda^*) \\ 0 & -D_\psi y^s(0, \lambda^*) & D_\lambda y^u(0, \lambda^*) - D_\lambda y^s(0, \lambda^*) \\ 0 & 0 & D_\lambda z^u(0, \lambda^*) - D_\lambda z^s(0, \lambda^*) \end{pmatrix}.$$

Since the matrix

$$\begin{pmatrix} D_\phi x^u(0, \lambda^*) & 0 \\ 0 & -D_\psi y^s(0, \lambda^*) \end{pmatrix}$$

is invertible, the implicit function theorem implies that the system

$$x^u(\phi, \lambda) - x^s(\psi, \lambda) = 0, \qquad y^u(\phi, \lambda) - y^s(\psi, \lambda) = 0$$

can be solved for (ϕ, ψ) in terms of λ. Then $L(\phi, \psi, \lambda) = 0$ if and only if

$$S(\lambda) := z^u(\phi(\lambda), \lambda) - z^s(\psi(\lambda), \lambda) = 0. \qquad \square$$

In order to state the following result, we add to the differential equation (3.1) the differential equation

$$\dot{\lambda} = 0, \tag{3.3}$$

thus obtaining a system on $\mathbb{R}^n \times \Lambda$. This system has the normally hyperbolic manifolds of equilibria

$$P_- = \{(u, \lambda) : u = u_-(\lambda)\} \text{ and } P_+ = \{(u, \lambda) : u = u_+(\lambda)\}.$$

$W^u(P_-)$ consists of all pairs (u, λ) such that u is in the unstable manifold of $u_-(\lambda)$ for (3.1); $W^s(P_-)$, $W^u(P_+)$, and $W^s(P_+)$ are defined analogously. $W^u(P_-)$ intersects $W^s(P_+)$ along $\Gamma^* \times \{\lambda^*\}$.

Proposition 3.2. *The following are equivalent.*

(a) Range $D_\phi u^u(0, \lambda^*) +$ *Range* $D_\psi u^s(0, \lambda^*) +$ *Range* $(D_\lambda u^u(0, \lambda^*) - D_\lambda u^s(0, \lambda^*)) = T_{u^*}\Sigma$.
(b) $DS(\lambda^*)$ *is surjective.*
(c) $W^u(P_-)$ *intersects* $W^s(P_+)$ *transversally along* $\Gamma^* \times \{\lambda^*\}$.

Proof. For simplicity, we assume in the proof that Λ is u_-s-space.

Now $DS(\lambda^*) = D_\lambda z^u(0, \lambda^*) - D_\lambda z^s(0, \lambda^*)$. Therefore, $DS(\lambda^*)$ is surjective if and only if (a) holds, so (a) and (b) are equivalent.

Statement (c) holds if and only if $W^u(P_-) \cap \Sigma \times \Lambda$ and $W^s(P_+) \cap \Sigma \times \Lambda$ meet transversally within $\Sigma \times \Lambda$ at (u^*, λ^*). $W^u(P_-) \cap \Sigma \times \Lambda$ and $W^s(P_+) \cap \Sigma \times \Lambda$ are parametrized by $(u^u(\phi, \lambda), \lambda)$ and $(u^s(\psi, \lambda), \lambda)$, respectively. Their tangent spaces at (u^*, λ^*) are the ranges of

$$\begin{pmatrix} D_\phi u^u(0, \lambda^*) & D_\lambda u^u(0, \lambda^*) \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} D_\psi u^s(0, \lambda^*) & D_\lambda u^s(0, \lambda^*) \\ 0 & I \end{pmatrix},$$

respectively. The sum of these tangent spaces can be written as the span of the column vectors in the matrix

$$\begin{pmatrix} D_\phi u^u(0, \lambda^*) & D_\lambda u^u(0, \lambda^*) - D_\lambda u^s(0, \lambda^*) & D_\psi u^s(0, \lambda^*) & D_\lambda u^s(0, \lambda^*) \\ 0 & 0 & 0 & I \end{pmatrix}. \tag{3.4}$$

This span is $T_{u^*}\Sigma \times T_{\lambda^*}\Lambda$ if and only if (a) holds. $\square$

4. Lemmas about shock waves

In this section we gather several lemmas.

Lemma 4.1. *Let* $(u_-^*, s^*, u_+^*, \Gamma^*)$ *be a regular shock wave. Let* V *be a* k*-dimensional subspace of* $\dot{u}_-$*-space with* $k < n$. *Then* $DR(u_-^*, s^*, u_+^*)$*, restricted to* $\{(\dot{u}_-, \dot{s}, \dot{u}_+) : \dot{u}_- \in V\}$*, is surjective. Moreover, let* K *denote the kernel of* $DR(u_-^*, s^*, u_+^*)$ *restricted to* $\{(\dot{u}_-, \dot{s}, \dot{u}_+) : \dot{u}_- \in V\}$*, which has dimension* $k + 1$. *Then the following are equivalent:*

(a) $(Df(u_-^*) - s^* I)^{-1}(u_+^* - u_-^*) \notin V$;
(b) K *projects regularly to* $\dot{u}_+$*-space.*

Proof. We have

$$DR(u_-^*, s^*, u_+^*)(\dot u_-, \dot s, \dot u_+) = (Df(u_+^*) - s^*I)\dot u_+ - (Df(u_-^*) - s^*I)\dot u_- - \dot s(u_+^* - u_-^*). \tag{4.1}$$

The first conclusion follows from invertibility of $Df(u_+^*) - s^*I$.

The vector $(\dot u_-, \dot s, \dot u_+) \in K$ if and only if $\dot u_- \in V$ and

$$\dot u_+ = (Df(u_+^*) - s^*I)^{-1}((Df(u_-^*) - s^*I)\dot u_- + \dot s(u_+^* - u_-^*)). \tag{4.2}$$

The second conclusion follows from this formula. □

Lemma 4.2. *Let $(u_-^*, s^*, u_+^*, \Gamma^*)$ be an undercompressive shock wave of type (k_-, k_+). Let $l = 1 + k_- - k_+ > 0$. Let V be a k-dimensional subspace of $\dot u_-$-space with $k < n$. Then the following are equivalent:*

(a) $DS(u_-^*, s^*)$ is surjective, and $V \times \dot s$-space is transverse to the kernel of $DS(u_-^*, s^*)$.
(b) $DG_T(u_-^*, s^*, u_+^*)$, restricted to $\{(\dot u_-, \dot s, \dot u_+) : \dot u_- \in V\}$, is surjective.

Moreover, if one of the above conditions holds, let K denote the kernel of $DG_T(u_-^*, s^*, u_+^*)$ restricted to $\{(\dot u_-, \dot s, \dot u_+) : \dot u_- \in V\}$, which has dimension $k + 1 - l$. Then the following are equivalent:

(c) If $(\dot u_-, \dot s)$ is a non-zero vector such that $\dot u_- \in V$ and $DS(u_-^*, s^*)(\dot u_-, \dot s) = 0$, then

$$(Df(u_-^*) - s^*I)\dot u_- + \dot s(u_+^* - u_-^*) \neq 0.$$

(d) K projects regularly to $\dot u_+$-space.

Proof. Let K_1 be the set of $(\dot u_-, \dot s)$ such that $\dot u_- \in V$ and $DS(u_-^*, s^*)(\dot u_-, \dot s) = 0$. Statements (a) and (b) are each equivalent to the assertion that K_1 has dimension $k+1-l$. The equivalence of statements (c) and (d) follows from (4.2). □

The following proposition follows from lemmas 4.1 and 4.2 and an application of the implicit function theorem to the equation $G_T(u_-, s, u_+) = 0$.

Proposition 4.3. *Let $(u_-^*, s^*, u_+^*, \Gamma^*)$ be a compressive or undercompressive shock wave of type $T = (k_-, k_+)$. Let M be a k-dimensional submanifold of u_--space through u_-^* with $k < n$. Let $V = T_{u_-^*}M$. If T is a compressive type, assume that V satisfies condition (1) of lemma 4.1; if T is an undercompressive type, assume that V satisfies conditions (1) and (3) of lemma 4.2. Let $l = 1 + k_- - k_+ \geqslant 0$. Let K denote the kernel of $DG_T(u_-^*, s^*, u_+^*)$ restricted to $\{(\dot u_-, \dot s, \dot u_+) : \dot u_- \in V\}$; K has dimension $k + 1 - l$. Let $\tilde K$ denote the projection of K onto $\dot u_+$-space. Then there is a $(k+1-l)$-dimensional submanifold $\tilde M$ of u_+-space through u_+^* such that (1) $\tilde K = T_{u_-^*}\tilde M$ and (2) for $u_- \in M$, there is a shock wave (u_-, s, u_+, Γ) of type T near $(u_-^*, s^*, u_+^*, \Gamma^*)$ if and only if $u_+ \in \tilde M$. Moreover, there is a smooth function $(u_-(u_+), s(u_+))$ defined on $\tilde M$ that gives the other end and speed of the connection.*

5. Structurally stable Riemann solutions

Recall that we are considering (1.1) with a fixed parabolic regularization (1.7).

Let $\sigma = (u_0, s_1, u_1, \ldots, u_{m-1}, s_m, u_m)$ (the u's and s's alternate), with each $u_i \in \mathbb{R}^n$, each $s_i \in \mathbb{R}$, and $s_1 < s_2 < \cdots < s_m$. Let $\hat\sigma = (\Gamma_1, \ldots, \Gamma_m)$ with each $\Gamma_i \subset \mathbb{R}^n$. Assume that each four-tuple $w_i = (u_{i-1}, s_i, u_i, \Gamma_i)$, $i = 1, \ldots, m$, is a regular shock wave. Then the pair of sequences $(\sigma, \hat\sigma)$ define a solution of the Riemann problem (1.1), (1.9), with $u_L = u_0$

and $u_R = u_m$, that consists of m shock waves. We shall refer to $(\sigma, \hat{\sigma})$ as a *Riemann solution*. Of course, there are other sorts of Riemann solutions, but we will not discuss them in this paper.

Let the type of w_i be T_i. Then the *type* of the Riemann solution is $(T_1, T_2, \ldots, T_m)$. The condition $s_1 < s_2 < \cdots < s_m$ and proposition 2.1 imply that if $T_i = (k_-^i, k_+^i)$ and $T_{i+1} = (k_-^{i+1}, k_+^{i+1})$, then

$$k_+^i \leqslant k_-^{i+1}. \tag{5.1}$$

Let $\sigma^* = (u_0^*, s_1^*, u_1^*, \ldots, u_{m-1}^*, s_m^*, u_m^*)$, let $\hat{\sigma}^* = (\Gamma_1^*, \ldots, \Gamma_m^*)$, and let $(\sigma^*, \hat{\sigma}^*)$ be a Riemann solution of type $(T_1, T_2, \ldots, T_m)$. Let $(u_0, s_1, u_1, \ldots, u_{m-1}, s_m, u_m)$ be a point near $(u_0^*, s_1^*, u_1^*, \ldots, u_{m-1}^*, s_m^*, u_m^*)$ in $\mathbb{R}^{mn+m+n}$. There is a corresponding m-tuple of connections $(\Gamma_1, \ldots, \Gamma_m)$ near $(\Gamma_1^*, \ldots, \Gamma_m^*)$ provided a system of $e_{T_1} + \cdots + e_{T_m}$ equations is satisfied. The system is

$$G(u_0, s_1, u_1, \ldots, u_{m-1}, s_m, u_m) := (G_{T_1}(u_0, s_1, u_1), \ldots, G_{T_m}(u_{m-1}, s_m, u_m)) = 0. \tag{5.2}$$

The map G goes from $\mathbb{R}^{mn+m+n}$ to $\mathbb{R}^{e_{T_1}+\cdots+e_{T_m}}$.

We shall say that the Riemann solution $(\sigma^*, \hat{\sigma}^*)$ is *structurally stable* provided $DG(u_0^*, s_1^*, u_1^*, \ldots, u_{m-1}^*, s_m^*, u_m^*)$, restricted to the $(mn+m-n)$-dimensional space of vectors $(\dot{u}_0, \dot{s}_1, \dot{u}_1, \ldots, \dot{u}_{m-1}, \dot{s}_m, \dot{u}_m)$ with $\dot{u}_0 = \dot{u}_m = 0$, is invertible. When this condition holds, the implicit function theorem implies that the equation $G = 0$ can be solved for $(s_1, u_1, \ldots, u_{m-1}, s_m)$ in terms of (u_0, u_m) near (u_0^*, u_m^*). Thus, for each (u_0, u_m) near (u_0^*, u_m^*), there is a Riemann solution $(\sigma, \hat{\sigma})$ near $(\sigma^*, \hat{\sigma}^*)$. The two Riemann solutions have the same type, and $(s_1, u_1, \ldots, u_{m-1}, s_m)$ depends smoothly on (u_0, u_m).

An obvious necessary condition for structural stability is that

$$\sum_{i=1}^{m} e_{T_i} = mn + m - n. \tag{5.3}$$

If $T = (k_-, k_+)$ is a shock wave type, define

$$\begin{aligned} \rho(T) &= n + 1 - e_T \\ &= \begin{cases} 1 & \text{if } T \text{ is an overcompressive or compressive type,} \\ k_+ - k_- < 1 & \text{if } T \text{ is an undercompressive type.} \end{cases} \end{aligned}$$

Then

$$\rho(T) = \min(1, k_+ - k_-), \tag{5.4}$$

and equation (5.3) holds if and only if

$$\sum_{i=1}^{m} \rho(T_i) = n. \tag{5.5}$$

Theorem 5.1. *Let $(\sigma^*, \hat{\sigma}^*)$ be a Riemann solution of type $(T_1, T_2, \ldots, T_m)$ with $T_i = (k_-^i, k_+^i)$. Then*

(a) $\sum_{i=1}^{m} \rho(T_i) \leqslant n$.

(b) $\sum_{i=1}^{m} \rho(T_i) = n$ *if and only if*

1. $k_-^1 = 0$;
2. $k_+^m = n$;
3. *no waves are overcompressive;*
4. *if* $T_i = (k_-^i, k_+^i)$ *and* $T_{i+1} = (k_-^{i+1}, k_+^{i+1})$, *then* $k_+^i = k_-^{i+1}$.

Proof. From (5.4) and (5.1),

$$\sum_{i=1}^{m}\rho(T_i) \leqslant \sum_{i=1}^{m}(k_+^i - k_-^i) = -k_-^1 + \sum_{i=1}^{m-1}(k_+^i - k_-^{i+1}) + k_+^m \leqslant k_+^m - k_-^1 \leqslant n. \tag{5.6}$$

Thus (1) is proved.

$\sum_{i=1}^{m}\rho(T_i) = n$ if and only if all inequalities in (5.6) are equalities. The first inequality is an equality if and only if $\rho(T_i) = k_+^i - k_-^i$ for all i, i.e. no waves are overcompressive. The second inequality is an equality if and only if $k_+^i = k_-^{i+1}$ for all i. The third inequality is an equality if and only if $k_+^m = n$ and $k_-^1 = 0$. Thus (2) is proved. □

Corollary 5.2. *Let* $(\sigma^*, \hat{\sigma}^*)$ *be a Riemann solution of type* $(T_1, T_2, \dots, T_m)$. *Then* $\sum_{i=1}^{m}\rho(T_i) = n$ *if and only if there is a sequence* $k_0, \dots, k_m$ *such that* $k_0 = 0$, $0 < k_i < n$ *for* $i = 1, \dots, m-1$, $k_m = n$, $k_i \leqslant k_{i-1} + 1$ *for* $i = 1, \dots, m$, *and* $T_i = (k_{i-1}, k_i)$ *for* $i = 1, \dots, m$.

In other words: k_0 is 0; at each stage, either k_i increases by one (compressive shock wave), or it stays the same or decreases (undercompressive shock wave); k_i never decreases to 0 (since no orbit can end at an equilibrium at which all eigenvalues have positive real part); when k_i reaches n, the sequence ends.

Assuming that the Riemann solution $(\sigma^*, \hat{\sigma}^*)$ satisfies the necessary condition (5.5) for structural stability, let us investigate when $DG(u_0^*, s_1^*, u_1^*, \dots, u_{m-1}^*, s_m^*, u_m^*)$, restricted to the appropriate subspace, is invertible.

Let $(\sigma^*, \hat{\sigma}^*)$ be a Riemann solution of type $(T_1, T_2, \dots, T_m)$ with $\sum_{i=1}^{m}\rho(T_i) = n$. Let $(k_0, \dots, k_m)$ be the sequence given by corollary 5.2. We shall inductively construct a sequence V_i, $i = 0, \dots, m-1$, such that each V_i is a k_i-dimensional subspace of u_i-space. Since $k_0 = 0$, let $V_0 = 0$. Suppose V_{i-1} is a k_{i-1}-dimensional subspace of u_{i-1}-space that satisfies condition (a) of lemma 4.1 when T_i is a compressive type, and conditions (a) and (c) of lemma 4.2 when T_i is an undercompressive type. Then the projection of the kernel of $DG_{T_i}(u_{i-1}^*, s_i^*, u_i^*)$, restricted to $\{(\dot{u}_{i-1}, \dot{s}_i, \dot{u}_i) : \dot{u}_{i-1} \in V_{i-1}\}$, onto $\dot{u}_i$-space has dimension $k_{i-1} + 1 = k_i$ in the compressive case, and dimension $k_{i-1} + 1 - l = k_i$ in the undercompressive case. We define the projection to be V_i. If V_i satisfies the appropriate transversality conditions, we can continue the construction. This motivates the hypotheses of the following theorem.

In order to state the theorem we first define a one-dimensional subspace $\tilde{V}_{m-1}$ of $\dot{u}_{m-1}$-space: $\tilde{V}_{m-1}$ is the projection onto $\dot{u}_{m-1}$-space of the kernel of $DR(u_{m-1}^*, s_m^*, u_m^*)$ restricted to $\{(\dot{u}_{m-1}, \dot{s}_m, \dot{u}_m) : \dot{u}_m = 0\}$.

Theorem 5.3. *Let* $(\sigma^*, \hat{\sigma}^*)$ *be a Riemann solution of type* $(T_1, T_2, \dots, T_m)$ *where* $\sum_{i=1}^{m}\rho(T_i) = n$. *Let* $(k_0, \dots, k_m)$ *be the sequence given by corollary 5.2. Assume that there are* k_i*-dimensional subspaces* V_i *of* $\dot{u}_i$*-space,* $i = 0, \dots, m-1$, *such that*

(a) $V_0 = 0$.
(b) For $i = 1, \dots, m-1$, V_{i-1} *satisfies condition (1) of lemma 4.1 when* T_i *is a compressive type, and conditions (a) and (c) of lemma 4.2 when* T_i *is an undercompressive type.*
(c) For $i = 1, \dots, m-1$, V_i *is the projection of the kernel of* $DG_{T_i}(u_{i-1}^*, s_i^*, u_i^*)$, *restricted to* $\{(\dot{u}_{i-1}, \dot{s}_i, \dot{u}_i) : \dot{u}_{i-1} \in V_{i-1}\}$, *onto* $\dot{u}_i$*-space.*
(d) V_{m-1} *is transverse to* $\tilde{V}_{m-1}$.

Then the Riemann solution $(\sigma^*, \hat{\sigma}^*)$ *is structurally stable. Such a sequence* $V_0, \dots, V_{m-1}$, *if it exists, is unique and can be constructed inductively. Moreover, if such a sequence* $V_0, \dots, V_{m-1}$ *does not exist, then* $(\sigma^*, \hat{\sigma}^*)$ *is not structurally stable.*

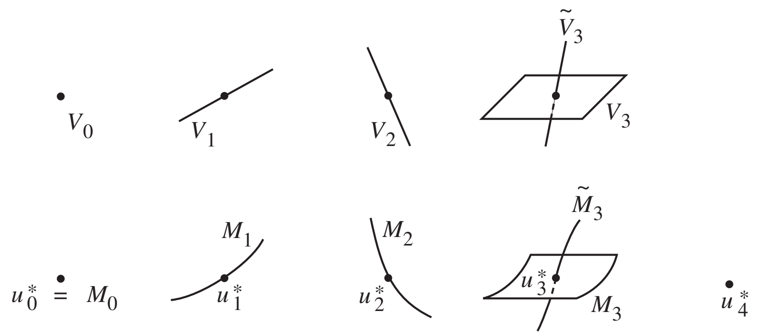

Figure 2. Example illustrating theorem 5.3 and corollary 5.4 with $n = 3$ and $m = 4$. In this example, $(k_0, k_1, k_2, k_3, k_4) = (0, 1, 1, 2, 3)$. The Riemann solution consists of a Lax one-shock from u_0^* to u_1^*, an undercompressive shock wave from u_1^* to u_2^*, and a Lax two-shock from u_2^* to u_3^* and a Lax three-shock from u_3^* to u_4^*. M_1 is the curve of points u_1 near u_1^* for which there is a Lax one-shock from u_0^* to u_1. For each $u_1 \in M_1$ there is a unique speed for which there is an undercompressive shock wave from u_1 to some u_2 near u_2^*. M_2 is the curve of all such u_2. For each $u_2 \in M_2$, there is a curve of points u_3 near u_3^* such that there is a Lax two-shock from u_2 to u_3. M_3 is the surface of all such u_3. $\tilde{M}_3$ is the curve of u_3 near u_3^* for which there is a Lax three-shock from u_3 to u_4^*. The V_i are the tangent spaces to the M_i.

Proof. If such a sequence $V_0, \ldots, V_{m-1}$ exists, then there are no non-zero vectors in the kernel of $DG(u_0^*, s_1^*, u_1^*, \ldots, u_{m-1}^*, s_m^*, u_m^*)$ restricted to the space of vectors $(\dot{u}_0, \dot{s}_1, \dot{u}_1, \ldots, \dot{u}_{m-1}, \dot{s}_m, \dot{u}_m)$ with $\dot{u}_0 = \dot{u}_m = 0$.

The spaces V_i are constructed inductively. We must have $k_1 = 1$, and V_0 automatically satisfies condition (a) of lemma 4.1 for waves of type (0,1). Therefore, V_1 can be constructed. If at any stage $i = 2, \ldots, m-1$, V_{i-1} fails to satisfy the appropriate conditions, we easily obtain a non-zero vector in the kernel of $DG(u_0^*, s_1^*, u_1^*, \ldots, u_{m-1}^*, s_m^*, u_m^*)$ restricted to the space of vectors $(\dot{u}_0, \dot{s}_1, \dot{u}_1, \ldots, \dot{u}_{m-1}, \dot{s}_m, \dot{u}_m)$ with $\dot{u}_0 = \dot{u}_m = 0$. If all V_i can be constructed but condition (d) fails, we again obtain such a vector. □

Corollary 5.4. *Assume the hypothesis of theorem 5.3. Then for each $j = 1, \ldots, m-1$, there is a k_j-dimensional manifold M_j through u_j^* in u_j-space, tangent there to V_j, and a mapping $(s_1, u_1, \ldots, u_{j-2}, s_{j-1})(u_j)$ defined on M_j, such that there is an admissible wave sequence $(u_0^*, s_1, u_1, \ldots, u_{j-1}, s_j, u_j)$ of type $(T_1, \ldots, T_j)$ near $(u_0^*, s_1^*, u_1^*, \ldots, u_{j-1}^*, s_j^*, u_j^*)$, with connecting orbits $(\Gamma_1, \ldots, \Gamma_j)$ near $(\Gamma_1^*, \ldots, \Gamma_j^*)$, if and only if $u_j \in M_j$ and*

$$(s_1, u_1, \ldots, u_{j-2}, s_{j-1}) = (s_1, u_1, \ldots, u_{j-2}, s_{j-1})(u_j).$$

The proof is by induction using proposition 4.3. To start the induction, let $M_0 = \{u_0^*\}$ and use proposition 4.3 to define M_1.

Theorem 5.3 and corollary 5.4 are illustrated in figure 2.

6. Dafermos regularization

Following [21], we convert the non-autonomous second-order ODE (1.15) into an autonomous first-order ODE by letting $v = \epsilon B(u)u'$ and treating ξ as a state variable:

$$\epsilon u' = B(u)^{-1} v, \tag{6.1}$$

$$\epsilon v' = (Df(u) - \xi I)B(u)^{-1} v, \tag{6.2}$$

$$\xi' = 1. \tag{6.3}$$

As an autonomous ODE, the system (6.1)–(6.3) is a singular perturbation problem written in the slow time η, with $d\xi/d\eta = 1$ (i.e. $\xi = \eta + \xi_0$). Here the prime symbol denotes a derivative with respect to η.

We let $\eta = \epsilon\tau$, and we use a dot to denote differentiation with respect to τ. System (6.1)–(6.3) becomes

$$\dot{u} = B(u)^{-1}v, \tag{6.4}$$

$$\dot{v} = (Df(u) - \xi I)B(u)^{-1}v, \tag{6.5}$$

$$\dot{\xi} = \epsilon. \tag{6.6}$$

System (6.4)–(6.6) is system (6.1)–(6.3) written in the fast time τ. The boundary conditions (1.12)–(1.13) become

$$(u, v, \xi)(-\infty) = (u_L, 0, -\infty), \qquad (u, v, \xi)(\infty) = (u_R, 0, \infty). \tag{6.7}$$

Setting $\epsilon = 0$ in (6.4)–(6.6) yields the fast limit system

$$\dot{u} = B(u)^{-1}v, \tag{6.8}$$

$$\dot{v} = (Df(u) - \xi I)B(u)^{-1}v, \tag{6.9}$$

$$\dot{\xi} = 0. \tag{6.10}$$

The set $v = 0$ is invariant under (6.4)–(6.6) for every ϵ. For a small $\delta > 0$, let

$$S_0 = \left\{(u, v, \xi)\ :\ \|u\| \leqslant \frac{1}{\delta}, v = 0, \ \text{ and }\ \xi \leqslant \lambda_1(u) - \delta\right\},$$

$$S_k = \left\{(u, v, \xi)\ :\ \|u\| \leqslant \frac{1}{\delta}, v = 0, \ \text{ and }\ \lambda_k(u) + \delta \leqslant \xi \leqslant \lambda_{k+1}(u) - \delta\right\},$$
$$k = 1, \ldots, n-1,$$

$$S_n = \left\{(u, v, \xi) : \|u\| \leqslant \frac{1}{\delta}, v = 0, \ \text{ and }\ \lambda_n(u) + \delta \leqslant \xi\right\}.$$

For the system (6.8)–(6.10), each S_k, $k = 1, \ldots, n-1$, is a compact $(n+1)$-dimensional normally hyperbolic manifold of equilibria [7, 10]. S_0 and S_n can be compactified at $\xi = -\infty$ and $\xi = \infty$, respectively, to produce compact $(n+1)$-dimensional normally hyperbolic manifold of equilibria [21]. For each $k = 0, \ldots, n$, every point of S_k has a stable manifold of dimension k and an unstable manifold of dimension $n - k$. Thus, for each $k = 0, \ldots, n$, the stable manifold of S_k for (6.8)–(6.10), which is the union of the stable manifolds of the equilibria that comprise S_k, has dimension $n + 1 + k$.

For $\epsilon > 0$, each S_k remains a locally normally hyperbolic invariant manifold [7]. It no longer consists of equilibria; in fact, the system (6.4)–(6.6) on the invariant manifold $v = 0$ is

$$\dot{u} = 0, \qquad \dot{\xi} = \epsilon.$$

Rewriting this system in the slow time η yields

$$u' = 0, \tag{6.11}$$

$$\xi' = 1. \tag{6.12}$$

Thus the orbits of the slow system on the invariant manifold $v = 0$ are the lines $u =$ constant.

Fix k, let M be a submanifold of u-space, and let $N = \{(u, 0, \xi) \in S_k : u \in M\}$. The set N is a locally invariant subset of S_k for each ϵ. Hence it has unstable and stable manifolds that depend smoothly on ϵ [7, 10]. We denote them $W^u(N, \epsilon)$ and $W^s(N, \epsilon)$. For $\epsilon = 0$, N consists of equilibria, and $W^u(N, 0)$ and $W^s(N, 0)$ are the unions of the unstable and stable manifolds of these equilibria.

Let $(\sigma^*, \hat{\sigma}^*)$ be a solution of the Riemann problem (1.1), (1.9) that satisfies the hypotheses of theorem 5.3. We have $\sigma^* = (u_0^*, s_1^*, u_1^*, \ldots, u_{m-1}^*, s_m^*, u_m^*)$ with $u_0^* = u_L$ and $u_m^* = u_R$, and $\hat{\sigma}^* = (\Gamma_1^*, \ldots, \Gamma_m^*)$. Let the type be $(T_1, T_2, \ldots, T_m)$, and let $(k(0), \ldots, k(m))$ be the sequence given by corollary 5.2. We assume $\delta > 0$ is chosen small enough that for each $i = 1, \ldots, m$, $(u_{i-1}^*, s_i^*) \in S_{k(i-1)}$ and $(u_i^*, s_i^*) \in S_{k(i)}$. For $i = 0, \ldots, m$, if we set $M = \{u_i^*\}$ and $k = k(i)$ in the above construction, we obtain the sets

$$A_i = \{(u, 0, \xi) \in S_{k(i)} \, : \, u = u_i^*\}.$$

Let

$$\begin{aligned}
\tilde{A}_0 &= \{(u_0^*, 0, \xi) \, : \, \xi \leqslant s_1^*\}, \\
\tilde{A}_i &= \{(u_i^*, 0, \xi) \, : \, s_i^* \leqslant \xi \leqslant s_{i+1}^*\}, \qquad i = 1, \ldots, m-1, \\
\tilde{A}_m &= \{(u_m^*, 0, \xi) \, : \, s_m^* \leqslant \xi\},
\end{aligned}$$

Because of the choice of δ, for each $i = 0, \ldots, m$, $\tilde{A}_i \subset A_i \subset S_{k(i)}$.

Let (u_-, s, u_+, Γ) be a regular shock wave of type (k_-, k_+). Corresponding to Γ is a connecting orbit $\tilde{\Gamma}$ of (6.8)–(6.10). For small δ, $\tilde{\Gamma}$ goes from $(u_-, 0, s) \in S_{k_-}$ to $(u_+, 0, s) \in S_{k_+}$, and

$$\tilde{\Gamma} = \{(u, v, \xi) \, : \, u \in \Gamma, v = f(u) - f(u_-) - s(u - u_-), \text{ and } \xi = s\}.$$

The singularly perturbed boundary value problem (6.4)–(6.7) has the singular solution

$$\tilde{A}_0 \cup \tilde{\Gamma}_1^* \cup \tilde{A}_1 \cup \cdots \cup \tilde{A}_{m-1} \cup \tilde{\Gamma}_m^* \cup \tilde{A}_m, \tag{6.13}$$

a union of orbits of the reduced slow system (6.11) and (6.12) and the limit fast system (6.8)–(6.10) (see figure 3). Notice that $\tilde{\Gamma}_i^*$ goes from $(u_{i-1}^*, 0, s_i^*)$ in $S_{k(i-1)}$ to $(u_i^*, 0, s_i^*)$ in $S_{k(i)}$.

For $\epsilon > 0$ we seek a solution of the boundary value problem (6.4)–(6.7) that is near this singular solution. To find it, we shall seek a solution that lies in $W^u(A_0, \epsilon) \cap W^s(A_m, \epsilon)$. Such a solution satisfies the boundary conditions (6.7). Both $W^u(A_0, \epsilon)$ and $W^s(A_m, \epsilon)$ have dimension $n + 1$. Thus they are expected to intersect in isolated curves, which are orbits of (6.4)–(6.6).

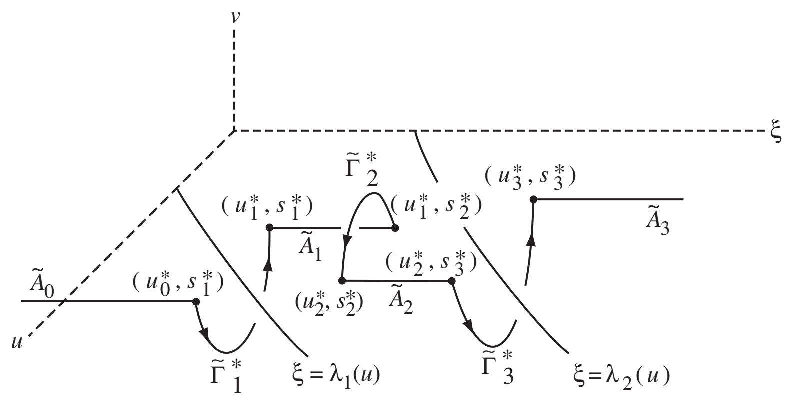

Figure 3. A singular solution with $n = 2$ and $m = 3$; u-space and v-space are pictured as one dimensional, although they, are of course, two dimensional. The solution has $(k(0), k(1), k(2), k(3)) = (0, 1, 1, 2)$. The Riemann solution consists of a Lax one-shock, an undercompressive shock wave, and a Lax two-shock.

Let $M_0 = \{u_0^*\}$ and let M_i, $i = 1, \ldots, m-1$, be the manifolds given by corollary 5.4. For $i = 0, \ldots, m-1$, let $N_i = \{(u, 0, \xi) \in S_{k(i)} : u \in M_i\}$. N_i has dimension $k(i) + 1$, and $W^u(N_i, \epsilon)$ has dimension $(k(i) + 1) + (n - k(i)) = n + 1$ for every i. Notice that $N_0 = A_0$.

Proposition 6.1. *For each* $i = 1, \ldots, m-1$,

(a) $W^u(N_{i-1}, 0)$ *is transverse to* $W^s(S_{k(i)}, 0)$ *along* $\tilde{\Gamma}_i^*$;
(b) $W^u(N_{i-1}, 0) \cap W^s(S_{k(i)}, 0)$ *consists of one orbit to each point* $(u, 0, s(u))$ *such that* $u \in M_i$ *and* $s(u)$ *is the speed of the corresponding connection.*

Proof. In (6.8)–(6.10) we let $w = f(u) - \xi u - v$, i.e. we make the invertible coordinate transformation

$$(u, v, \xi) \to (u, w, \xi) = (u, f(u) - \xi u - v, \xi). \tag{6.14}$$

Then, in $uw\xi$ coordinates, the system becomes

$$\dot{u} = B(u)^{-1}(f(u) - \xi u - w), \tag{6.15}$$

$$\dot{w} = 0, \tag{6.16}$$

$$\dot{\xi} = 0. \tag{6.17}$$

We may regard (6.15)–(6.17) as a family of differential equations on u-space, with (w, ξ) as a vector of parameters. We compare (6.15)–(6.17) to the system

$$\dot{u} = B(u)^{-1}(f(u) - f(u_-) - s(u - u_-)), \tag{6.18}$$

$$\dot{u}_- = 0, \tag{6.19}$$

$$\dot{s} = 0, \tag{6.20}$$

which was studied in sections 3 and 4. As in those sections, we regard (6.18)–(6.20) as a family of differential equations on u-space, with (u_-, s) as a vector of parameters. The two systems are related by the parameter transformation

$$w = f(u_-) - su_-, \tag{6.21}$$

$$\xi = s. \tag{6.22}$$

This transformation is locally invertible near any (u_-^*, s^*) where $Df(u_-^*) - s^*I$ is invertible.

We consider (6.18)–(6.20) with

$$\Lambda = \{(u_-, s) : (u_-, 0, s) \in N_i\}.$$

We let U denote an open neighbourhood of (u_{i-1}^*, s_i^*) in u_-s-space. For each $(u_-, s) \in U$, there is an equilibrium $u_+(u_-, s)$ of (6.18) near u_i^*. In uu_-s-space, let

$$P_- = \{(u_-, u_-, s) : (u_-, s) \in \Lambda\},$$

$$P_+ = \{(u_+, u_-, s) : (u_-, s) \in \Lambda \text{ and } u_+ = u_+(u_-, s)\},$$

$$\tilde{P}_+ = \{(u_+, u_-, s) : (u_-, s) \in U \text{ and } u_+ = u_+(u_-, s)\}.$$

The transformation

$$(u, u_-, s) \to (u, w, \xi) = (u, f(u_-) - su_-, s), \tag{6.23}$$

followed by the inverse of (6.14), takes P_- to N_{i-1}, and it takes $\tilde{P}_+$ to a neighbourhood of $(u_i^*, 0, s_i^*)$ in $S_{k(i)}$.

In the compressive case, the tangent spaces to $W^u(u_{i-1}^*, u_{i-1}^*, s_i^*)$ and $W^s(u_i^*, u_{i-1}^*, s_i^*)$ are transverse within u-space along Γ_i^*. Therefore, $W^u(u_{i-1}^*, u_{i-1}^*, s_i^*) \times \{(u_{i-1}^*, s_i^*)\}$ is transverse to $W^s(\tilde{P}_+)$ along $\Gamma^* \times \{(u_{i-1}^*, s_i^*)\}$ in uu_-s-space, and hence $W^u(P_-)$ is transverse to $W^s(\tilde{P}_+)$. Using the parameter transformations (6.23) and the inverse of (6.14), we see that (a) holds.

In the undercompressive case, $DS(u^*_{i-1}, s^*_i)$, restricted to the tangent space to Λ, is surjective by statement (a) of lemma 4.2. Then by proposition 3.2, $W^u(P_-)$ is transverse to $W^s(P_+)$ within $\mathbb{R}^n \times \Lambda \times \mathbb{R}$. Therefore, $W^u(P_-)$ is transverse to $W^s(\tilde{P}_+)$ in uu_-s-space, so again (1) holds.

Conclusion (b) follows from proposition 4.3 and the parameter transformations. □

Theorem 6.2. *Let $(\sigma^*, \hat{\sigma}^*)$ be a solution of the Riemann problem (1.1), (1.9) that satisfies the hypotheses of theorem 5.3. Then for each small $\epsilon > 0$, there is a solution of the boundary value problem (6.4)–(6.7) near the singular solution (6.13).*

Proof. Let $\epsilon > 0$ be small. We seek a solution in $W^u(A_0, \epsilon) \cap W^s(A_m, \epsilon)$. We will follow $W^u(A_0, \epsilon)$ along the flow. We will show by induction that for $i = 0, \dots, m-1$, $W^u(A_0, \epsilon)$ passes near $(u^*_i, 0, s^*_{i+1})$ C^1-close to $W^u(N_i, 0)$. This is true for $i = 0$, since $A_0 = N_0$ and $W^u(A_0, \epsilon)$ is C^1-close to $W^u(A_0, 0)$ [7, 10].

Let i be a number between 1 and $m-1$. Assume that $W^u(A_0, \epsilon)$ passes near $(u^*_{i-1}, 0, s^*_i)$ C^1-close to $W^u(N_{i-1}, 0)$. By proposition 6.1, $W^u(N_{i-1}, 0)$ is transverse to $W^s(S_{k(i)}, 0)$ along $\tilde{\Gamma}^*_i$, which connects $(u^*_{i-1}, 0, s^*_i)$ to $(u^*_i, 0, s^*_i)$. According to proposition 6.1, the projection of $W^u(N_{i-1}, 0) \cap W^s(S_{k(i)}, 0)$ to $S_{k(i)}$ along its stable foliation is

$$\{(u, 0, s(u)) : u \in M_i \text{ and } s(u) \text{ is the speed of the connection}\}.$$

This manifold is not parallel to the orbit of the fast flow through $(u^*_i, 0, s^*_i)$, which is the line A_i. By the exchange lemma [11] (see figure 4), $W^u(A_0, \epsilon)$ arrives at $(u^*_i, 0, s^*_{i+1})$ C^1-close to $W^u(N_i, 0)$.

We conclude that $W^u(A_0, \epsilon)$ passes near $(u^*_{m-1}, 0, s^*_m)$ C^1-close to $W^u(N_{m-1}, 0)$. Now the exchange lemma applied to $W^s(A_m, \epsilon)$ flowing backward shows that it passes near $(u^*_{m-1}, 0, s^*_m)$ C^1-close to $W^s(\tilde{N}_{m-1}, 0)$, where the tangent space to $\tilde{N}_{m-1}$ at $(u^*_{m-1}, 0, s^*_m)$ is $\tilde{V}_{m-1} \times \{0\} \times \mathbb{R}$. Since $W^u(N_{m-1}, 0)$ and $W^s(\tilde{N}_{m-1}, 0)$ meet transversally at $(u^*_{m-1}, 0, s^*_m)$ (the intersection is A_{m-1}), $W^u(A_0, \epsilon)$ and $W^s(A_m, \epsilon)$ meet transversally near there. The intersection is the desired solution. □

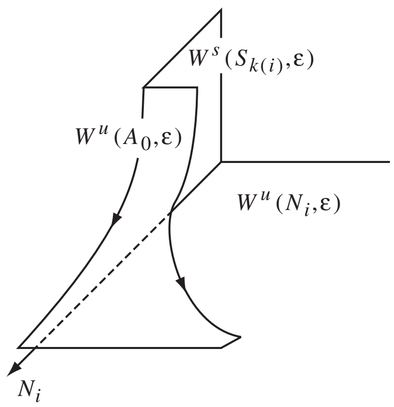

Figure 4. Geometry of the exchange lemma.

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