STEAM CONDENSATION WAVES
IN WATER-SATURATED POROUS ROCK

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ABSTRACT. We formulate balance laws governing condensation of steam injected into a linear porous medium containing water. Heat losses to the outside are neglected. Longitudinal heat conduction and capillary effects are taken into account. The condensation process is modeled by a rate equation based on a simple heat-transfer model. We study the condensation front as a traveling wave, under the approximation that pressure variations are negligible within the front. We find this traveling wave for the case of injection of high-quality steam using a combination of phase-plane analysis and numerical calculation of orbits.

INTRODUCTION

Steam injection is an effective technique to restore groundwater aquifers contaminated with non-aqueous-phase liquids (NAPL's) such as hydrocarbon fuels and halogenated hydrocarbons [17]. It is also one of the most effective methods to recover oil from medium to heavy oil reservoirs [16]. The main feature of steam injection is the steam condensation front (SCF), which marks the boundary between the upstream zone at boiling temperature and the downstream liquid zone below the boiling temperature. The main result of this work is the analysis of the structure of the SCF in the absence of NAPL's.

There is an abundant engineering literature on numerical ([17], [20]), experimental ([9]), and theoretical ([1], [8], [11], [12], [13], [18], [19], [21], [22], [24], [25], [26]) modeling of steam injection for clean-up and oil recovery. The main focus of the experimental and theoretical work is the internal structure of the steam condensation front. They are reviewed in [5].

In the latter work, a classification of Riemann solutions for steam injection in water is also presented. However, the fact that the steam condensation front is the small "diffusion" limit of a traveling wave is assumed in [5], rather than proved. The purpose of the present work is to show that this assumption is correct, for the particular case where the steam condensation front arises in the injection of steam with little water. In this case, the SCF ceases to obey Lax's entropy condition at the injection end. In other words, in the associated ODE, the left equilibrium has the extra technical difficulty of being non-hyperbolic.

In Section 1, the physical model is summarized. (The interested reader can find a detailed description in [5].) It is described mathematically by balance equations of mass and thermal energy, which are rewritten into a form suitable for analysis.

In Section 2, we derive the ordinary differential equation boundary value problem for the traveling wave that represents the SCF. In Section 3, some useful facts about the traveling wave ODE are proved. In Section 4, we show that existence of a solution to the ODE

boundary value problem is plausible. Section 5 proves the existence of a solution; however, one fact needed in the proof is verified numerically rather than analytically. Appendix A gives notation and values for the physical quantities appearing in the model. In Appendix B we calculate the speed of the steam condensation front.

1. PHYSICAL AND MATHEMATICAL MODEL

1.1. **Physical model.** We consider linear steam displacement in a homogeneous reservoir of constant permeability and porosity. The reservoir is initially saturated with water. The pressure gradients $\partial p_w/\partial x$, $\partial p_g/\partial x$ driving the fluids are small with respect to the prevailing system pressure p divided by the length of the reservoir. In particular, within the short steam condensation zone pressure variations are negligible. Hence we disregard the effect of pressure variation on the density of the fluids and on their thermodynamic properties. The reservoir is horizontal, so gravitational effects can be ignored. A steam-water mixture is injected at constant rate and constant composition. Transverse heat losses are disregarded.

The relevant effects of temperature on the fluid properties, *e.g.* steam density ρ_g, the water density ρ_w, water viscosity μ_w, and steam viscosity μ_g are taken into account (see also Appendix A). Darcy's Law determines the fluid motion. The (non-linear) temperature dependence of enthalpies and of the evaporation heat are also taken into account (see also Appendix A). Capillary pressure as well as an effective longitudinal heat conduction term are included.

We have chosen to describe condensation in terms of a steam mass condensation rate equation. The mass condensation rate q is always positive when the temperature drops below the boiling temperature T^b as long as not all steam has condensed, that is $S_w < 1$.

The stated conditions can be considered representative of steam injection in the subsurface for remediation of contaminated sites. As steam is injected the reservoir is heated. Depending on the proportions of steam and water in the injected mixture, we can distinguish three regimes, which differ in the structure of Riemann solutions. In two of the three regimes, there is an *SCF*. To the right of the *SCF*, there is pure water.

Each of the enthalpies per unit mass $h_w(T)$, $h_r(T)$, $h_g(T)$ $[J/Kg]$ is defined with respect to the enthalpy at the reference temperature T^0. In particular,

$$h_w(T^0) = h_r(T^0) = 0. \tag{1.1}$$

The enthalpy of steam is subdivided into a sensible part, (called sensible heat) $h_g^s(T)$ and a latent part (called latent heat) $h_g^l(T^0)$, *i.e.*, $h_g(T) = h_g^s(T) + h_g^l(T^0)$. We have $h_g^s(T^0) = 0$ and

$$h_g^l(T^0) = \bar{\Lambda}, \tag{1.2}$$

where $\bar{\Lambda}$ is the evaporation heat or the latent heat per unit mass.

The enthalpies per unit volume are $H_w(T) = \rho_w(T)h_w(T)$, $H_r(T) = \rho_r(T)h_r(T)$, $H_g^s(T) = \rho_g(T)h_g^s(T)$, $[J/m^3]$, which vanish at T^0, the initial reservoir temperature, and

$$H_g^l(T) = \rho_g(T)\bar{\Lambda}. \tag{1.3}$$

The enthalpies as functions of temperature are summarized in Appendix A for convenience.

We assume Darcy's law without gravity for the flow rates u_w, u_g of water and steam:

$$u_w = -\frac{kk_{rw}}{\mu_w}\frac{\partial p_w}{\partial x}, \qquad u_g = -\frac{kk_{rg}}{\mu_g}\frac{\partial p_g}{\partial x}. \tag{1.4}$$

Expressions for the relative permeabilities k_{rw}, k_{rg} are given in Eqs. A.10 and A.11 and the difference $p_g - p_w = P_c$ (also called the capillary pressure) is given by Eq. A.12.

We use the Brooks-Corey power law dependence of the capillary pressure on saturation (see Appendix A).

The water mass source term is taken as

$$q = \begin{cases} q_b(T - T^b)(S_w - 1) & \text{if } T \le T^b \text{ and } 0 \le S_w \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

This term is motivated by the idea that the condensation rate is determined by a "driving force" which is proportional to its departure from equilibrium $T = T^b$ (see also [15]) and the amount of gas $(1 - S_w)$ present. The value of q_b is considered to be very large.

1.2. **The model equations.** The mass balance equation of liquid water and steam read as follows:

$$\frac{\partial(\varphi\rho_w S_w)}{\partial t} + \frac{\partial(\rho_w u_w)}{\partial x} = q, \tag{1.6}$$

$$\frac{\partial(\varphi\rho_g S_g)}{\partial t} + \frac{\partial(\rho_g u_g)}{\partial x} = -q. \tag{1.7}$$

The rock porosity φ is assumed to be constant. The densities of pure liquid water and pure steam are ρ_w and ρ_g. Water density is more or less constant, while steam density depends on pressure in our situation where pressure variation effects other than driving the flow are neglected. We include longitudinal heat conduction, but neglect heat losses to the surrounding rock, in the energy balance equation given below. By our assumption of almost constant pressure we ignore adiabatic compression and decompression effects. Thus the energy balance is (see [3], Table 10.4-1)

$$\frac{\partial}{\partial t}(H_r + \varphi S_w H_w + \varphi S_g H_g) + \frac{\partial}{\partial x}(u_w H_w + u_g H_g) = \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right). \tag{1.8}$$

Here κ is the composite conductivity of the rock–fluid system [2], given by an equation such as

$$\kappa = \kappa_r + \varphi(S_w \kappa_w + S_g \kappa_g). \tag{1.9}$$

It turns out that $\kappa > \kappa_r > 0$ is all we need from the formula (1.9). Equations (1.6), (1.7), and (1.8) are the basic governing equations for the flow.

Equations (1.6) and (1.7) are combined with the heat balance equation (1.8), where we also use separation in sensible and latent quantities, to obtain:

$$\frac{\partial}{\partial t}\left(H_r + \varphi S_w H_w + \varphi S_g H_g^s\right) + \frac{\partial}{\partial x}\left(u_w H_w + u_g H_g^s\right) - \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right) =$$

$$= -\frac{\partial}{\partial t}\left(\varphi S_g H_g^l\right) - \frac{\partial}{\partial x}\left(u_g H_g^l\right) = -\frac{\partial}{\partial t}\left(\varphi S_g \rho_g \bar{\Lambda}\right) - \frac{\partial}{\partial x}\left(u_g \rho_g \bar{\Lambda}\right) =$$

$$= -\bar{\Lambda}\left(\frac{\partial}{\partial t}\left(\varphi S_g \rho_g\right) + \frac{\partial}{\partial x}\left(u_g \rho_g\right)\right).$$

Using Eq. (1.7), this yields

$$\frac{\partial}{\partial t}\left(H_r + \varphi S_w H_w + \varphi S_g H_g^s\right) + \frac{\partial}{\partial x}\left(u_w H_w + u_g H_g^s\right) = q\bar{\Lambda} + \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right). \tag{1.10}$$

Let us define the fractional flow functions for water and steam:

$$f_w = \frac{k_{rw}/\mu_w}{k_{rw}/\mu_w + k_{rg}/\mu_g}, \qquad f_g = \frac{k_{rg}/\mu_g}{k_{rw}/\mu_w + k_{rg}/\mu_g}. \tag{1.11}$$

The capillary pressure

$$P_c = P_c(S_w) = p_g - p_w, \tag{1.12}$$

which is given by Equation (A.12), is a strictly monotone decreasing function; it appears in the definition of the capillary diffusion coefficient

$$\Omega = -f_w\frac{kk_{rg}}{\mu_g}\frac{dP_c}{dS_w} \geq 0. \tag{1.13}$$

We notice that Ω vanishes precisely at water saturations $S_w = S_{wc}$ and $S_w = 1$ ($f_w(S_w) = 0$ for $S_w \leq S_{wc}$). We will invoke the presence of diffusion due to physical effects other than capillary pressure to add a small positive quantity to Ω so it does not vanish near $S_w = 1$.

Using Darcy's law (1.4) and the definition of P_c given in Eq. (1.12), one can easily show from Eqs. (1.4) and (1.13) that

$$u_w = uf_w - \Omega\frac{\partial S_w}{\partial x}, \qquad u_g = uf_g - \Omega\frac{\partial S_g}{\partial x}, \tag{1.14}$$

where

$$u = u_w + u_g \tag{1.15}$$

is the *total* or *Darcy velocity* and Ω acts as a saturation-dependent capillary diffusion coefficient.

Substituting (1.14) into Equations (1.6), (1.7) and (1.10) leads to

$$\varphi\frac{\partial\left(\rho_w S_w\right)}{\partial t} + \frac{\partial\left(\rho_w u f_w\right)}{\partial x} = q + \frac{\partial}{\partial x}\left(\rho_w\Omega\frac{\partial S_w}{\partial x}\right), \tag{1.16}$$

$$\varphi\frac{\partial\left(\rho_g S_g\right)}{\partial t} + \frac{\partial\left(\rho_g u f_g\right)}{\partial x} = -q + \frac{\partial}{\partial x}\left(\rho_g\Omega\frac{\partial S_g}{\partial x}\right), \tag{1.17}$$

$$\frac{\partial}{\partial t}\left(H_r + \varphi H_w S_w + \varphi H_g^s S_g\right) + \frac{\partial}{\partial x}\left(u\left(H_w f_w + H_g^s f_g\right)\right) =$$
$$= q\bar{\Lambda} + \frac{\partial}{\partial x}\left(\left(H_w - H_g^s\right)\Omega\frac{\partial S_w}{\partial x}\right) + \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right), \tag{1.18}$$

where on the right-hand side of the Eq. (1.18) we used that $\partial S_w/\partial x = -\partial S_g/\partial x$. The governing system of equations can be rewritten as (1.16)–(1.18).

2. Blow-up of the steam condensation zone

In this section, we assume that time and distance have been scaled so that the diffusion terms and the heat condensation terms balance. Therefore the steam condensation zone has width $O(1)$; it will be studied as a traveling wave extending from $-\infty$ to $+\infty$. In the steam condensation zone capillary pressure effects, conduction, mass and heat transfer terms all have to be taken into account. However, because this zone is thin relative to the reservoir, we are always justified in ignoring pressure variation effects on densities within the zone.

For the analysis of traveling waves, it is advantageous to define the cumulative condensation distribution $Q(x,t)$ and cumulative condensation $Q^+(t)$ by

$$
Q(x,t) = \int_{-\infty}^{x} q(x',t)dx', \qquad Q^+(t) = \int_{-\infty}^{+\infty} q(x',t)dx'. \tag{2.1}
$$

We look for solutions of Eq. (2.1) and (1.16)–(1.18) that depend only on $\xi = (x - v^{SCF})t$, where v^{SCF} is the speed of the steam condensation traveling wave. Such a solution must satisfy the system

$$
\frac{dQ}{d\xi} = q, \tag{2.2}
$$

$$
-\varphi v^{SCF}\frac{d\left(\rho_w S_w\right)}{d\xi} + \frac{d\left(\rho_w u f_w\right)}{d\xi} = \frac{dQ}{d\xi} + \frac{d}{d\xi}\left(\rho_w \Omega \frac{dS_w}{d\xi}\right), \tag{2.3}
$$

$$
-\varphi v^{SCF}\frac{d\left(\rho_g S_g\right)}{d\xi} + \frac{d\left(\rho_g u f_g\right)}{d\xi} = -\frac{dQ}{d\xi} + \frac{d}{d\xi}\left(\rho_g \Omega \frac{dS_g}{d\xi}\right), \tag{2.4}
$$

$$
-v^{SCF}\frac{d}{d\xi}\left(H_r + \varphi H_w S_w + \varphi H_g^s S_g\right) + \frac{d}{d\xi}\left(u\left(H_w f_w + H_g^s f_g\right)\right) =
$$
$$
= \bar{\Lambda}\frac{dQ}{d\xi} + \frac{d}{d\xi}\left(\left(H_w - H_g^s\right)\Omega\frac{dS_w}{d\xi}\right) + \frac{d}{d\xi}\left(\kappa\frac{dT}{d\xi}\right). \tag{2.5}
$$

We look for solutions $(Q(\xi), S_w(\xi), S_g(\xi), T(\xi))$ of Eqs. (2.2)–(2.5) that join two equilibria. More precisely

$$
\lim_{\xi \to -\infty}(Q(\xi), S_w(\xi), S_g(\xi), T(\xi)) = (Q^-, S_w^-, S_g^-, T^-),
$$

$$
\lim_{\xi \to \infty}(Q(\xi), S_w(\xi), S_g(\xi), T(\xi)) = (Q^+, S_w^+, S_g^+, T^+),
$$

$$
\lim_{\xi \to \pm\infty}\left(\frac{dQ}{d\xi}, \frac{dS_w}{d\xi}, \frac{dS_g}{d\xi}, \frac{dT}{d\xi}\right) = 0,
$$

with $Q^- = 0$, S_w^- and u^- given, $S_g^- = 1 - S_w^-$, $T^- = T^b$, Q^+ to be determined, $S_w^+ = 1$, $S_g^+ = 0$, $T^+ = T^0$. Since steam is injected at the left, we assume $u^- > 0$.

We integrate Eqs. (2.3)–(2.5) from $-\infty$ to ξ, and use Eq. (1.5), obtaining

$$\frac{dQ}{d\xi} = \begin{cases} q_b(T - T^b)(S_w - 1) & \text{for} \quad T \leq T^b, \quad 0 \leq S_w \leq 1; \\ 0 & \text{otherwise.} \end{cases} \quad , \tag{2.6}$$

$$\rho_w \Omega \frac{dS_w}{d\xi} = (-Q - \varphi v^{SCF} \rho_w S_w + \rho_w u f_w) - (-\varphi v^{SCF} \rho_w^- S_w^- + \rho_w^- u^- f_w^-), \tag{2.7}$$

$$-\rho_g \Omega \frac{dS_g}{d\xi} = (-Q + \varphi v^{SCF} \rho_g S_g - \rho_g u f_g) - (\varphi v^{SCF} \rho_g^- S_g^- - \rho_g^- u^- f_g^-), \tag{2.8}$$

$$(H_w - H_g^s)\Omega \frac{dS_w}{d\xi} + \kappa \frac{dT}{d\xi} = -v^{SCF}(H_r + \varphi H_w S_w + \varphi H_g^s S_g) + u(H_w f_w + H_g^s f_g) - \bar{\Lambda} Q$$

$$+ v^{SCF}(H_r^- + \varphi H_w^- S_w^- + \varphi H_g^{s-} S_g^-) - u^-(H_w^- f_w^- + H_g^{s-} f_g^-). \tag{2.9}$$

To take advantage of proportionality between certain quantities, we introduce

$$\widetilde{u} := \frac{u}{u^-}, \quad \widetilde{v}^{SCF} := \frac{\varphi v^{SCF}}{u^-}, \quad \widetilde{u}^+ := \frac{u^+}{u^-}, \quad \widetilde{Q}^+ := \frac{Q^+}{u^-}, \quad \widetilde{q}_b := \frac{q_b}{u^-}, \quad \widetilde{\Omega} := \frac{\Omega}{u^-}, \quad \widetilde{\kappa} := \frac{\kappa}{u^-}. \tag{2.10}$$

Since $S_w^+ = 1$ and $S_g^+ = 0$, we see that $f_w^+ = 1$ and $f_g^+ = 0$. By letting ξ tend to $+\infty$ in Eq. (2.7), we obtain

$$\widetilde{Q}^+ = \widetilde{u}^+ \rho_w^+ - \rho_w^- f_w^- - \widetilde{v}^{SCF}(\rho_w^+ - \rho_w^- S_w^-). \tag{2.11}$$

Similarly, Eq. (2.8) yields

$$\widetilde{Q}^+ = \rho_g^- \left(f_g^- - \widetilde{v}^{SCF} S_g^- \right). \tag{2.12}$$

Since $T^+ = T^0$, we use Eq. (1.1) in Eq. (2.9) to obtain the Rankine–Hugoniot condition

$$\bar{\Lambda} \widetilde{Q}^+ = -(H_w^- f_w^- + H_g^{s-} f_g^-) + \widetilde{v}^{SCF} \left(H_r^-/\varphi + H_w^- S_w^- + H_g^{s-} S_g^- \right). \tag{2.13}$$

Eqs. (2.11), (2.12) and (2.13) are a linear system in the unknowns $\widetilde{v}^{SCF}$, $\widetilde{u}^+$ and $\widetilde{Q}^+$. Since $S_w + S_g = 1$, one of the Eqs. (2.7)–(2.9) can be replaced by an equation for the total velocity $\widetilde{u}$. To eliminate one of the Eqs. (2.7)–(2.9), we divide Eqs. (2.7), (2.8) by the densities, and obtain in the new notation (2.10):

$$\widetilde{\Omega} \frac{dS_w}{d\xi} = \left(-\frac{\widetilde{Q}}{\rho_w} - \widetilde{v}^{SCF} S_w + \widetilde{u} f_w \right) - \frac{\rho_w^-}{\rho_w} \left(-\widetilde{v}^{SCF} S_w^- + f_w^- \right), \tag{2.14}$$

$$-\widetilde{\Omega} \frac{dS_g}{d\xi} = \left(-\frac{\widetilde{Q}}{\rho_g} + \widetilde{v}^{SCF} S_g - \widetilde{u} f_g \right) - \frac{\rho_g^-}{\rho_g} \left(\widetilde{v}^{SCF} S_g^- - f_g^- \right). \tag{2.15}$$

Since $S_w + S_g = 1$, $\frac{dS_w}{d\xi} = -\frac{dS_g}{d\xi}$. Therefore, subtracting Eq. (2.15) from Eq. (2.14) we obtain $\widetilde{u} = \widetilde{u}(\widetilde{Q}, T)$ as

$$\widetilde{u} = \widetilde{Q} \left(\frac{1}{\rho_w} - \frac{1}{\rho_g} \right) + \widetilde{v}^{SCF} + \frac{\rho_w^-}{\rho_w} \left(-\widetilde{v}^{SCF} S_w^- + f_w^- \right) + \frac{\rho_g^-}{\rho_g} \left(-\widetilde{v}^{SCF} S_g^- + f_g^- \right). \tag{2.16}$$

Remark 1. *Letting $\xi \to +\infty$ in Eq. (2.16), we obtain $\widetilde{u}^+$ as*

$$\widetilde{u}^+ = \widetilde{Q}^+ \left(\frac{1}{\rho_w^+} - \frac{1}{\rho_w^-} \right) + \widetilde{v}^{SCF} + \left(\rho_w^- h_w^+ - \rho_g^- h_g^+ \right) \left(f_w^- - \widetilde{v}^{SCF} S_w^- \right) + \rho_g^- h_g^+ \left(1 - \widetilde{v}^{SCF} \right).$$

(2.17)

This equation can also be obtained from Eqs. (2.11) and (2.12).

Multiplying Eq. (2.14) by H_w, Eq. (2.15) by H_g^s, and subtracting we obtain

$$\begin{aligned}
\left(H_w - H_g^s \right) \widetilde{\Omega} \frac{dS_w}{d\xi} &= -\widetilde{Q} \left(h_w - h_g^s \right) - \widetilde{v}^{SCF} \left(H_w S_w + H_g^s S_g \right) \\
&+ \widetilde{u} \left(H_w f_w + H_g^s f_g \right) - \rho_w^- h_w \left(-\widetilde{v}^{SCF} S_w^- + f_w^- \right) + \rho_g^- h_g^s \left(\widetilde{v}^{SCF} S_g^- - f_g^- \right).
\end{aligned}$$

(2.18)

Subtracting this equation from Eq. (2.9) divided by u^- we obtain

$$\widetilde{\kappa} \frac{\partial T}{\partial \xi} = \widetilde{Q} \left(h_w - h_g^s - \bar{\Lambda} \right) - \widetilde{v}^{SCF} \left(\rho_w^- \left(h_w - h_w^- \right) S_w^- + \rho_g^- \left(h_g^s - h_g^{s-} \right) S_g^- \right)$$

$$- \widetilde{v}^{SCF} (H_r - H_r^-)/\varphi + \left(\rho_w^- \left(h_w - h_w^- \right) f_w^- + \rho_g^- \left(h_g^s - h_g^{s-} \right) f_g^- \right).$$

(2.19)

Subtracting Eq. (2.7) from Eq. (2.8) we obtain

$$\begin{aligned}
(\rho_w - \rho_g) \widetilde{\Omega} \frac{dS_w}{d\xi} &= - \widetilde{v}^{SCF} (\rho_w S_w + \rho_g S_g) + \widetilde{u} (\rho_w f_w + \rho_g f_g) \\
&+ \widetilde{v}^{SCF} (\rho_w^- S_w^- + \rho_g^- S_g^-) - (\rho_w^- f_w^- + \rho_g^- f_g^-).
\end{aligned}$$

(2.20)

The system of ODE's that we shall study consists of Eqs. (2.6), (2.19) and (2.20), in the independent variables $(\widetilde{Q}, T, S_w)$. To put these equations in , we divide Eq. (2.19) by $\widetilde{\kappa}$ and Eq. (2.20) by $(\rho_w - \rho_g) \widetilde{\Omega}$, and in Eq. (2.20) we substitute Eq. (2.16) for $\widetilde{u}$. We obtain

$$\frac{d\widetilde{Q}}{d\xi} = \begin{cases} q_b (T - T^b)(S_w - 1) & \text{if } T \leq T^b \text{ and } 0 \leq S_w \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

(2.21)

$$\frac{dT}{d\xi} = \widetilde{Q} A(T) + B(T),$$

(2.22)

$$\frac{dS_w}{d\xi} = \widetilde{Q} C(T, S_w) + D(T, S_w),$$

(2.23)

with

$$\widetilde{\kappa}A(T) = h_w - h_g^s - \bar{\Lambda},$$
(2.24)

$$\widetilde{\kappa}B(T) = -\widetilde{v}^{SCF}\left(\rho_w^-(h_w - h_w^-)S_w^- + \rho_g^-(h_g^s - h_g^{s-})S_g^-\right)$$

$$- \widetilde{v}^{SCF}(H_r - H_r^-)/\varphi + \left(\rho_w^-\left(h_w - h_w^-\right)f_w^- + \rho_g^-\left(h_g^s - h_g^{s-}\right)f_g^-\right),$$
(2.25)

$$(\rho_w - \rho_g)\widetilde{\Omega}C(T, S_w) = \left(\frac{1}{\rho_w} - \frac{1}{\rho_g}\right)(\rho_w f_w + \rho_g f_g),$$
(2.26)

$$(\rho_w - \rho_g)\widetilde{\Omega}D(T, S_w) = -\widetilde{v}^{SCF}(\rho_w S_w + \rho_g S_g)$$

$$+ \left(\widetilde{v}^{SCF} + \frac{\rho_w^-}{\rho_w}\left(-\widetilde{v}^{SCF}S_w^- + f_w^-\right) + \frac{\rho_g^-}{\rho_g}\left(-\widetilde{v}^{SCF}S_g^- + f_g^-\right)\right)(\rho_w f_w + \rho_g f_g)$$

$$+ \widetilde{v}^{SCF}(\rho_w^- S_w^- + \rho_g^- S_g^-) - (\rho_w^- f_w^- + \rho_g^- f_g^-).$$
(2.27)

The right side of Eq. (2.24) has the meaning of minus the heat released at condensation. Physics dictates that this heat is positive, so we assume

$$A(T) < 0 \text{ for } T^0 \leq T \leq T^b.$$
(2.28)

Physics also dictates that the enthalpies are strictly increasing functions of T, so we assume

$$B(T) \text{ is a strictly decreasing function of } T \text{ for } T^0 \leq T \leq T^b.$$
(2.29)

3. Properties of the traveling wave ODE

In Eqs. (2.24)–(2.27), we recall that $T^- = T^b$, S_w^- is given, and $S_g^- = 1 - S_w^-$. We then define $\widetilde{v}^{SCF}$ by Eq. (B.4). $\widetilde{Q}^+$ is given by Eq. (2.12).

As explained in [5], the steam condensation front has the following properties when pure or almost pure steam is injected in water:

I. The SCF is preceded by a rarefaction wave adjacent at its left. In other words, it is a *left-characteristic* shock, i.e.

$$\widetilde{v}^{SCF} = \frac{\partial f_w}{\partial S_w}(T^-, S_w^-).$$
(3.1)

II. For the SCF between S_w^- and S_w^+, the condensation is complete, i.e., $S_w^+ = 1$ and $T^+ = T^0$. (This was noted in the previous section.)

III. One can easily show based on I, II, and the $\mathcal{S}$-shape of f_w which arises from the behavior of k_w and k_g, that

$$\left(\frac{\partial^2 f_w}{\partial S_w^2}\right)^- > 0.$$
(3.2)

IV. One can easily show based on I, II, the $\mathcal{S}$-shape of f_w, and the dependence of μ_g and μ_w on temperature, that

$$(f_w - f_w^-) - \widetilde{v}^{SCF}(S_w - S_w^-) > 0 \quad \text{for } S_w^- < S_w < 1$$
(3.3)

(Oleĭnik's inequality), and

$$-\widetilde{v}^{SCF}S_g^- + f_g^- > 0.$$
(3.4)

Another fact that we will use quite often is that by the definition of liquid and gas phases

$$\rho_w - \rho_g > 0.$$

away from the critical point.

Theorem 3.1. *The system* (2.21)–(2.23) *has the following properties:*

(1) $A(T) < 0$ *for* $T^0 \leq T \leq T^b$.
(2) $B(T)$ *is a strictly decreasing function of* T *for* $T^0 \leq T \leq T^b$, *and* $B(T^b) = 0$.
(3) $\widetilde{Q}^+ A(T^0) + B(T^0) = 0$.
(4) $\widetilde{Q}^+ A(T) + B(T) < 0$ *for* $T^0 < T \leq T^b$.
(5) $C(T, S_w) < 0$ *for* $T^0 \leq T \leq T^b$ *and* $S_w^- \leq S_w \leq 1$.
(6) $D(T^b, S_w^-) = 0$.
(7) $\frac{\partial D}{\partial S_w}(T^b, S_w^-) = 0$.
(8) $\frac{\partial^2 D}{\partial S_w^2}(T^b, S_w^-) > 0$.
(9) $D(T^b, S_w) > 0$ *for* $S_w^- < S_w \leq 1$.
(10) $\widetilde{Q}^+ C(T, 1) + D(T, 1) = 0$ *for* $T^0 \leq T \leq T^b$.
(11) $\frac{\partial D}{\partial T}(T^b, S_w^-) > 0$.

Notice that by (2), (3), (6) and (10), there are equilibria $E^- = (0, T^b, S_w^-)$ and $E^+ = (\widetilde{Q}^+, T^0, 1)$.

Proof. (1) This was assumed in (2.28).

(2) This follows from the assumption (2.29) and substitution of $T = T^- = T^b$ into Eq. (2.25).

(3) $\widetilde{Q}^+ A(T^0) + B(T^0) = \widetilde{Q}^+ A(T^+) + B(T^+) = 0$ by Eq. (2.13).

(4) This is a consequence of (2) and (3).

(5) $C(T, S_w) < 0$ because in Eq. (2.26), $\rho_w - \rho_g$ and $\frac{1}{\rho_w} - \frac{1}{\rho_g}$ have opposite sign.

(6) $D(T^b, S_w^-) = 0$ follows from substituting $T = T^- = T^b$ into Eq. (2.27) and using both $S_w + S_g = 1$ and $f_w + f_g = 1$.

(7) Differentiating Eq. (2.27) with respect to S_w, we obtain, with $D = D(T, S_w)$,

$$(\rho_w - \rho_g)\frac{\partial \widetilde{\Omega}}{\partial S_w}D + (\rho_w - \rho_g)\widetilde{\Omega}\frac{\partial D}{\partial S_w} = -\widetilde{v}^{SCF}(\rho_w - \rho_g)$$

$$+ \left(\widetilde{v}^{SCF} + \rho_w^- h_w\left(-\widetilde{v}^{SCF}S_w^- + f_w^-\right) + \rho_g^- h_g\left(-\widetilde{v}^{SCF}S_g^- + f_g^-\right)\right)(\rho_w - \rho_g)\frac{\partial f_w}{\partial S_w}. \quad (3.5)$$

Hence using (6) we obtain

$$\left((\rho_w - \rho_g)\widetilde{\Omega}\frac{\partial D}{\partial S_w}\right)^- = \left(\left(\frac{\partial f_w}{\partial S_w}\right)^- - \widetilde{v}^{SCF}\right)(\rho_w^- - \rho_g^-) = 0. \quad (3.6)$$

The second equality in Eq. (3.6) follows from property I of the SCF (Eq. (3.1)).

(8) The second derivative of Eq. (2.27) relative to S_w can be obtained from Eq. (3.5) as

$$(\rho_w - \rho_g)\frac{\partial^2 \widetilde{\Omega}}{\partial S_w^2}D + 2(\rho_w - \rho_g)\frac{\partial \widetilde{\Omega}}{\partial S_w}\frac{\partial D}{\partial S_w} + (\rho_w - \rho_g)\widetilde{\Omega}\frac{\partial^2 D}{\partial S_w^2} =$$

$$\left(\widetilde{v}^{SCF} + \frac{\rho_w^-}{\rho_w}\left(-\widetilde{v}^{SCF}S_w^- + f_w^-\right) + \frac{\rho_g^-}{\rho_g}\left(-\widetilde{v}^{SCF}S_g^- + f_g^-\right)\right)(\rho_w - \rho_g)\frac{\partial^2 f_w}{\partial S_w^2}. \quad (3.7)$$

At E^- the first term and the second term on the left are zero because $(\partial D/\partial S_w)^- = D^- = 0$. Thus Eq. (3.7) at E^- can be simplified to

$$\left((\rho_w - \rho_g)\widetilde{\Omega}\frac{\partial^2 D}{\partial S_w^2}\right)^- = \left((\rho_w - \rho_g)\frac{\partial^2 f_w}{\partial S_w^2}\right)^- > 0. \tag{3.8}$$

The inequality in Eq. (3.8) follows from Eq. (3.2).

(9) Substitution of $T = T^-$ in Eq. (2.27) leads to

$$(\rho_w^- - \rho_g^-)\widetilde{\Omega}\left(T^-, S_w\right)D(T^-, S_w) =$$
$$- \widetilde{v}^{SCF}(\rho_w^- S_w + \rho_g^- S_g) + (\rho_w^- f_w + \rho_g^- f_g) + \widetilde{v}^{SCF}(\rho_w^- S_w^- + \rho_g^- S_g^-) - (\rho_w^- f_w^- + \rho_g^- f_g^-). \tag{3.9}$$

Using $S_g = 1 - S_w$, $f_g = 1 - f_w$ this can be rewritten as

$$(\rho_w^- - \rho_g^-)\widetilde{\Omega}\left(T^-, S_w\right)D(T^-, S_w) = \left(\rho_w^- - \rho_g^-\right)\left((f_w - f_w^-) - \widetilde{v}^{SCF}(S_w - S_w^-)\right) > 0.$$

The inequality follows from Eq. (3.3).

(10) This is a tedious but straightforward calculation starting from Eqs. (2.8), (2.16) and (2.9) with $S_w = 1$ and $\widetilde{Q} = \widetilde{Q}^+$.

(11) The derivative of Eq. (2.27) relative to T can be written as

$$\left(\widetilde{\Omega}D\frac{d}{dT}(\rho_w - \rho_g) + (\rho_w - \rho_g)D\frac{d\widetilde{\Omega}}{dT} + (\rho_w - \rho_g)\widetilde{\Omega}\frac{dD}{dT}\right)^- =$$

$$- \widetilde{v}^{SCF}S_g^-\left(\frac{d\rho_g}{dT}\right)^- + \rho_g^-\left(\frac{d(\rho_g^{-1})}{dT}\right)^-\left(-\widetilde{v}^{SCF}S_g^- + f_g^-\right)(\rho_w f_w + \rho_g f_g)^-$$

$$+ \left(\widetilde{v}^{SCF} + \frac{\rho_w^-}{\rho_w}(-\widetilde{v}^{SCF}S_w^- + f_w^-) + \frac{\rho_g^-}{\rho_g}(-\widetilde{v}^{SCF}S_g^- + f_g^-)\right)^-\left(f_g\frac{d\rho_g}{dT}\right)^-. \tag{3.10}$$

The first and second term on the left are zero because $D^- = 0$, and therefore we obtain

$$\left((\rho_w - \rho_g)\widetilde{\Omega}\frac{dD}{dT}\right)^- = -\widetilde{v}^{SCF}S_g\left(\frac{d\rho_g}{dT}\right)^-$$

$$- \frac{1}{\rho_g^-}\left(\frac{d\rho_g}{dT}\right)^-\left(-\widetilde{v}^{SCF}S_g^- + f_g^-\right)(\rho_w f_w + \rho_g f_g)^- + \left(f_g\frac{d\rho_g}{dT}\right)^-, \tag{3.11}$$

which can be rearranged to

$$\left((\rho_w - \rho_g)\widetilde{\Omega}\frac{dD}{dT}\right)^- = \left(f_g - \widetilde{v}^{SCF}S_g\right)^-\left(\rho_g^- - (\rho_w f_w + \rho_g f_g)^-\right)\frac{1}{\rho_g^-}\left(\frac{d\rho_g}{dT}\right)^-. \tag{3.12}$$

Using $f_g^- = 1 - f_w^-$ in the second parenthesis, as well as the law of ideal gases for $d\rho_g/dT$, we obtain

$$\left(\widetilde{\Omega}\frac{dD}{dT}\right)^- = -\left(f_g - \widetilde{v}^{SCF}S_g\right)^- f_w^-\left(\frac{1}{\rho_g}\frac{d\rho_g}{dT}\right)^- = \frac{1}{T^-}\left(f_g - \widetilde{v}^{SCF}S_g\right)^- f_w^- > 0. \tag{3.13}$$

The inequality follows from Eq. (3.4).   □

We shall need one more property of the system (2.21)–(2.23):

(12) $\widetilde{q_b}(T - T^b)(S_w - 1)C + (\widetilde{Q}A + B)(\widetilde{Q}\frac{\partial C}{\partial T} + \frac{\partial D}{\partial T}) < 0$ at all points $(\widetilde{Q}, T, S_w)$ such that $0 < \widetilde{Q} \leq \widetilde{Q}^+$, $T_o \leq T < T^b$, $S_w^- < S_w < 1$, and $\dot{S_w} = \widetilde{Q}C(T, S_w) + D(T, S_w) = 0$.

In order to justify assuming (12), we make the following observations. The first term in the sum is negative, since $C < 0$ by (5). The quantity $\widetilde{Q}A + B$ is nonpositive by (4). If we solve for $\widetilde{Q}$ on the surface where $\widetilde{Q}C + D = 0$ and substitute into $\widetilde{Q}\frac{\partial C}{\partial T} + \frac{\partial D}{\partial T}$, we see that it suffices to verify the inequality $D\frac{\partial C}{\partial T} - C\frac{\partial D}{\partial T} > 0$. Using the formulas and values of physical quantities given in Appendix A, we have plotted $D\frac{\partial C}{\partial T} - C\frac{\partial D}{\partial T}$ as a function of T and S_w, and verified numerically that it is always positive.

4. Invariant manifolds

In the following section, we shall see that (i) at the equilibrium E^- there is a one-dimensional stable manifold and a two-dimensional center-unstable manifold, and (ii) at the equilibrium E^+ there is a two-dimensional stable or center-stable manifold and a one-dimensional unstable manifold. Because the center-unstable manifold of E^- and the stable or center-stable manifold of E^+ are both two-dimensional, it is reasonable to expect that they intersect. The intersection would be a connection from E^- to E^+ that represents the traveling wave we seek. See Fig. 4.1.
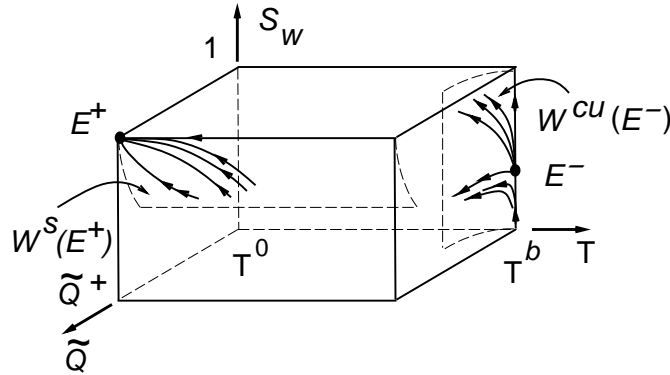

FIGURE 4.1. Invariant manifolds when E^+ has a two-dimensional stable manifold.

5. Existence of a connection

To simplify the notation in this section, in Eqs. (2.21)–(2.23) we replace $\widetilde{Q}$ by Q and S_w by S, and we remove the tildes and w subscripts from related terms. We shall restrict our attention to the set

$$R = \{(Q, T, S) : 0 \leq Q \leq Q^+, \ T^0 \leq T \leq T^b, \ 0 \leq S \leq 1\}.$$

Thus we consider the system

$$\dot{Q} = q_b(T - T^b)(S - 1), \tag{5.1}$$

$$\dot{T} = QA(T) + B(T), \tag{5.2}$$

$$\dot{S} = QC(T, S) + D(T, S) \tag{5.3}$$

on R. From Theorem 3.1, there are equilibria at $E^- = (0, T^b, S^-)$ and $E^+ = (Q^+, T^0, 1)$.

Theorem 5.1. *If the system* (5.1)–(5.3) *satisfies property (12) of Section 3, then there is a connection in* R *from* E^- *to* E^+.

Proof. We shall frequently use properties (1)–(11) of Theorem 3.1.

Using (7), we see that the linearization of Eqs. (5.1)–(5.3) at E^- has the matrix

$$\begin{pmatrix} 0 & q_b(S^- - 1) & 0 \\ A(T^b) & B'(T^b) & 0 \\ C(T^b, S^-) & \frac{\partial D}{\partial T}(T^b, S^-) & 0 \end{pmatrix}. \tag{5.4}$$

The characteristic equation is

$$\lambda(\lambda^2 - B'(T^b)\lambda - q_b(S^- - 1)A(T^b)) = 0.$$

Thus there is a 0 eigenvalue with eigenvector $(0, 0, 1)$. By (1), $q_b(S^- - 1)A(T^b) > 0$, so there are also a negative eigenvalue and a positive eigenvalue. Let $W^c(E^-)$, $W^u(E^-)$ and $W^{cu}(E^-)$ denote the one-dimensional center, one-dimensional unstable, and two-dimensional center-unstable manifolds of E^- respectively.

In R:

1. $\dot{Q} > 0$ unless $T = T^b$ or $S = 1$. There $\dot{Q} = 0$.
2. $\dot{T}$ depends only on Q and T. It has the sign of $QA(T) + B(T)$
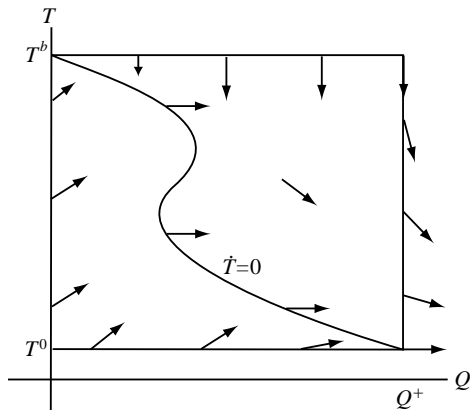3. $\dot{S}$ has the sign of $QC(T, S) + D(T, S)$.

See Figures 5.1 and 5.2.

FIGURE 5.1. The vector field for fixed S, $0 \leq S < 1$, projected onto the QT-plane. Notice that $\dot{Q} > 0$ for $T^0 \leq T < T^b$; and $\dot{T} = 0$ on the curve $Q(T) = -\frac{B(T)}{A(T)}$, a smooth curve with $Q(T^0) = Q^+$, $Q(T^b) = 0$, and $0 < Q(T) < Q^+$ for $T^0 < T < T^b$ (a consequence of properties (1)–(4)). Also notice that by property (1), $\dot{T} < 0$ to the right of this curve and $\dot{T} > 0$ to the left of this curve. For $S = 1$ the picture is different, since $\dot{Q} = 0$.

For $i, j = +, -$, define

$$R_j^i = \{(Q, T, S) \in \text{Int}\,R : \dot{T} \text{ has sign } i, \ \dot{S} \text{ has sign } j\}.$$

By Lemma 5.2 below, at E^- an eigenvector v for the positive eigenvalue points into R_-^-. Let Γ denote the branch of $W^u(E^-)$ that is tangent to v. Then near E^-, Γ lies in R_-^-.
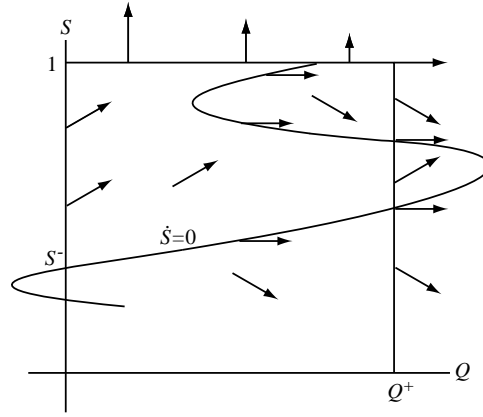
FIGURE 5.2. The vector field for fixed T, $T^0 \leq T < T^b$, projected onto the QS-plane. Notice that $\dot{Q} > 0$ for $0 \leq S < 1$, and $\dot{S} = 0$ on the curve $Q_T(S) = -\frac{D(T,S)}{C(T,S)}$, which by property (5) is smooth at least for $S^- - \epsilon \leq S \leq 1$. By properties (6)–(9), for $T = T^b$, this curve has $Q > 0$ except at $S = S^-$. where it has second order contact with the S-axis. However, little can be said about it for other values of T. Also notice that by property (5), $\dot{S} < 0$ to the right of this curve and $\dot{S} > 0$ to the left of this curve. For $T = T^b$ the picture is different, since $\dot{Q} = 0$.

The line $Q = 0$, $T = T^b$ is invariant under Eqs. (5.1)–(5.3) and can be taken to be $W^c(E^-)$. Thus $W^{cu}(E^-)$, which is tangent at E^- to the plane spanned by $(0,0,1)$ and v, includes this line and Γ. From properties (7) and (8), all solutions in $R \cap W^{cu}(E^-)$ lie in the sector of $W^{cu}(E^-)$ between Γ and the portion of the line that is above E^-, which we denote L. See Figure 5.3.
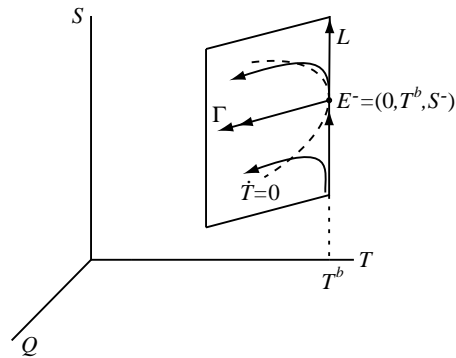

FIGURE 5.3. Part of $W^{cu}(E^-)$. The set $\dot{S} = 0$ meets $W^{cu}(E^-)$ in a curve, shown dashed, which has second order contact with the line $Q = 0$, $T = T^b$ at E^-. Near E^-, Γ lies in the intersection of $W^{cu}(E^-)$ and R_-^-. Also, near E^-, the portion of $W^{cu}(E^-)$ in $\mathrm{Int}\,R$ that is outside the dashed curve is in R_+^-.

Let us parameterize the solution curves in this sector of $W^{cu}(E^-)$ as $\Gamma(\theta)$, $0 \leq \theta \leq 1$, with $\Gamma(0) = \Gamma$ and $\Gamma(1) = L$. For $0 < \theta < 1$, as $t \to -\infty$, $\Gamma(\theta)$ approaches E^- tangent to L to infinite order. Thus all $\Gamma(\theta)$ with $0 < \theta < 1$ lie in R_+^- for large negative t. See Figure 5.3.

In the interior of R, $\dot{Q} > 0$. Thus, as t increases, all $\Gamma(\theta)$ with $0 < \theta < 1$ either leave R_+^- through its boundary or approach an invariant set contained in its boundary.

The boundary of R_+^- is contained in the union of the eight sets $Q = 0$, $Q = Q^+$, $T = T^0$, $T = T^b$, $S = 0$, $S = 1$, $\dot{T} = 0$, and $\dot{S} = 0$.

Since (i) $E^- = (0, T^b, S^-)$, (ii) all $\Gamma(\theta)$ with $0 < \theta < 1$ approach E^- as $t \to -\infty$, and (iii) all have $\dot{Q} > 0$, $\dot{T} < 0$, and $\dot{S} > 0$ while they are in the interior of R_+^-, we conclude that they cannot leave R_+^- through $Q = 0$, $T = T^b$, or $S = 0$. Also, properties (1) and (3) imply that the portion of the set $T = T^0$ with $Q < Q^+$ has $\dot{T} > 0$ and hence does not contain any boundary points of R_+^-. In addition, no solution can leave R_+^- through the portion of the set $\dot{T} = 0$ in the interior of R, because of the direction of the vectors on this set. See Figure 5.1.

We conclude that, as t increases, all $\Gamma(\theta)$ with $0 < \theta < 1$ leave R_+^- through the union of the sets $Q = Q^+$, $\dot{S} = 0$, and $S = 1$. Now no solution can leave R_+^- through the line segment $Q = Q^+$, $T = T^0$, $0 \le S < 1$ because on this set $\dot{Q} > 0$ and $\dot{T} = 0$. See the point (Q^+, T^0) in Figure 5.1. Also, because the S-coordinate of E^- is S^-, no solution $\Gamma(\theta)$ with $0 < \theta < 1$ can leave R_+^- through a point (θ, T, S) with $S \le S^-$. Combining this information with that in the previous paragraph, we have the following picture. For $T^0 < T < T^b$ and $S^- < S < 1$, define

$$\tilde{Q}(T, S) = \min(-\frac{D(T, S)}{C(T, S)}, Q^+).$$

Let $U = \{(T, S) : T^0 < T < T^b, S^- < S < 1, \text{ and } \tilde{Q}(T, S) > 0\}$, an open subset of TS-space. Then all $\Gamma(\theta)$ with $0 < \theta < 1$ leave the interior of R_+^- through one of the following three disjoint sets:

1. $G_1 = \{(Q, T, S) : (T, S) \in U \text{ and } Q = \tilde{Q}(T, S)\}$,
2. $G_2 = \{(Q, T, S) : 0 < Q < Q^+, T^0 < T < T^b, S = 1\}$,
3. $G_3 = \{(Q, T, S) : Q = Q^+, T^0 \le T < T^b, S = 1\}$.

The first two sets are two-dimensional. G_3 is a one-dimensional set in the common boundary of G_1 and G_2.

Let I_j denote the set of θ in $0 < \theta < 1$ such that $\Gamma(\theta)$ leaves the interior of R_+^- through G_j. On G_2, $\dot{S} > 0$. On G_1, where $Q = Q^+$, $\dot{Q} > 0$; and where $Q = -\frac{D(T,S)}{C(T,S)}$, $\dot{S} = 0$, so by the assumption of the theorem,

$$\nabla \dot{S} \cdot (\dot{Q}, \dot{T}, \dot{S}) = (C, Q\frac{\partial C}{\partial T} + \frac{\partial D}{\partial T}, Q\frac{\partial C}{\partial S} + \frac{\partial D}{\partial S}) \cdot (\dot{Q}, \dot{T}, 0)$$
$$= q_b(S - 1)(T - T^b)C + (QA + B)(Q\frac{\partial C}{\partial T} + \frac{\partial D}{\partial T})$$

is negative. It follows easily that I_j is open for $j = 1, 2$. The sets I_1 and I_2 are disjoint and nonempty: I_1 includes θ near 0, and because of property (9), I_2 includes θ near 1. Since the interval $0 < \theta < 1$ is connected, it cannot be the union of disjoint open sets. Therefore there exists θ_0 in this interval such that $\Gamma(\theta_0)$ leaves the interior of R_+^- through G_3 or is asymptotic to an invariant set in G_3.

On G_3, $\dot{Q} = \dot{S} = 0$. In fact, G_3 consists of (1) the equilibrium E^+ and (2) the open line segment $Q = Q^+$, $T^0 < T < T^b$, $S = 1$, on which $\dot{T} < 0$ by property (4). This segment is a proper subset of a solution curve. Thus $\Gamma(\theta_0)$ cannot leave the interior of R_+^- through G_3, and the only invariant subset of G_3 to which it can be asymptotic is E^+. $\qquad\square$

We now state and prove Lemma 5.2. To simplify the notation, let

$$q = q_b(S^- - 1) < 0,$$
$$a = A(T^b) < 0,$$
$$b = B'(T^b),$$
$$c = C(T^b, S^-) < 0,$$
$$d = \frac{\partial D}{\partial T}(T^b, S^-) > 0,$$
$$\alpha = \sqrt{b^2 + 4aq}.$$

Lemma 5.2. *The linearization of Eqs. (5.1)–(5.3) at E^- has an eigenvector v for the positive eigenvalue that points into R_-^-.*

Proof. With the above notation, Eq. (5.4) becomes the matrix

$$\begin{pmatrix} 0 & q & 0 \\ a & b & 0 \\ c & d & 0 \end{pmatrix}.$$

The positive eigenvalue is

$$\frac{b + \alpha}{2} > 0. \tag{5.5}$$

An eigenvector is

$$v = (\frac{b + \alpha}{2}, \frac{b^2 + \alpha b + 2aq}{2q}, *),$$

where the last component is not important. Now the expression $b^2 + \alpha b + 2aq$ is positive for $b = 0$, and the equation $b^2 + \alpha b + 2aq = 0$ implies, after a little algebra, that $a^2q^2 = 0$. Since this is false, we have

$$b^2 + \alpha b + 2aq > 0 \text{ for all } b. \tag{5.6}$$

From Eqs. (5.5) and (5.6), the first two components of v have signs $(+, -)$. Therefore the vector v, based at $E^- = (0, T^b, S^-)$, points into the interior of R.

To see that v points into the interior of R_-^-, we regard $\dot{T}$ and $\dot{S}$ as functions of (Q, T, S), and we check that at E^-, $\nabla \dot{T} \cdot v < 0$ and $\nabla \dot{S} \cdot v < 0$.

At E^-,

$$\nabla \dot{T} \cdot v = (a, b, 0) \cdot (\frac{b + \alpha}{2}, \frac{b^2 + \alpha b + 2aq}{2q}, *) = \frac{1}{2q}(3abq + a\alpha q + b^3 + \alpha b^2).$$

For $b = 0$, this expression is negative. Setting this expression equal to 0 implies, after some algebra, that $a^3 q^3 = 0$. Since this is false, $\nabla \dot{T} \cdot v < 0$ for all b.

At E^-,

$$\nabla \dot{S} \cdot v = (c, d, 0) \cdot (\frac{b + \alpha}{2}, \frac{b^2 + \alpha b + 2aq}{2q}, *) = \frac{c(b + \alpha)q + d(b^2 + \alpha b + 2aq)}{2q}.$$

From $c < 0$, $q < 0$, $d > 0$, Eqs. (5.5) and (5.6), this expression is negative. $\square$

The linearization of Eqs. (5.1)–(5.3) at E^+ has the matrix

$$\begin{pmatrix} 0 & 0 & q_b(T^0 - T^b) \\ A(T^0) & Q^+ A'(T^0) + B'(T^0) & 0 \\ C(T^0, 1) & 0 & Q^+ \frac{\partial C}{\partial S}(T^0, 1) + \frac{\partial D}{\partial S}(T^0, 1) \end{pmatrix}.$$

The characteristic equation is

$$(\lambda - Q^+ A'(T^0) - B'(T^0))(\lambda^2 - (Q^+ \frac{\partial C}{\partial S}(T^0, 1) + \frac{\partial D}{\partial S}(T^0, 1))\lambda - q_b(T^0 - T^b)C(T^0, 1)) = 0.$$

The eigenvalues are $Q^+ A'(T^0) - B'(T^0)$ and two whose product is $-q_b(T^0 - T^b)C(T^0, 1)$. Assumptions (3) and (4) imply that $Q^+ A'(T^0) - B'(T^0) \leq 0$. Since $q_b(T^0 - T^b)C(T^0, 1) > 0$, the other two eigenvalues are one positive and one negative. Thus at E^+ there is a one-dimensional stable manifold and a two-dimensional stable or center-stable manifold. This fact was used in Section 4.

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APPENDIX A. PHYSICAL QUANTITIES, SYMBOLS AND VALUES

In this Appendix we summarize the values and units of the various physical quantities used in the computation and empirical expressions for the various parameter functions. For convenience we express the heat capacity of the rock C_r^p in terms of energy per unit volume of *porous medium* per unit temperature *i.e.* the factor $1 - \varphi$ is included in the rock density. All other densities are expressed in terms of mass per unit volume of the phase.

Temperature-dependent properties of steam and water

We use reference [21] to obtain all the temperature-dependent properties below. The water and steam densities used to obtain the enthalpies are defined at the bottom. First we obtain the boiling point T^b at the given pressure p, *i.e.*

$$T^b = 280.034 + \ell(14.0856 + \ell(1.38075 + \ell(-0.101806 + 0.019017\ell))), \qquad (A.1)$$

where $\ell = \log(p)$ and p is the pressure in [k Pa]. The evaporation heat [J/kg] is given as a function of the temperature T at which the evaporation occurs. We use atmospheric pressure ($p = 101.325$ [k Pa]) in our computations, to make the example representative of subsurface contaminant cleaning.

The liquid water enthalpy $h_w(T)$ [J/kg] as a function of temperature is approximated by

$$h_w(T) = 2.36652 \times 10^7 - 3.66232 \times 10^5 T + 2.26952 \times 10^3 T^2 - 7.30365 T^3 + 1.30241 \times 10^{-2} T^4$$
$$- 1.22103 \times 10^{-5} T^5 + 4.70878 \times 10^{-9} T^6. \qquad (A.2)$$

The steam enthalpy h_g [J/kg] as a function of temperature is approximated by

$$h_g(T) = -2.20269 \times 10^7 + 3.65317 \times 10^5 T - 2.25837 \times 10^3 T^2 + 7.3742 T^3$$
$$- 1.33437 \times 10^{-2} T^4 + 1.26913 \times 10^{-5} T^5 - 4.9688 \times 10^{-9} T^6. \qquad (A.3)$$

For the latent heat h_g^l [J/kg] or evaporation heat $\Lambda(T)$ we obtain

$$h_g^l(T) = \left(7.1845 \times 10^{12} + 1.10486 \times 10^{10}T - 8.8405 \times 10^{7}T^2 + 1.6256 \times 10^{5}T^3 - 121.377T^4\right)^{\frac{1}{2}}. \tag{A.4}$$

The sensible heat of steam $H_g^s(T)$ in [J/m^3] is given as

$$H_g^s(T) = \rho_g \left(h_g(T) - h_w\left(T^0\right) - \Lambda\left(T^0\right)\right). \tag{A.5}$$

We also use the temperature-dependent steam viscosity

$$\mu_g(T) = -5.46807 \times 10^{-4} + 6.89490 \times 10^{-6}T - 3.39999 \times 10^{-8}T^2 + 8.29842 \times 10^{-11}T^3$$

$$-9.97060 \times 10^{-14}T^4 + 4.71914 \times 10^{-17}T^5. \tag{A.6}$$

The temperature-dependent water viscosity μ_w is approximated by

$$\mu_w(T) = -0.0123274 + \frac{27.1038}{T} - \frac{23527.5}{T^2} + \frac{1.01425 \times 10^7}{T^3} - \frac{2.17342 \times 10^9}{T^4} + \frac{1.86935 \times 10^{11}}{T^5}. \tag{A.7}$$

For the steam density as a function of temperature $T[K]$ we use a different expression than [21] because our interest is a steam density at constant pressure, which is not necessarily in equilibrium with liquid water. We use:

$$\rho_g(T) = p\frac{M_{H_2O}}{ZRT} \tag{A.8}$$

where p is the pressure, R=8.31 [J/mol K] and Z is the Z-factor (see e.g. [6]). For the atmospheric pressures of interest here the Z-factor is close to unity so we use $Z = 1$. The liquid water density as a function of the temperature $T[K]$ is given as

$$\rho_w(T) = 3786.31 - 37.2487T + 0.196246T^2 - 5.04708 \times 10^{-4}T^3$$

$$+6.29368 \times 10^{-7}T^4 - 3.08480 \times 10^{-10}T^5. \tag{A.9}$$

Table 1. Summary of physical parameters and variables			
Physical quantity	*Symbol*	*Value*	*Unit*
Water, steam fractional functions	f_w, f_g	Eq. (1.11).	[m³/m³]
Porous rock permeability	k	1.0×10^{-12}	[m³]
Water, steam relative permeabilities	k_{rw}, k_{rg}	Eqs. (A.10), (A.11).	[m³/m³]
Pressure	p	1.0135×10^5	[Pa]
Mass condensation rate	q	Eq. (1.5).	[kg /(m³s)]
Mass condensation rate coefficient	q_b	0.01	[kg /(m³sK)]
Water, steam phase velocity	u_w, u_g	Eq. (1.4).	[m³/(m²s)]
Total Darcy velocity	u	$u_w + u_g$, Eq. (1.15).	[m³/(m²s)]
SCF velocity	v^{SCF}	Eq. (B.4).	[m/s]
Effective rock heat capacity	C_r^p	2.029×10^6	[J/(m³K)]
Steam enthalpy per unit mass	h_g	Eq. (A.3)	[J/kg]
Rock enthalpy per unit mass	h_r	$C_r^p(T - T^0)/\rho_r$	[J/kg]
Water enthalpy per unit mass	h_w	(A.2)	[J/kg]
Steam enthalpy per unit volume	H_g	$\rho_g(T)h_g(T)$	[J/m³]
Steam sensible heat per unit volume	H_g^s	Eq. (A.5)	[J/m³]
Steam latent heat per unit volume	H_g^l	$\rho_g(T)\bar{\Lambda}$	[J/m³]
Rock enthalpy per unit volume	H_r	$C_r^p(T - T^0)$	[J/m³]
Water enthalpy per unit volume	H_w	$\rho_w(T)h_w(T)$	[J/m³]
Steam total enthalpy at boil. temp.	H_g^B	Eq. (B.5).	[J/m³]
Capillary pressure	P_c	Eq. (A.12).	[Pa]
Condensation rate parameter	q_b	Eq. (1.5)	[kg /m³/K]
Cumulative mass condensation	Q	Eq. (2.1).	[kg /m³]
Water, steam saturations	S_w, S_g	Dependent variables.	[m³/m³]
Connate water saturation	S_{wc}	0.15	[m³/m³]
Temperature	T	Dependent variable.	[K]
Reservoir temperature	T^0	293.	[K]
Boiling point of water–steam	T^b	Eq. (A.1).	[K]
Water, steam thermal conductivity	κ_w, κ_g	0.652, 0.0208	[W/(mK)]
Rock, composite thermal conductivity	κ_r, κ	1.83, Eq. (1.9).	[W/(mK)]
Water, steam viscosity	μ_w, μ_g	Eq. (A.7), Eq. (A.6).	[Pa s]
Rock density	ρ_s	2650	[kg/m³]
Effective rock density	ρ_r	$(1 - \varphi)\,\rho_s$;	[kg/m³]
Water, steam densities	ρ_w, ρ_g	Eq. (A.9), Eq. (A.8).	[kg/m³]
Interfacial tension	σ_{wg}	58×10^{-3}	[N/m]
Rock porosity	φ	0.38	[m³/m³]
Water evaporation heat	$\bar{\Lambda}$	Eq. (1.3).	[J/kg]
Capillary diffusion coefficient	Ω	Eq. (1.13), Eq. (1.14).	[m³/m³]

A.1. **Constitutive relations.** The relative permeability functions k_{rw} and k_{rg} are considered to be power functions of their respective saturations [7], *i.e.*,

$$k_{rw} = \begin{cases} \left(\frac{S_w - S_{wc}}{1 - S_{wc}}\right)^{n_w} & \text{for} \quad S_w \geq S_{wc}, \\ 0 & \text{for} \quad 0 \leq S_w \leq S_{wc}, \end{cases} \qquad (A.10)$$

$$k_{rg} = \left(\frac{S_g}{1 - S_{wc}}\right)^{n_g}. \qquad (A.11)$$

For the computations we take $n_w = 4, = n_g = 2$. The connate water saturation S_{wc} is given in the table.

The capillary pressure is of the Brooks-Corey type. We use the dimensionless capillary pressure $P_c(S_w = 0.5)/\left(\sigma_{wg}\sqrt{\varphi/k}\right) = 0.5$ which is appropriate for many sandstones. The capillary pressure between steam and water is given by the empirical expression which combines Leverett's approach to non-dimensionalize the capillary pressure [10] with the semi-empirically determined saturation dependence suggested in [4]:

$$P_c = \sigma_{wg}\gamma\sqrt{\frac{\varphi}{k}} \left(\frac{\frac{1}{2} - S_{wc}}{1 - S_{wc}}\right)^{\frac{1}{\lambda_s}} \left(\frac{S_w - S_{wc}}{1 - S_{wc}}\right)^{-\frac{1}{\lambda_s}}, \qquad (A.12)$$

where γ is a parameter that in many cases assumes values between 0.3 and 0.7. We use $\gamma = 0.5$ and $\lambda_s = \frac{1}{2}$. Finally $\sigma_{wg} = 0.058$ N/m is the water-vapor interfacial tension. We disregard its temperature dependence and use the value at the boiling point (see [23], p. F-45).

Appendix B. Steam condensation front velocity

Equating Q^+ from Eqs. (2.11), (2.12), we obtain:

$$\left(\rho_w f_w + \rho_g f_g\right)^- - \tilde{v}^{SCF}\left(\rho_w S_w + \rho_g S_g\right)^- = \rho_w^+\left(\tilde{u}^+ - \tilde{v}^{SCF}\right). \qquad (B.1)$$

Similarly, we obtain from a Hugoniot condition derived from Eq.eq:enbal :(1.8)

$$\left(H_w f_w + H_g f_g\right)^- - \tilde{v}^{SCF}\left(H_r + \varphi H_w S_w + \varphi H_g S_g\right)^- = \tilde{u}^+ H_w^+ - \tilde{v}^{SCF}\left(H_r^+ + \varphi H_w^+\right) = 0. \qquad (B.2)$$

The right hand side term of Eq. (B.2) vanishes because $T^+ = T^0$, see Eq. (1.1). For the SCF velocity it follows from Eqs. (B.1), (B.2) that

$$\tilde{u}^+ = \left(\frac{\rho_g^-}{\rho_w^+}f_g^- + \frac{\rho_w^-}{\rho_w^+}f_w^-\right) - \tilde{v}^{SCF}\left(\frac{\rho_g^-}{\rho_w^+}S_g^- + \frac{\rho_w^-}{\rho_w^+}S_w^- - 1\right), \qquad (B.3)$$

$$\frac{\varphi v^{SCF}}{u^-} = \tilde{v}^{SCF} = \frac{H_w^- f_w^- + H_g^B f_g^-}{H_r^- + \varphi H_w^- S_w^- + \varphi H_g^B S_g^-}, \qquad (B.4)$$

where we use the nomenclature that follows from Eq. (1.3):

$$H_g^B \equiv H_g(T^-) = H_g^s(T^-) + H_g^l = H_g^{s-} + \rho_g^-\bar{\Lambda}. \qquad (B.5)$$

From Eqs. (B.3) and (B.4), we can write $\tilde{u}^+$:

$$\widetilde{u}^+ = \left(\frac{\rho_w^-}{\rho_w^+} f_w^- + \frac{\rho_g^-}{\rho_w^+} f_g^- \right) - \left(\frac{\rho_w^-}{\rho_w^+} S_w^- + \frac{\rho_g^-}{\rho_w^+} S_g^- - 1 \right) \frac{H_w^- f_w^- + H_g^B f_g^-}{H_r^- + \varphi H_w^- S_w^- + \varphi H_g^B S_g^-}. \tag{B.6}$$

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