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Heteroclinic Orbits in Slow–Fast Hamiltonian Systems with Slow Manifold Bifurcations

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Abstract Motivated by a problem in which a heteroclinic orbit represents a moving interface between ordered and disordered crystalline states, we consider a class of slow–fast Hamiltonian systems in which the slow manifold loses normal hyperbolicity due to a transcritical or pitchfork bifurcation as a slow variable changes. We show that under assumptions appropriate to the motivating problem, a singular heteroclinic solution gives rise to a true heteroclinic solution. In contrast to previous approaches to such problems, our approach uses blow-up of the bifurcation manifold, which allows geometric matching of inner and outer solutions.

Keywords Geometric singular perturbation theory · Blow-up · Pitchfork bifurcation · Transcritical bifurcation

Mathematics Subject Classification (2000) 34E15 · 34C37 · 37G99

1 Introduction

In [18], the authors consider a multi-order-parameter phase field model developed by Braun et al. [4] for the description of crystalline interphase boundaries. Assuming large anisotropy, a parameter ϵ is small. The authors are led to consider a second-order system of the form

$$x_{\tau\tau} = g_x(x, y), \tag{1.1}$$
$$\epsilon^2 y_{\tau\tau} = g_y(x, y); \tag{1.2}$$

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a heteroclinic orbit of this system that connects certain equilibria represents a moving interface between ordered and disordered states. The system (1.1), (1.2) can be regarded as Hamiltonian with potential energy $-g(x, y)$; the two equilibria are of course assumed to have equal potential energy.

The mathematical interest of the problem stems from the fact that, when converted to a 4-dimensional first order system, the 2-dimensional slow manifold undergoes a pitchfork bifurcation along a line as a slow variable changes (and thus is not actually a manifold, although we will refer to it as one). This bifurcation results in loss of normal hyperbolicity of the slow manifold along the line. The limiting problem has a heteroclinic solution that connects equilibria on the slow manifold, but it passes through the bifurcation line and in fact has a corner there due to the pitchfork bifurcation. The question is whether, for small $\epsilon > 0$, there is a true heteroclinic solution near this singular solution.

This question has been resolved in the affirmative by Fife [10], who used a shooting argument, and by the authors of [18] based on the contraction mapping principle. Key to the analysis of [18] is an inner solution, used near the bifurcation point. The existence of the inner solution, which is a solution with appropriate limiting behavior at $\pm\infty$ of a nonautonomous second-order ODE, had already been shown in [1]. The paper [18] is mostly devoted to matching of the inner and outer solutions.

In this paper we consider from the viewpoint of geometric singular perturbation theory [9,13,14] a class of problems that includes the one studied in [18]. In problems involving loss of normal hyperbolicity of the slow manifold, geometric singular perturbation theory typically does not help to find and study the inner solution, but it often does simplify matching of inner and outer solutions. Matching becomes a geometric problem of identifying the behavior of certain manifolds of solutions, which can often be resolved using standard results.

Using the blow-up procedure for dealing with loss of normal hyperbolicity of the slow manifold [8,15], we blow up the bifurcation line to a 3-sphere crossed with a line. The inner solution corresponds to a solution of the blown-up system that connects two equilibria on one of these spheres. Within the sphere, one equilibrium has a two-dimensional center-unstable manifold, and the other has a two-dimensional center-stable manifold. Results in [1] on the linearization of the nonautonomous second-order ODE along the inner solution imply that these manifolds intersect transversally along the inner solution. We show that this implies that for small $\epsilon > 0$, the 2-dimensional unstable and stable manifolds of the equilibria to be connected meet transversally within a 3-dimensional energy surface. This result yields existence of the desired heteroclinic solution.

We begin in Sects. 2 and 3 by describing a slightly simplified version of the system studied in [18]. Then in Sect. 4 we describe a more general version of this system, in which the 2-dimensional slow manifold undergoes a pitchfork bifurcation along a line as a slow variable changes; we give assumptions and state our result. The result is proved in Sects. 5 to 9. In Sect. 10 we state the analogous result for transcritical bifurcation of the slow manifold, and indicate where changes in the proof are required.

2 Model System

We consider the second-order system (1.1), (1.2), with

$$g(x, y) = h(x) - \frac{1}{2}xy^2 + \frac{1}{4}y^4. \tag{2.1}$$

The function h is C^2 and will shortly be described more precisely.

Write (1.1), (1.2) as a first-order system (the slow system):

$$u_{1\tau} = u_2, \tag{2.2}$$

$$u_{2\tau} = g_x(u_1, u_3) = h'(u_1) - \frac{1}{2}u_3^2, \tag{2.3}$$

$$\epsilon u_{3\tau} = u_4, \tag{2.4}$$

$$\epsilon u_{4\tau} = g_y(u_1, u_3) = u_3^3 - u_1 u_3. \tag{2.5}$$

In (2.2)–(2.5) let $\tau = \epsilon\sigma$. We obtain the fast system:

$$u_{1\sigma} = \epsilon u_2, \tag{2.6}$$

$$u_{2\sigma} = \epsilon g_x(u_1, u_3) = \epsilon\left(h'(u_1) - \frac{1}{2}u_3^2\right), \tag{2.7}$$

$$u_{3\sigma} = u_4, \tag{2.8}$$

$$u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3. \tag{2.9}$$

For each ϵ, (2.6)–(2.9) has the first integral

$$H(u_1, u_2, u_3, u_4) = \frac{1}{2}u_2^2 + \frac{1}{2}u_4^2 - g(u_1, u_3).$$

Indeed, for $\epsilon > 0$, (2.6)–(2.9) is a Hamiltonian system, with Hamiltonian H, in the sense that there is a nonsingular skew-symmetric matrix

$$J = \begin{pmatrix} 0 & \epsilon & 0 & 0 \\ -\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

such that $(u_{1\sigma} \quad u_{2\sigma} \quad u_{3\sigma} \quad u_{4\sigma})^\top = J\left(\frac{\partial H}{\partial u_1} \quad \frac{\partial H}{\partial u_2} \quad \frac{\partial H}{\partial u_3} \quad \frac{\partial H}{\partial u_4}\right)^\top$.

The equilibria of (2.6)–(2.9) for $\epsilon > 0$ are:

- $(u_1, 0, 0, 0)$ with $h'(u_1) = 0$.
- $(u_1, 0, u_3, 0)$ with $u_1 = u_3^2$ and $h'(u_1) - \frac{1}{2}u_1 = 0$.

We make the following assumptions on h. Let

$$\ell(x) = \begin{cases} h'(x) & \text{if } x \le 0, \\ h'(x) - \frac{1}{2}x & \text{if } x > 0. \end{cases}$$

Then we assume:

(1) There are exactly three points x_-, x_0, x_+, with $x_- < 0 < x_0 < x_+$, at which $\ell(x) = 0$.
(2) $\ell'(x_-) = h''(x_-) > 0$.
(3) $\ell'(x_0) = h''(x_0) - \frac{1}{2} < 0$.
(4) $\ell'(x_+) = h''(x_+) - \frac{1}{2} > 0$.
(5) $\int_{x_-}^{x_+} \ell(x)\,dx = 0$.
(6) $h(x_-) = 0$.

See Fig. 1.

With these assumptions the equilibria of (2.6)–(2.9) for $\epsilon > 0$ are

$$(x_-, 0, 0, 0), \quad \left(x_0, 0, \pm x_0^{\frac{1}{2}}, 0\right), \quad \left(x_+, 0, \pm x_+^{\frac{1}{2}}, 0\right),$$

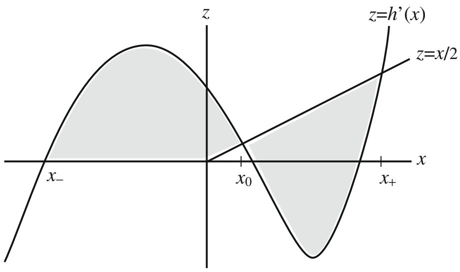

Fig. 1 Graph of $z = h'(x)$. The shaded areas are equal

and possibly others with $u_2 = u_3 = u_4 = 0$ and $u_1 > 0$.

Our goal is to show that for small $\epsilon > 0$, there is a heteroclinic solution of (2.6)–(2.9) from $(x_-, 0, 0, 0)$ to $\left(x_+, 0, x_+^{\frac{1}{2}}, 0\right)$. There is another from $(x_-, 0, 0, 0)$ to $\left(x_+, 0, -x_+^{\frac{1}{2}}, 0\right)$, but we shall not consider it.

Lemma 2.1 $h(x_+) = \frac{1}{4}x_+^2$.

Proof

$$0 = \int_{x_-}^{x_+} \ell(x)\mathrm{d}x = \int_{x_-}^{0} h'(x)\mathrm{d}x + \int_{0}^{x_+} \left(h'(x) - \frac{1}{2}x\right)\mathrm{d}x$$
$$= h(0) + \left(h(x_+) - \frac{1}{4}x_+^2 - h(0)\right) = h(x_+) - \frac{1}{4}x_+^2.$$
□

Using assumption (6) and Lemma 2.1, we see that

$$H(x_-, 0, 0, 0) = H\left(x_+, 0, x_+^{\frac{1}{2}}, 0\right) = 0.$$

It is easy to check that for $\epsilon > 0$, $(x_-, 0, 0, 0)$ and $\left(x_+, 0, x_+^{\frac{1}{2}}, 0\right)$ are hyperbolic equilibria of (2.6)–(2.9) with two negative eigenvalues and two positive eigenvalues. The heteroclinic solution that we will find corresponds to an intersection of the 2-dimensional manifolds $W_\epsilon^u(x_-, 0, 0, 0)$ and $W_\epsilon^s\left(x_+, 0, x_+^{\frac{1}{2}}, 0\right)$ that is transverse within the 3-dimensional manifold $H^{-1}(0)$ (which is indeed a manifold away from equilibria).

3 Fast Limit and Slow Systems

Setting $\epsilon = 0$ in (2.6)–(2.9), we obtain the fast limit system:

$$u_{1\sigma} = 0, \tag{3.1}$$
$$u_{2\sigma} = 0, \tag{3.2}$$
$$u_{3\sigma} = u_4, \tag{3.3}$$
$$u_{4\sigma} = g_y(u_1, u_3) = -u_1 u_3 + u_3^3. \tag{3.4}$$

The equilibria of (3.1)–(3.4) are the 2-dimensional plane

$$E = \{(u_1, u_2, 0, 0) : u_1 \text{ and } u_2 \text{ arbitrary}\}$$

and the 2-dimensional parabolic cylinder

$$F = \{(u_1, u_2, u_3, 0) : u_1 = u_3^2,\ u_2 \text{ and } u_3 \text{ arbitrary}\}.$$

These two surfaces meet along the u_2-axis. See Fig. 3.

The matrix of the linearization of (3.1)–(3.4) is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ g_{xy}(u_1, u_3) & 0 & g_{yy}(u_1, u_3) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -u_3 & 0 & -u_1 + 3u_3^2 & 0 \end{pmatrix}. \tag{3.5}$$

Therefore equilibria (u_1, u_2, u_3, u_4) of (3.1)–(3.4) are normally hyperbolic if and only if $g_{yy}(u_1, u_3) = -u_1 + 3u_3^2 > 0$, in which case there is one positive and one negative eigenvalue, in addition to two zero eigenvalues. Hence (3.1)–(3.4) has three manifolds of normally hyperbolic equilibria:

$$\begin{aligned} E_- &= \{(u_1, u_2, 0, 0) : u_1 < 0 \text{ and } u_2 \text{ arbitrary}\}, \\ F_- &= \left\{\left(u_1, u_2, -u_1^{\frac{1}{2}}, 0\right) : u_1 > 0 \text{ and } u_2 \text{ arbitrary}\right\}, \\ F_+ &= \left\{\left(u_1, u_2, u_1^{\frac{1}{2}}, 0\right) : u_1 > 0 \text{ and } u_2 \text{ arbitrary}\right\}. \end{aligned}$$

In addition, there is a manifold of equilibria $E_+ = \{(u_1, u_2, 0, 0) : u_1 > 0 \text{ and } u_2 \text{ arbitrary}\}$ that is not normally hyperbolic. See Fig. 3.

Setting $\epsilon = 0$ in (2.2)–(2.5), we obtain the slow limit system:

$$u_{1\tau} = u_2, \tag{3.6}$$

$$u_{2\tau} = g_x(u_1, u_3) = -\frac{1}{2}u_3^2 + h'(u_1), \tag{3.7}$$

$$0 = u_4, \tag{3.8}$$

$$0 = g_y(u_1, u_3) = -u_1 u_3 + u_3^3. \tag{3.9}$$

E_-, E_+, F_-, and F_+ are manifolds of solutions of the algebraic Eqs. 3.8, 3.9. Equations 3.6, 3.7 then give the slow system on these manifolds. We are interested only in the slow system on E_- and F_+.

Slow system on E_- ($u_1 < 0$, u_2 arbitrary):

$$u_{1\tau} = u_2, \tag{3.10}$$

$$u_{2\tau} = g_x(u_1, 0) = h'(u_1). \tag{3.11}$$

Slow system on F_+ ($u_1 > 0$, u_2 arbitrary):

$$u_{1\tau} = u_2, \tag{3.12}$$

$$u_{2\tau} = g_x\left(u_1, u_1^{\frac{1}{2}}\right) = -\frac{1}{2}u_1 + h'(u_1). \tag{3.13}$$

Phase portraits of (3.10), (3.11) and (3.12), (3.13), both extended to $u_1 = 0$, are shown in Fig. 2. Note:

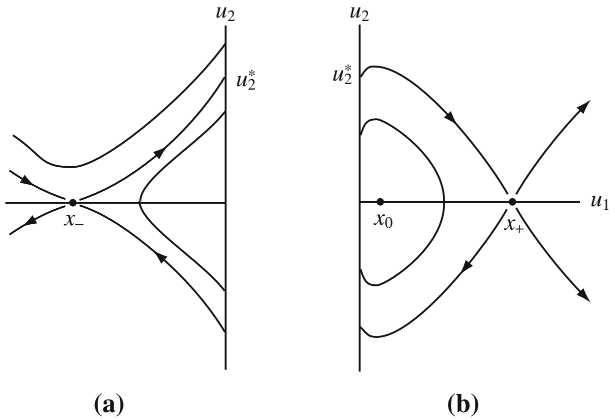

Fig. 2 **a** Phase portrait of (3.10)–(3.11), $u_1 \leq 0$. **b** Phase portrait of (3.12)–(3.13), $u_1 \geq 0$

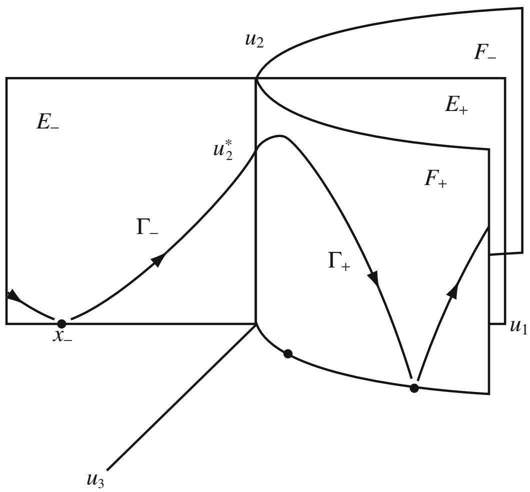

Fig. 3 Phase portrait of the slow limit system (3.6)–(3.9) on E_- and F_+. The space $u_4 = 0$ is shown

(1) For (3.10), (3.11), $(x_-, 0)$ is a hyperbolic saddle, and a branch of its unstable manifold meets the u_2 axis at a point $(0, u_2^*)$.
(2) For (3.12), (3.13), $(x_+, 0)$ is a hyperbolic saddle, and a branch of its stable manifold meets the u_2 axis at the same point $(0, u_2^*)$.

We must have $0 = H(0, u_2^*, 0, 0) = \frac{1}{2}(u_2^*)^2 - h(0)$, so $u_2^* = (2h(0))^{\frac{1}{2}}$.

Figure 3 shows the phase portrait of the slow limit system (3.6)–(3.9) on E_- and F_+. Let

- Γ_- denote the set of $(u_1, u_2, 0, 0)$ such that $x_- < u_1 < 0$ and (u_1, u_2) is in the unstable manifold of $(x_-, 0)$ for (3.10), (3.11);
- Γ_+ denote the set of $\left(u_1, u_2, u_1^{\frac{1}{2}}, 0\right)$ such that $0 < u_1 < x_+$ and (u_1, u_2) is in the stable manifold of $(x_+, 0)$ for (3.12), (3.13).

The following result is proved in [10] and [18]:

Theorem 3.1 *For small* $\epsilon > 0$*, there is a heteroclinic solution of* (2.6)–(2.9) *from* $(x_-, 0, 0, 0)$ *to* $\left(x_+, 0, x_+^{\frac{1}{2}}, 0\right)$ *that is close to* $\Gamma_- \cup \{(0, u_2^*, 0, 0)\} \cup \Gamma_+$.

In the following sections we shall prove a somewhat more general result.

4 General System

We consider the system

$$\bar{x}_{\bar{\tau}\bar{\tau}} = \bar{g}_{\bar{x}}(\bar{x}, \bar{y}, \epsilon), \tag{4.1}$$

$$\epsilon^2 \bar{y}_{\bar{\tau}\bar{\tau}} = \bar{g}_{\bar{y}}(\bar{x}, \bar{y}, \epsilon), \tag{4.2}$$

where $\bar{g}$ is smooth and

$$\bar{g}(\bar{x}, \bar{y}, \epsilon) = \bar{h}(\bar{x}, \epsilon) + \bar{y}\bar{k}(\bar{x}, \bar{y}, \epsilon). \tag{4.3}$$

We assume:

(P1) At $(\bar{x}, \bar{y}, \epsilon) = (0, 0, 0)$, $\quad \bar{k} = \bar{k}_{\bar{x}} = \bar{k}_{\bar{y}} = \bar{k}_{\epsilon} = \bar{k}_{\bar{y}\bar{y}} = 0, \quad \bar{k}_{\bar{x}\bar{y}} < 0, \quad \bar{k}_{\bar{y}\bar{y}\bar{y}} > 0.$ (4.4)

Hence

$$\bar{k}(\bar{x}, \bar{y}, 0) = \lambda \bar{x}^2 - \frac{\mu}{2}\bar{x}\bar{y} + \frac{\nu}{4}\bar{y}^3 + \cdots \tag{4.5}$$

with λ arbitrary, $\mu > 0$, and $\nu > 0$; "…" represents other cubic terms and higher order terms.

A simple change of variables allows us to assume $\mu = \nu = 1$. More precisely, the change of variables

$$\bar{x} = \frac{\mu}{\nu}x, \quad \bar{y} = \frac{\mu}{\nu}y, \quad \bar{\tau} = \frac{\nu^{\frac{1}{2}}}{\mu}\tau$$

converts (4.1), (4.2) to

$$x_{\tau\tau} = g_x(x, y, \epsilon), \tag{4.6}$$

$$\epsilon^2 y_{\tau\tau} = g_y(x, y, \epsilon), \tag{4.7}$$

with

$$g(x, y, \epsilon) = h(x, \epsilon) + yk(x, y, \epsilon),$$

where $g(x, y, \epsilon) = \frac{\nu^3}{\mu^4}\bar{g}\left(\frac{\mu}{\nu}x, \frac{\mu}{\nu}y, \epsilon\right)$, $h(x, \epsilon) = \frac{\nu^3}{\mu^4}\bar{h}\left(\frac{\mu}{\nu}x, \epsilon\right)$, and $k(x, y, \epsilon) = \frac{\nu^2}{\mu^3}\bar{k}\left(\frac{\mu}{\nu}x, \frac{\mu}{\nu}y, \epsilon\right)$; hence

$$k(x, y, 0) = \omega x^2 - \frac{1}{2}xy + \frac{1}{4}y^3 + \cdots,$$

with $\omega = \frac{\lambda}{\mu}$; "…" represents other cubic terms and higher order terms. For the model system, the rescaling is not necessary; h is independent of ϵ and satisfies the assumptions given in Sect. 2, and $k(x, y, \epsilon) = -\frac{1}{2}xy + \frac{1}{4}y^3$.

We note that

$$g_x(x, y, 0) = h_x(x, 0) + yk_x(x, y, 0) = h_x(x, 0) + y\left(2\omega x - \frac{1}{2}y + \cdots\right), \tag{4.8}$$

$$g_y(x, y, 0) = k(x, y, 0) + yk_y(x, y, 0) = \omega x^2 - xy + y^3 + \cdots; \tag{4.9}$$

in (4.8), "…" represents higher order terms; in (4.9), "…" represents other cubic terms and higher order terms.

Write (4.6), (4.7) as a first-order system (the slow system):

$$u_{1\tau} = u_2, \tag{4.10}$$

$$u_{2\tau} = g_x(u_1, u_3, \epsilon), \tag{4.11}$$

$$\epsilon u_{3\tau} = u_4, \tag{4.12}$$

$$\epsilon u_{4\tau} = g_y(u_1, u_3, \epsilon). \tag{4.13}$$

In (4.10)–(4.13) let $\tau = \epsilon\sigma$. We obtain the fast system:

$$u_{1\sigma} = \epsilon u_2, \tag{4.14}$$

$$u_{2\sigma} = \epsilon g_x(u_1, u_3, \epsilon), \tag{4.15}$$

$$u_{3\sigma} = u_4, \tag{4.16}$$

$$u_{4\sigma} = g_y(u_1, u_3, \epsilon). \tag{4.17}$$

For each ϵ, (4.14)–(4.17) has the first integral

$$H(u_1, u_2, u_3, u_4) = \frac{1}{2}u_2^2 + \frac{1}{2}u_4^2 - g(u_1, u_3, \epsilon), \tag{4.18}$$

and is Hamiltonian in the sense described in Sect. 2.

Setting $\epsilon = 0$ in (4.14)–(4.17), we obtain the fast limit system:

$$u_{1\sigma} = 0, \tag{4.19}$$

$$u_{2\sigma} = 0, \tag{4.20}$$

$$u_{3\sigma} = u_4, \tag{4.21}$$

$$u_{4\sigma} = g_y(u_1, u_3, 0). \tag{4.22}$$

There exist $u_e > 0$ and $u_f > 0$ such that this system has two 2-dimensional manifolds of equilibria (see [11, Ch. 1]), defined as follows:

$$E = \{(u_1, u_2, u_3, 0) : |u_1| < u_e, u_2 \text{ arbitrary}, u_3 = e(u_1) = \omega u_1 + \cdots\},$$
$$F = \{(u_1, u_2, u_3, 0) : |u_3| < u_f, u_2 \text{ arbitrary}, u_1 = f(u_3) = u_3^2 + \cdots\}.$$

E and F meet along the u_2-axis. For the model system, we can take $u_e = u_f = \infty$; $e(u_1) = 0$ and $f(u_3) = u_3^2$.

The matrix of the linearization of (4.19)–(4.22) is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ g_{xy}(u_1, u_3, 0) & 0 & g_{yy}(u_1, u_3, 0) & 0 \end{pmatrix}. \tag{4.23}$$

Therefore equilibria (u_1, u_2, u_3, u_4) of (4.19)–(4.22) are normally hyperbolic if and only if $g_{yy}(u_1, u_3, 0) > 0$, in which case there is one positive and one negative eigenvalue, in addition to two zero eigenvalues.

We have $g_{yy}(x, y, 0) = -x + 3y^2 + \cdots$. Therefore, on E, $g_{yy}(u_1, e(u_1), 0) = -u_1 + \cdots$, and on F, $g_{yy}(f(u_3), u_3, 0) = -(u_3^2 + \cdots) + 3u_3^2 + \cdots = 2u_3^2 + \cdots$. We define E_- (respectively E_+) to be the subset of E with $u_1 < 0$ (respectively $u_1 > 0$), and F_- (respectively F_+) to be the subset of F with $u_3 < 0$ (respectively $u_3 > 0$). E_-, F_-, and F_+ are normally hyperbolic for (4.19)–(4.22).

The equation $u_1 = f(u_3)$ can be solved for u_3 as a smooth function of w with $w^2 = u_1$: $u_3 = m(w) = w + \cdots$. Then there exists $u_m > 0$ such that

$$F_- = \left\{(u_1, u_2, u_3, 0) : 0 < u_1 < u_m, u_2 \text{ arbitrary, and } u_3 = m\left(-u_1^{\frac{1}{2}}\right) = -u_1^{\frac{1}{2}} + \cdots\right\},$$
$$F_+ = \left\{(u_1, u_2, u_3, 0) : 0 < u_1 < u_m, u_2 \text{ arbitrary, and } u_3 = m\left(u_1^{\frac{1}{2}}\right) = u_1^{\frac{1}{2}} + \cdots\right\}.$$

For the model system, we can take $u_m = \infty$, and $m(w) = w$, so F_- is given by $u_3 = -u_1^{\frac{1}{2}}$, and F_+ is given by $u_3 = u_1^{\frac{1}{2}}$.

Setting $\epsilon = 0$ in (4.10)–(4.13), we obtain the slow limit system:

$$u_{1\tau} = u_2, \tag{4.24}$$
$$u_{2\tau} = g_x(u_1, u_3, 0), \tag{4.25}$$
$$0 = u_4, \tag{4.26}$$
$$0 = g_y(u_1, u_3, 0). \tag{4.27}$$

E_-, E_+, F_-, and F_+ are manifolds of solutions of the system of Eqs. 4.26, 4.27. Equations (4.24), (4.25) then give the slow system on these manifolds. We are interested only in the slow system on E_- and F_+.

Slow system on E_-:

$$u_{1\tau} = u_2, \tag{4.28}$$
$$u_{2\tau} = g_x(u_1, e(u_1), 0), \tag{4.29}$$

with $-u_e < u_1 < 0$ and u_2 arbitrary.

Slow system on F_+:

$$u_{1\tau} = u_2, \tag{4.30}$$
$$u_{2\tau} = g_x\left(u_1, m\left(u_1^{\frac{1}{2}}\right), 0\right), \tag{4.31}$$

with $0 < u_1 < u_m$ and u_2 arbitrary.

We assume:

(P2) The system (4.28), (4.29) has a hyperbolic saddle equilibrium $(x_-, 0)$ with $-u_e < x_- < 0$. One branch B_- of its unstable manifold arrives at a point $(0, u_2^*)$ on the u_2-axis with $u_2^* > 0$.

(P3) The system (4.30), (4.31) has a hyperbolic saddle equilibrium $(x_+, 0)$ with $0 < x_+ < u_m$. One branch B_+ of its stable manifold arrives at the same point $(0, u_2^*)$ on the u_2-axis.

It follows that:

(1) System (4.14)–(4.17) has the smooth families of equilibria $p_-(\epsilon)$ and $p_+(\epsilon)$, with $p_-(0) = (x_-, 0, e(x_-), 0)$ and $p_+(0) = \left(x_+, 0, m\left(x_+^{\frac{1}{2}}\right), 0\right)$. For $\epsilon > 0$ these equilibria are hyperbolic, with two eigenvalues with positive real part and two with negative real part. For the model system $p_\pm(\epsilon)$ are independent of ϵ.

(2) $H(p_-(0)) = H(p_+(0))$.

Remark 4.1 Conversely, it is easy to show that (P2) and (P3) follow from (1), (2), if we further assume that $g(x, y, 0) > g(x_-, e(x_-), 0)$ on $\{(x, e(x)),\ x_- < x \le 0\} \cup \{\left(x, m(x^{\frac{1}{2}})\right),\ 0 \le x < x_+\}$ (see [2]). Note that $(x_-, e(x_-))$, $\left(x_+, m\left(x_+^{\frac{1}{2}}\right)\right)$ are equal nondegenerate minima of $g(\cdot, \cdot, 0)$.

We assume:

(P4) $H(p_-(\epsilon)) = H(p_+(\epsilon))$ for $\epsilon \ge 0$.

Let

- Γ_- denote the set of $(u_1, u_2, e(u_1), 0)$ such that $(u_1, u_2) \in B_-$;
- Γ_+ denote the set of $\left(u_1, u_2, m\left(u_1^{\frac{1}{2}}\right), 0\right)$ such that $(u_1, u_2) \in B_+$.

We shall prove

Theorem 4.2 *Assume (P1)–(P4). Then for small $\epsilon > 0$, there is a heteroclinic solution of (4.14)–(4.17) from $p_-(\epsilon)$ to $p_+(\epsilon)$ that is close to $\Gamma_- \cup \{(0, u_2^*, 0, 0)\} \cup \Gamma_+$.*

5 Blow-Up

To the fast system (4.14)–(4.17) we append the equation $\epsilon_\sigma = 0$, obtaining the following system on $u_1u_2u_3u_4\epsilon$-space:

$$u_{1\sigma} = \epsilon u_2, \tag{5.1}$$
$$u_{2\sigma} = \epsilon g_x(u_1, u_3, \epsilon), \tag{5.2}$$
$$u_{3\sigma} = u_4, \tag{5.3}$$
$$u_{4\sigma} = g_y(u_1, u_3, \epsilon), \tag{5.4}$$
$$\epsilon_\sigma = 0. \tag{5.5}$$

In $u_1u_2u_3u_4\epsilon$-space we shall blow up the u_2-axis, which consists of equilibria of (5.1)–(5.4) with $\epsilon = 0$ that are not normally hyperbolic within $u_1u_2u_3u_4$-space, to the product of u_2-space with a 3-sphere. The 3-sphere is a blow-up of the origin in $u_1u_3u_4\epsilon$-space.

The blowup transformation is a map Φ from *blow-up space* $\mathbb{R} \times S^3 \times [0, \infty)$ to $u_1u_2u_3u_4\epsilon$-space defined as follows. Let $(u_2, (\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\epsilon}), \bar{r})$ be a point of $\mathbb{R} \times S^3 \times [0, \infty)$; we have $\bar{u}_1^2 + \bar{u}_3^2 + \bar{u}_4^2 + \bar{\epsilon}^2 = 1$. Then the blow-up transformation Φ is given by

$$u_1 = \bar{r}^2\bar{u}_1, \quad u_2 = u_2, \quad u_3 = \bar{r}\bar{u}_3, \quad u_4 = \bar{r}^2\bar{u}_4, \quad \epsilon = \bar{r}^3\bar{\epsilon}. \tag{5.6}$$

Under this transformation the system (5.1)–(5.5) pulls back to a vector field X on blow-up space for which the *blow-up cylinder* $\bar{r} = 0$ consists entirely of equilibria. The vector field we shall study is $\tilde{X} = \bar{r}^{-1}X$. Division by $\bar{r}$ desingularizes the vector field on the blow-up cylinder but leaves it invariant.

Note that

$$\Phi^{-1}(E_- \times \{0\}) = \Phi^{-1}\{(u_1, u_2, u_3, 0, 0) : -u_e < u_1 < 0, u_2 \text{ arbitrary}, u_3 = e(u_1)\}$$

is a 2-dimensional submanifold of blowup space. Let $\tilde{E}_-$ denote $\Phi^{-1}(E_- \times \{0\})$ together with its limit points in the blow-up cylinder $\bar{r} = 0$. We define $\tilde{E}_+$, $\tilde{F}_-$, and $\tilde{F}_+$ analogously.

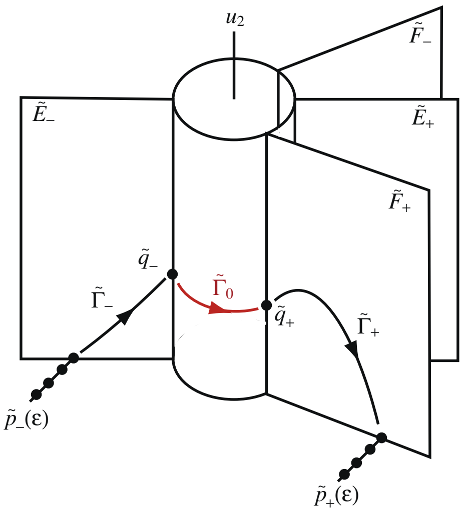

Fig. 4 Blow-up space and the singular connecting orbit. Note that the blow-up has separated the four manifolds $\tilde{E}_\pm$ and $\tilde{F}_\pm$

Let $\tilde{p}_-(\epsilon)$ (respectively $\tilde{p}_+(\epsilon)$) be the unique point in blow-up space that corresponds to $p_-(\epsilon)$ (respectively $p_+(\epsilon)$); we calculate the coordinates of these points using (5.6). In order to prove Theorem 3.1, we wish to show that for small $\epsilon > 0$ there is an integral curve of X from $\tilde{p}_-(\epsilon)$ to $\tilde{p}_+(\epsilon)$. Equivalently, we shall show that for small $\epsilon > 0$ there is an integral curve of $\tilde{X}$ from $\tilde{p}_-(\epsilon)$ to $\tilde{p}_+(\epsilon)$.

Corresponding to $\Gamma_\pm$ are curves $\tilde{\Gamma}_\pm$ in blow-up space. We shall see that:

- $\tilde{\Gamma}_-$ approaches a point $\tilde{q}_- = (u_2^*, \hat{q}_-, 0)$ on the blow-up cylinder.
- $\tilde{\Gamma}_+$ approaches a point $\tilde{q}_+ = (u_2^*, \hat{q}_+, 0)$ on the blow-up cylinder.
- On the blow-up cylinder, each 3-sphere $u_2 =$ constant is invariant.

Proposition 5.1 *There is an integral curve $\tilde{\Gamma}_0$ of $\tilde{X}$ from $\tilde{q}_-$ to $\tilde{q}_+$ that lies in the 3-dimensional hemisphere given by $u_2 = u_2^*$, $\bar{r} = 0$, $\bar{\epsilon} > 0$.*

Theorem 5.2 *For small $\epsilon > 0$, there is an integral curve $\tilde{\Gamma}(\epsilon)$ of $\tilde{X}$ from $\tilde{p}_-(\epsilon)$ to $\tilde{p}_+(\epsilon)$. As $\epsilon \to 0$, $\tilde{\Gamma}(\epsilon) \to \tilde{\Gamma}_- \cup \{\tilde{q}_-\} \cup \tilde{\Gamma}_0 \cup \{\tilde{q}_+\} \cup \tilde{\Gamma}_+$.*

See Fig. 4. $\tilde{\Gamma}_-$ and $\tilde{\Gamma}_+$ are *outer solutions*; $\tilde{\Gamma}_0$ is the *inner solution*. The union in Theorem 5.2 is the *singular solution*. Theorem 5.2 implies Theorem 4.2. The next four sections are devoted to the proofs of Proposition 5.1 and Theorem 5.2.

6 Charts

We shall need three charts on open subsets of blow-up space.

6.1 Chart for $\bar{\epsilon} > 0$

On the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\bar{\epsilon} > 0$, this chart uses the coordinates $b_1 = \bar{u}_1\bar{\epsilon}^{-\frac{2}{3}}$, u_2, $b_3 = \bar{u}_3\bar{\epsilon}^{-\frac{1}{3}}$, $b_4 = \bar{u}_4\bar{\epsilon}^{-\frac{2}{3}}$, and $r = \bar{r}\bar{\epsilon}^{\frac{1}{3}}$. Thus we have

$$u_1 = r^2 b_1, \quad u_2 = u_2, \quad u_3 = r b_3, \quad u_4 = r^2 b_4, \quad \epsilon = r^3, \tag{6.1}$$

with $r \geq 0$. After division by r, the system (5.1)–(5.5) becomes

$$b_{1s} = u_2, \tag{6.2}$$

$$u_{2s} = r^2 g_x(r^2 b_1, r b_3, r^3), \tag{6.3}$$

$$b_{3s} = b_4, \tag{6.4}$$

$$b_{4s} = r^{-3} g_y(r^2 b_1, r b_3, r^3) = -b_1 b_3 + b_3^3 + \mathcal{O}(r), \tag{6.5}$$

$$r_s = 0. \tag{6.6}$$

In calculating b_{4s} we have used the assumption $k_\epsilon(0,0,0) = 0$; without this assumption, b_{4s} would include a constant term. For the model system, $u_{2s} = r^2(h'(r^2 b_1) - \frac{1}{2} r^2 b_3^2)$ and $b_{4s} = -b_1 b_3 + b_3^3$.

Since $r = \bar{r}\bar{\epsilon}^{\frac{1}{3}}$, (6.2)–(6.6) on $\{(b_1, u_2, b_3, b_4, r) : r \geq 0\}$ represents the vector field

$$r^{-1} X = \bar{r}^{-1} \bar{\epsilon}^{-\frac{1}{3}} X = \epsilon^{-\frac{1}{3}} \tilde{X}$$

on the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\bar{\epsilon} > 0$.

We remark that multiplication of a vector field by a nonzero function is equivalent to a reparameterization of time, so in (6.2)–(6.6) we have used a new symbol, s, for time. We will try to be careful about this where it is important.

System (6.2)–(6.6) is used to construct the inner solution. The advantage of the blow-up method is that there are two additional charts that can be used to match it with the outer solutions.

6.2 Chart for $\bar{u}_1 < 0$

On the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\bar{u}_1 < 0$, this chart uses the coordinates $v = \bar{r}(-\bar{u}_1)^{\frac{1}{2}}$, u_2, $a_3 = \bar{u}_3(-\bar{u}_1)^{-\frac{1}{2}}$, $a_4 = -\bar{u}_4 \bar{u}_1^{-1}$, and $\delta = \bar{\epsilon}(-\bar{u}_1)^{-\frac{3}{2}}$. Thus we have

$$u_1 = -v^2, \quad u_2 = u_2, \quad u_3 = v a_3, \quad u_4 = v^2 a_4, \quad \epsilon = v^3 \delta, \tag{6.7}$$

with $v \geq 0$. After division by v, the system (5.1)–(5.5) becomes

$$v_t = -\frac{1}{2} v \delta u_2, \tag{6.8}$$

$$u_{2t} = v^2 \delta g_x(-v^2, v a_3, v^3 \delta), \tag{6.9}$$

$$a_{3t} = a_4 + \frac{1}{2} \delta u_2 a_3, \tag{6.10}$$

$$a_{4t} = v^{-3} g_y(-v^2, v a_3, v^3 \delta) + \delta u_2 a_4 = a_3 + a_3^3 + \delta u_2 a_4 + \mathcal{O}(v), \tag{6.11}$$

$$\delta_t = \frac{3}{2} \delta^2 u_2. \tag{6.12}$$

For the model system, $u_{2t} = v^2 \delta(h'(-v^2) - \frac{1}{2} v^2 a_3^2)$, and $a_{4t} = a_3 + a_3^3 + \delta u_2 a_4$.

Since $v = \bar{r}(-\bar{u}_1)^{\frac{1}{2}}$, (6.8)–(6.12) on $\{(v, u_2, a_3, a_4, \delta) : v \geq 0\}$ represents the vector field

$$v^{-1} X = \bar{r}^{-1} (-\bar{u}_1)^{-\frac{1}{2}} X = (-\bar{u}_1)^{-\frac{1}{2}} \tilde{X}$$

on the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\bar{u}_1 < 0$.

6.3 Chart for $\bar{u}_1 > 0$

On the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\bar{u}_1 > 0$, this chart uses the coordinates $w = \bar{r}\bar{u}_1^{\frac{1}{2}}$, u_2, $c_3 = \bar{u}_3\bar{u}_1^{-\frac{1}{2}}$, $c_4 = \bar{u}_4\bar{u}_1^{-1}$, and $\gamma = \bar{\epsilon}\bar{u}_1^{-\frac{3}{2}}$. Thus we have

$$u_1 = w^2, \quad u_2 = u_2, \quad u_3 = wc_3, \quad u_4 = w^2 c_4, \quad \epsilon = w^3\gamma, \tag{6.13}$$

with $w \geq 0$. After division by w, the system (5.1)–(5.5) becomes

$$w_t = \frac{1}{2} w\gamma u_2, \tag{6.14}$$

$$u_{2t} = w^2\gamma g_x(w^2, wc_3, w^3\gamma), \tag{6.15}$$

$$c_{3t} = c_4 - \frac{1}{2}\gamma u_2 c_3, \tag{6.16}$$

$$c_{4t} = w^{-3} g_y(w^2, wc_3, w^3\gamma) - \gamma u_2 c_4 = -c_3 + c_3^3 - \gamma u_2 c_4 + \mathcal{O}(w), \tag{6.17}$$

$$\gamma_t = -\frac{3}{2}\gamma^2 u_2. \tag{6.18}$$

For the model system, $u_{2t} = w^2\gamma(h'(w^2) - \frac{1}{2}w^2 c_3^2)$, and $c_{4t} = -c_3 + c_3^3 - \gamma u_2 c_4$.

Since $w = \bar{r}\bar{u}_1^{\frac{1}{2}}$, (6.14)–(6.18) on $\{(w, u_2, c_3, c_4, \gamma) : w \geq 0\}$ represents the vector field

$$w^{-1}X = \bar{r}^{-1}\bar{u}_1^{-\frac{1}{2}}X = \bar{u}_1^{-\frac{1}{2}}\tilde{X}$$

on the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\bar{u}_1 > 0$.

6.4 Important Sets and First Integral

1. Note that

$$\Phi^{-1}\{(u_1, u_2, u_3, u_4, \epsilon) : \epsilon = 0 \text{ and } u_1^2 + u_3^2 + u_4^2 > 0\}$$
$$= \{(u_2, (\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\epsilon}), \bar{r}) : \bar{\epsilon} = 0 \text{ and } \bar{r} > 0\}.$$

   It is invariant under $\tilde{X}$; hence so is its closure, which is the subset $\bar{\epsilon} = 0$ of blow-up space. In the chart for $\bar{u}_1 < 0$ (respectively $\bar{u}_1 > 0$), the corresponding invariant set is $\delta = 0$ (respectively $\gamma = 0$).
2. In the chart for $\bar{u}_1 < 0$, $\tilde{E}_-$ corresponds to

$$E_{-a} = \left\{(v, u_2, v^{-1}e(-v^2), 0, 0) : 0 \leq v < u_e^{\frac{1}{2}} \text{ and } u_2 \text{ arbitrary}\right\}$$
$$= \left\{(v, u_2, -\omega v + \cdots, 0, 0) : 0 \leq v < u_e^{\frac{1}{2}} \text{ and } u_2 \text{ arbitrary}\right\}.$$

   For the model system, $E_{-a} = \{(v, u_2, 0, 0, 0) : 0 \leq v\}$. For the system (6.8)–(6.12), E_{-a} is a 2-dimensional manifold of equilibria that is normally hyperbolic within the space $\delta = 0$. Note that normal hyperbolicity is *not* lost at $v = 0$, which corresponds to $u_1 = 0$.
   Similarly, in the chart for $\bar{u}_1 > 0$, $\tilde{E}_+$, $\tilde{F}_-$, and $\tilde{F}_+$ correspond respectively to

   - $E_{+c} = \left\{(w, u_2, w^{-1}e(w^2), 0, 0) : 0 \leq w < u_e^{\frac{1}{2}} \text{ and } u_2 \text{ arbitrary}\right\}$
     $= \left\{(w, u_2, \omega w + \cdots, 0, 0) : 0 \leq w < u_e^{\frac{1}{2}} \text{ and } u_2 \text{ arbitrary}\right\};$

- $F_{-c} = \left\{(w, u_2, w^{-1}m(-w), 0, 0) : 0 \leq w < u_m^{\frac{1}{2}} \text{ and } u_2 \text{ arbitrary}\right\}$
  $= \left\{(w, u_2, -1 + \cdots, 0, 0) : 0 \leq w < u_m^{\frac{1}{2}} \text{ and } u_2 \text{ arbitrary}\right\}$;
- $F_{+c} = \left\{(w, u_2, w^{-1}m(w), 0, 0) : 0 \leq w < u_m^{\frac{1}{2}} \text{ and } u_2 \text{ arbitrary}\right\}$
  $= \left\{(w, u_2, 1 + \ldots, 0, 0) : 0 \leq w < u_m^{\frac{1}{2}} \text{ and } u_2 \text{ arbitrary}\right\}$.

For the model system, $E_{+c} = \{(w, u_2, 0, 0, 0) : w \geq 0\}$, $F_{-c} = \{(w, u_2, -1, 0, 0) : w \geq 0\}$, and $F_{+c} = \{(w, u_2, 1, 0, 0) : w \geq 0\}$. For the system (6.14)–(6.18), each is a 2-dimensional manifold of equilibria. F_{-c} and F_{+c} are normally hyperbolic within the space $\gamma = 0$. Normal hyperbolicity is *not* lost at $w = 0$, which corresponds to $u_1 = 0$.

3. In the chart for $\bar{u}_1 < 0$, $\tilde{p}_-(\epsilon)$ corresponds to a unique point $p_{-a}(\delta)$, and $\tilde{\Gamma}_-$ corresponds to

$$\Gamma_{-a} = \{(v, u_2, a_3, 0, 0) : (-v^2, u_2, va_3, 0) \in \Gamma_-\}.$$

As $v \to 0$, Γ_{-a} approaches $(0, u_2^*, 0, 0, 0)$. $\tilde{\Gamma}_-$ ends at $\tilde{q}_- = (u_2^*, \hat{q}_-, 0)$, $\hat{q}_- = (-1, 0, 0, 0)$.
In the chart for $\bar{u}_1 > 0$, $\tilde{p}_+(\epsilon)$ corresponds to a unique point $p_{+c}(\gamma)$, and $\tilde{\Gamma}_+$ corresponds to

$$\Gamma_{+c} = \{(w, u_2, c_3, 0, 0) : (w^2, u_2, wc_3, 0) \in \Gamma_+\}.$$

As $w \to 0$, Γ_{+c} approaches $(0, u_2^*, 1, 0, 0)$. Let $\alpha = \frac{1}{2}(5^{\frac{1}{2}} - 1)$. $\tilde{\Gamma}_+$ ends at $\tilde{q}_+ = (u_2^*, \hat{q}_+, 0)$, $\hat{q}_+ = (\alpha, \alpha^{\frac{1}{2}}, 0, 0)$.

4. The first integral H for (5.1)–(5.5) becomes:
   - In blow-up space: $\tilde{H}(u_2, (\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\epsilon}), \bar{r}) = \frac{1}{2}u_2^2 + \frac{1}{2}\bar{r}^4\bar{u}_4^2 - g(\bar{r}^2\bar{u}_1, \bar{r}\bar{u}_3, \bar{r}^3\bar{\epsilon})$.
   - In the chart for $\bar{\epsilon} > 0$: $H_b(b_1, u_2, b_3, b_4, r) = \frac{1}{2}u_2^2 + \frac{1}{2}r^4b_4^2 - g(r^2b_1, rb_3, r^3)$.
   - In the chart for $\bar{u}_1 < 0$: $H_a(v, u_2, a_3, a_4, \delta) = \frac{1}{2}u_2^2 + \frac{1}{2}v^4a_4^2 - g(-v^2, va_3, v^3\delta)$.
   - In the chart for $\bar{u}_1 > 0$: $H_c(w, u_2, c_3, c_4, \gamma) = \frac{1}{2}u_2^2 + \frac{1}{2}w^4c_4^2 - g(w^2, wc_3, w^3\gamma)$.

$\tilde{H}$ is a first integral for X and hence for $\tilde{X}$. H_b, H_a, and H_c are first integrals for (6.2)–(6.6), (6.8)–(6.12), and (6.14)–(6.18) respectively. Therefore:

- In blow-up space, each 3-sphere u_2 is constant, $\bar{r} = 0$ is invariant for $\tilde{X}$.
- In the chart for $\bar{\epsilon} > 0$, each space $u_2 =$ constant, $r = 0$ is invariant for (6.2)–(6.6).
- In the chart for $\bar{u}_1 < 0$, each space $u_2 =$ constant, $v = 0$ is invariant for (6.8)–(6.12).
- In the chart for $\bar{u}_1 > 0$, each space $u_2 =$ constant, $w = 0$ is invariant for (6.14)–(6.18).

Of course the last three conclusions also follow easily from inspection of the systems themselves.

7 Proof of Proposition 5.1

Let $\hat{X}$ denote the restriction of the vector field $\tilde{X}$ to the invariant 3-sphere $M = \{u_2^*\} \times S^3 \times \{0\}$, $S^3 = \{(\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\epsilon}) : \bar{u}_1^2 + \bar{u}_3^2 + \bar{u}_4^2 + \bar{\epsilon}^2 = 1\}$. From Sect. 6 we have the following charts on M.

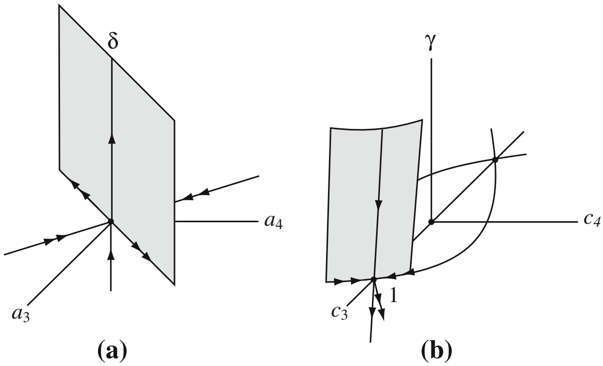

Fig. 5 Phase portraits of **a** (7.4)–(7.6) and **b** (7.7)–(7.9). $W^{cu}(0,0,0)$ and $W^{cs}(1,0,0)$ are shaded

Chart on the open subset of M with $\bar{\epsilon} > 0$: $b_1 = \bar{u}_1\bar{\epsilon}^{-\frac{2}{3}}, b_3 = \bar{u}_3\bar{\epsilon}^{-\frac{1}{3}}, b_4 = \bar{u}_4\bar{\epsilon}^{-\frac{2}{3}}$. In this chart, the vector field $\bar{\epsilon}^{-\frac{1}{3}}\hat{X}$ is

$$b_{1s} = u_2^*, \tag{7.1}$$
$$b_{3s} = b_4, \tag{7.2}$$
$$b_{4s} = -b_1b_3 + b_3^3. \tag{7.3}$$

Chart on the open subset of M with $\bar{u}_1 < 0$: $a_3 = \bar{u}_3(-\bar{u}_1)^{-\frac{1}{2}}, a_4 = -\bar{u}_4\bar{u}_1^{-1}, \delta = \bar{\epsilon}(-\bar{u}_1)^{-\frac{3}{2}}$. In this chart, the vector field $(-\bar{u}_1)^{-\frac{1}{2}}\hat{X}$ is

$$a_{3t} = a_4 + \frac{1}{2}\delta u_2^* a_3, \tag{7.4}$$
$$a_{4t} = a_3 + a_3^3 + \delta u_2^* a_4, \tag{7.5}$$
$$\delta_t = \frac{3}{2}\delta^2 u_2^*. \tag{7.6}$$

Chart on the open subset of M with $\bar{u}_1 > 0$: $c_3 = \bar{u}_3\bar{u}_1^{-\frac{1}{2}}, c_4 = \bar{u}_4\bar{u}_1^{-1}, \gamma = \bar{\epsilon}\bar{u}_1^{-\frac{3}{2}}$. In this chart, the vector field $\bar{u}_1^{-\frac{1}{2}}\hat{X}$ is

$$c_{3t} = c_4 - \frac{1}{2}\gamma u_2^* c_3, \tag{7.7}$$
$$c_{4t} = -c_3 + c_3^3 - \gamma u_2^* c_4, \tag{7.8}$$
$$\gamma_t = -\frac{3}{2}\gamma^2 u_2^*. \tag{7.9}$$

The system (7.4)–(7.6) has an equilibrium at the origin with 2-dimensional center-unstable manifold. Abusing notation a little, we let $W^{cu}(0,0,0)$ denote the part of this manifold with $\delta \geq 0$. Then $W^{cu}(0,0,0)$ consists of all solutions of (7.4)–(7.6) that approach $(0,0,0)$ as $t \to -\infty$. See Fig. 5a.

The system (7.7)–(7.9) has equilibria at $(-1,0,0)$, $(0,0,0)$, and $(1,0,0)$. The equilibrium at $(1,0,0)$ has a 2-dimensional center-stable manifold. Continuing to abuse notation, we let $W^{cs}(1,0,0)$ denote the part of this manifold with $\gamma \geq 0$. Then $W^{cs}(1,0,0)$ consists of all solutions of (7.7)–(7.9) that approach $(1,0,0)$ as $t \to \infty$. See Fig. 5b.

Let $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ denote the corresponding manifolds in M, extended to be invariant. $W^{cu}(\hat{q}_-)$ (respectively $W^{cs}(\hat{q}_+)$) consists of all integral curves of $\hat{X}$ that approach

$\hat{q}_-$ (respectively $\hat{q}_+$) as $\xi \to -\infty$ (respectively $\xi \to \infty$). To prove Proposition 5.1 we will find an integral curve of $\hat{X}$ in $W^{cu}(\hat{q}_-) \cap W^{cs}(\hat{q}_+)$.

The solution of (7.1) with $b_1(0) = 0$ is $b_1 = u_2^* s$. Substituting into (7.3) and combining (7.2) and (7.3) into a second-order equation, we obtain

$$b_{3ss} = -u_2^* s b_3 + b_3^3. \tag{7.10}$$

By [1,18], [3, p. 100], (7.10) has a solution $b_3(s)$ with $b_{3s} > 0$ such that

(S1) $b_3(s) = \mathcal{O}\left(|s|^{-\frac{1}{4}} e^{-\frac{2}{3}(u_2^*)^{\frac{1}{2}}|s|^{\frac{3}{2}}}\right)$ as $s \to -\infty$,

(S2) $b_3(s) = (u_2^* s)^{\frac{1}{2}} + \mathcal{O}(s^{-\frac{5}{2}})$ as $s \to \infty$,

(S3) $b_{3s}(s) \leq C|s|^{-\frac{1}{2}}$, $s \neq 0$.

$(u_2^* s, b_3(s), b_{3s}(s))$ is a solution of (7.1)–(7.3). We claim that it represents an intersection of $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ in the portion of M with $\bar{\epsilon} > 0$.

The change of coordinates between the chart for $\bar{\epsilon} > 0$ and the chart for $\bar{u}_1 < 0$ is

$$a_3 = b_3(-b_1)^{-\frac{1}{2}}, \quad a_4 = -b_4 b_1^{-1}, \quad \delta = (-b_1)^{-\frac{3}{2}}.$$

Using (S1), (S3), and $b_1 = u_2^* s$, we see:

$$(a_3(s), a_4(s), \delta(s)) \to (0, 0, 0) \text{ as } s \to -\infty. \tag{7.11}$$

The change of coordinates between the chart for $\bar{\epsilon} > 0$ and the chart for $\bar{u}_1 > 0$ is

$$c_3 = b_3(b_1)^{-\frac{1}{2}}, \quad c_4 = b_4 b_1^{-1}, \quad \gamma = (b_1)^{-\frac{3}{2}}. \tag{7.12}$$

Using (S2), (S3), and $b_1 = u_2^* s$, we see:

$$(c_3(s), c_4(s), \gamma(s)) \to (1, 0, 0) \text{ as } s \to \infty. \tag{7.13}$$

Equations (7.11) and (7.13) yield the result.

Remark 7.1 $(a_3(s), a_4(s), \delta(s))$ is not a solution of (7.4)–(7.6): it is an integral curve of $\epsilon^{-\frac{1}{3}}\hat{X}$ in (a_3, a_4, δ) coordinates, whereas solutions of (7.4)–(7.6) represent integral curves of $(-\bar{u}_1)^{-\frac{1}{2}}\hat{X}$. However, the reparameterization of time $t = -\frac{2}{3}(u_2^*)^{\frac{1}{2}}(-s)^{\frac{3}{2}}$ converts $(a_3(s), a_4(s), \delta(s))$, $s < 0$, to a solution of (7.4)–(7.6), which we denote $(a_3(t), a_4(t), \delta(t))$.

Similarly, the reparameterization of time $t = \frac{2}{3}(u_2^*)^{\frac{1}{2}} s^{\frac{3}{2}}$ converts $(c_3(s), c_4(s), \gamma(s))$, $s > 0$, to a solution of (7.7)–(7.9), which we denote $(c_3(t), c_4(t), \delta(t))$.

Remark 7.2 Using (S1), (S2), and the previous remark, one can show that $a_3(t)$ is $\mathcal{O}(|t|^{-\frac{1}{2}}e^t)$ as $t \to -\infty$, but $c_3(t) - 1$ is only $\mathcal{O}(t^{-2})$ as $t \to \infty$. The reason for the difference can be seen in Fig. 5: in the chart on M for $\bar{u}_1 < 0$, a center manifold for the origin is the δ-axis; in the chart on M for $\bar{u}_1 > 0$, center manifolds of $(1, 0, 0)$ are skew curves. Since solutions in $W^{cu}(0, 0, 0)$ approach the center manifold exponentially as $t \to -\infty$, and the eigenvalue is 1, we see that $a_3(t)$ (and $a_4(t)$) should be $\mathcal{O}(e^t)$ as $t \to -\infty$. No such conclusion can be drawn about $c_3(t)$ and $c_4(t)$.

8 Transversality

Let $\hat{\Gamma}_0 = \{(\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\epsilon}) : (u_2^*, (\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\epsilon}), 0) \in \tilde{\Gamma}_0\}$, where $\tilde{\Gamma}_0$ is given by Proposition 5.1.

Proposition 8.1 *$W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ intersect transversally within M along $\hat{\Gamma}_0$.*

Note that $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ have dimension 2 and M has dimension 3, so a transverse intersection is 1-dimensional, in this case $\hat{\Gamma}_0$.

Proof The equilibrium $(0, 0, 0)$ of (7.4)–(7.6) has one negative eigenvalue, one positive eigenvalue, and one zero eigenvalue, and the vector field is quadratic on the center manifold. It follows that solutions $(a_3(t), a_4(t), \delta(t))$ that approach $(0, 0, 0)$ as $t \to -\infty$ are $\mathcal{O}(|t|^{-1})$, and solutions of the linearized system along $(a_3(t), a_4(t), \delta(t))$ that do not grow exponentially as $t \to -\infty$ are $\mathcal{O}(t^{-2})$. There is a 2-dimensional space of such solutions of the linearized system, which we denote $E^{cu}_a(t)$.

Similarly, the equilibrium $(1, 0, 0)$ of (7.7)–(7.9) has one negative eigenvalue, one positive eigenvalue, and one zero eigenvalue, and the vector field is quadratic on the center manifold. It follows that solutions $(c_3(t), c_4(t), \gamma(t))$ that approach $(1, 0, 0)$ as $t \to \infty$ are $\mathcal{O}(t^{-1})$, and solutions of the linearized system along $(c_3(t), c_4(t), \gamma(t))$ that do not grow exponentially as $t \to \infty$ are $\mathcal{O}(t^{-2})$. There is a 2-dimensional space of such solutions of the linearized system, which we denote $E^{cs}_c(t)$.

Each of these spaces corresponds to a space of solutions of the linearization of $\hat{X}$ along $\hat{\Gamma}_0$. More precisely, if $\hat{\Gamma}_0$ is the set of points in the solution $y(\xi)$ of $y_\xi = \hat{X}(y)$ on M, then the linearized equation $Y_\xi = D\hat{X}(y(\xi))Y$ has a 2-dimensional space of solutions $E^{cu}(\xi)$ (respectively $E^{cs}(\xi)$) consisting of vectors tangent to $W^{cu}(\hat{q}_-)$ (respectively $W^{cs}(\hat{q}_+)$) at $y(\xi)$. These spaces correspond respectively to $E^{cu}_a(t)$ in the chart for $\bar{u}_1 < 0$ and to $E^{cs}_c(t)$ in the chart for $\bar{u}_1 > 0$. To prove the result we must show that $E^{cu}(\xi) \cap E^{cs}(\xi)$ is 1-dimensional (multiples of $y_\xi(\xi)$).

In the chart on M for $\bar{u}_1 > 0$, $\hat{\Gamma}_0$ corresponds to a solution $(c_3(t), c_4(t), \gamma(t))$ of (7.7)–(7.9). After a shift in time, we may assume $\gamma(t) = \frac{2}{3u_2^* t}$. The linearization of (7.7)–(7.9) along this solution is

$$\begin{pmatrix} C_{3t} \\ C_{4t} \\ \Gamma_t \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}u_2^*\gamma(t) & 1 & -\frac{1}{2}u_2^* c_3(t) \\ 3c_3(t)^2 - 1 & -u_2^*\gamma(t) & -u_2^* c_4(t) \\ 0 & 0 & -3u_2^*\gamma(t) \end{pmatrix} \begin{pmatrix} C_3 \\ C_4 \\ \Gamma \end{pmatrix}. \tag{8.1}$$

The inverse of the coordinate change (7.12) is

$$b_1 = \gamma^{-\frac{2}{3}}, \quad b_3 = \gamma^{-\frac{1}{3}} c_3, \quad b_4 = \gamma^{-\frac{2}{3}} c_4. \tag{8.2}$$

Under this coordinate change, (7.7)–(7.9) becomes

$$b_{1t} = u_2^* b_1^{-\frac{1}{2}}, \tag{8.3}$$

$$b_{3t} = b_1^{-\frac{1}{2}} b_4, \tag{8.4}$$

$$b_{4t} = b_1^{-\frac{1}{2}} (b_3^3 - b_1 b_3). \tag{8.5}$$

This system is just $b_1^{-\frac{1}{2}}$ times (7.1)–(7.3). Applying (8.2) to $(c_3(t), c_4(t), \gamma(t))$ yields a solution $(b_1(t), b_3(t), b_4(t))$ of (8.3)–(8.5) with $b_1(t) = \left(\frac{3}{2}u_2^* t\right)^{\frac{2}{3}}$. The linearization of (8.2) along $(c_3(t), c_4(t), \gamma(t))$, which is

$$\begin{pmatrix} B_1 \\ B_3 \\ B_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{2}{3}\gamma(t)^{-\frac{5}{3}} \\ \gamma(t)^{-\frac{1}{3}} & 0 & -\frac{1}{3}\gamma(t)^{-\frac{4}{3}} c_3(t) \\ 0 & \gamma(t)^{-\frac{2}{3}} & -\frac{2}{3}\gamma(t)^{-\frac{5}{3}} c_4(t) \end{pmatrix} \begin{pmatrix} C_3 \\ C_4 \\ \Gamma \end{pmatrix}, \tag{8.6}$$

converts (8.1) into the linearization of (8.3)–(8.5) along $(b_1(t), b_3(t), b_4(t))$, namely

$$\begin{pmatrix} B_{1t} \\ B_{3t} \\ B_{4t} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}u_2^* b_1(t)^{-\frac{3}{2}} & 0 & 0 \\ -\frac{1}{2}b_1(t)^{-\frac{3}{2}}b_4(t) & 0 & b_1(t)^{-\frac{1}{2}} \\ -\frac{1}{2}b_1(t)^{-\frac{1}{2}}b_3(t) - \frac{1}{2}b_1(t)^{-\frac{3}{2}}b_3(t)^3 & b_1(t)^{-\frac{1}{2}}(3b_3(t)^2 - b_1) & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_3 \\ B_4 \end{pmatrix}. \tag{8.7}$$

The change of coordinates (8.6) converts $E_c^{cs}(t)$ into a 2-dimensional space of solutions of (8.7) that we denote $E_b^{cs}(t)$.

One solution of (8.7) is $(B_1(t), B_3(t), B_4(t)) = (b_{1t}(t), b_{3t}(t), b_{4t}(t))$; it lies in $E_b^{cs}(t)$. There is a complementary 2-dimensional space of solutions of (8.7) with $B_1(t) = 0$ and $(B_3, B_4)(t)$ a solution of

$$\begin{pmatrix} B_{3t} \\ B_{4t} \end{pmatrix} = \begin{pmatrix} 0 & b_1(t)^{-\frac{1}{2}} \\ b_1(t)^{-\frac{1}{2}}(3b_3(t)^2 - b_1) & 0 \end{pmatrix} \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}. \tag{8.8}$$

If $E^{cu}(\xi) \cap E^{cs}(\xi)$ is not 1-dimensional, then $E^{cu}(\xi) = E^{cs}(\xi)$ and corresponds, in the chart for $\bar{\epsilon} > 0$, to $E_b^{cs}(t)$. Since two 2-dimensional spaces in a 3-dimensional space have nontrivial intersection, $E_b^{cs}(t)$ includes solutions with $B_1(t) = 0$.

If $(C_3(t), C_4(t), \Gamma(t)) \in E_c^{cs}(t)$, then $C_3(t), C_4(t)$, and $\Gamma(t)$ are $\mathcal{O}(t^{-2})$. Also, $c_3(t)$ is $\mathcal{O}(1)$, and $c_4(t)$ and $\gamma(t)$ are $\mathcal{O}(t^{-1})$. Therefore we see from (8.6) that if $(B_1(t), B_3(t), B_4(t)) \in E_b^{cs}(t)$, then $B_1(t)$ is $\mathcal{O}\left(t^{-\frac{1}{3}}\right)$, $B_3(t)$ is $\mathcal{O}\left(t^{-\frac{2}{3}}\right)$, and $B_4(t)$ is $\mathcal{O}\left(t^{-\frac{4}{3}}\right)$.

The rescaling of time $t = \frac{2}{3}(u_2^*)^{\frac{1}{2}}s^{\frac{3}{2}}$ converts (8.3)–(8.5) to (7.1)–(7.3), for which $b_1 = u_2^* s$, and converts (8.8) to

$$\begin{pmatrix} B_{3s} \\ B_{4s} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3b_3(s)^2 - u_2^* s & 0 \end{pmatrix} \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}. \tag{8.9}$$

Thus a solution $(0, B_3(t), B_4(t))$ of (8.7) in $E_b^{cs}(t)$ becomes $(0, B_3(s), B_4(s))$ with $(B_3(s), B_4(s))$ a solution of (8.9); moreover, $B_3(s)$ is $\mathcal{O}(s^{-1})$ and $B_4(s)$ is $\mathcal{O}(s^{-2})$.

A similar argument in the coordinate system for $\bar{u}_1 < 0$ leads to the following conclusion:

- $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ intersect transversally along $\hat{\Gamma}_0$ if and only if (8.9) has no nontrivial solution $(B_3(s), B_4(s))$ on $\mathbb{R}$ such that B_3 is $\mathcal{O}(|s|^{-1})$, and B_4 is $\mathcal{O}(s^{-2})$.

Now (8.9) is equivalent to the second order linear system

$$B_{3ss} = (3b_3(s)^2 - u_2^* s)B_3. \tag{8.10}$$

By [1, Theorem 2], (8.10) has no nontrivial solutions in L^2, hence no solution for which $B_3(s)$ is $\mathcal{O}(|s|^{-1})$. This completes the proof. □

Remark 8.2 Using Remark 7.2, one can obtain better estimates for $(B_3(s), B_4(s))$ as $s \to -\infty$, but they are not needed.

9 Proof of Theorem 5.2

Let N_ϵ denote the set of points in blow-up space at which $\tilde{H} = H(p_-(\epsilon))$ and $\bar{r}^3\bar{\epsilon} = \epsilon$. Away from equilibria of $\tilde{X}$, each N_ϵ is a manifold of dimension 3.

For the vector field $\tilde{X}$ and $\epsilon > 0$, the equilibria $\tilde{p}_-(\epsilon)$ and $\tilde{p}_+(\epsilon)$ have 2-dimensional unstable and stable manifolds. We will prove the theorem by showing that for small $\epsilon > 0$, $W^u(\tilde{p}_-(\epsilon))$ and $W^s(\tilde{p}_+(\epsilon))$ have a nonempty intersection that is transverse within N_ϵ. It will be clear that the intersection approaches $\tilde{\Gamma}_- \cup \tilde{\Gamma}_0 \cup \tilde{\Gamma}_+$ as $\epsilon \to 0$.

In the chart on blow-up space for $\bar{u}_1 < 0$, the system (6.8)–(6.12) has the 2-dimensional manifold of equilibria E_{-a}. The linearization of (6.8)–(6.12) at a point y of E_{-a}, taking into account that $va_3 = u_3 = \mathcal{O}(u_1) = \mathcal{O}(v^2)$, so that $a_3 = \mathcal{O}(v)$, and that $a_4 = \delta = 0$, has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \mathcal{O}(v) \\ 0 & 0 & 0 & 0 & \mathcal{O}(v^2) \\ 0 & 0 & 0 & 1 & \mathcal{O}(v) \\ * & 0 & v^{-2}g_{yy}(-v^2, va_3, 0) & 0 & g_{y\epsilon}(-v^2, va_3, 0) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues are $\pm v^{-1}(g_{yy}(-v^2, va_3, 0))^{\frac{1}{2}} = \pm(1+3a_3^2)^{\frac{1}{2}} + \cdots$ with multiplicity one and 0 with multiplicity three. The eigenvectors for $\pm v^{-1}(g_{yy}(-v^2, va_3, 0))^{\frac{1}{2}}$ span a_3a_4-space, so the generalized eigenspace $V_a(y)$ for the 0 eigenvalues is complementary to a_3a_4-space. From [7] we deduce that given any compact subset K of E_{-a}, there are a 3-dimensional normally hyperbolic invariant manifold S_- and a number $\delta_0 > 0$ such that $K \subset S_-$, $T_yS_- = V_a(y)$ for all $y \in K$, and S_- is a smooth graph over $\{(v, u_2, \delta) : (v, u_2, v^{-1}e(-v^2), 0, 0) \in K \text{ and } |\delta| < \delta_0\}$. For the model system, $S_- \subset vu_2\delta$-space. In the usual fibration of $W^u(S_-)$, each point of S_- has a 1-dimensional unstable fiber.

The equations for S_- have the form

$$a_3 = v^{-1}e(-v^2) + \mathcal{O}(\delta), \quad a_4 = \mathcal{O}(\delta).$$

Substituting these expressions into (6.8), (6.9), (6.12), we obtain the restriction of (6.8)–(6.12) to S_-. Once the system is restricted to S_-, each equation has a factor of δ, so we can divide by δ, obtaining

$$v_t = -\frac{1}{2}vu_2, \tag{9.1}$$

$$u_{2t} = v^2 g_x(-v^2, e(-v^2) + \mathcal{O}(v\delta), v^3\delta), \tag{9.2}$$

$$\delta_t = \frac{3}{2}\delta u_2. \tag{9.3}$$

Note that on the invariant plane $\delta = 0$, (9.1), (9.2) with $v > 0$ is just (4.28), (4.29) with $u_1 < 0$ after making the change of coordinates $u_1 = -v^2$ and dividing by v^2.

We shall use a $\breve{}$ to designate objects related to (9.1)–(9.3). From assumption (P2), (9.1)–(9.3) has a curve of equilibria $\breve{p}_{-a}(\delta)$, with $\breve{p}_{-a}(\delta) = (v, u_2, \delta)$, $v = (-x_-)^{\frac{1}{2}} + \mathcal{O}(\delta)$, $u_2 = 0$; this curve consists of equilibria that are normally hyperbolic for (9.1)–(9.3), with one positive eigenvalue and one negative eigenvalue. Of course, the coordinates of $\breve{p}_{-a}(\delta)$ are just the $vu_2\delta$-coordinates of $p_{-a}(\delta)$. Note that $p_{-a}(\delta) \in S_-$. Also, the line $\{(v, u_2, \delta) : v = \delta = 0, u_2 \neq 0\}$ consists of equilibria; by direct calculation, they are also normally hyperbolic for (9.1)–(9.3), with one positive eigenvalue and one negative eigenvalue. See Fig. 6.

Let $\breve{\Gamma}_{-a} = \{(v, u_2, 0) : (v, u_2, v^{-1}e(-v^2), 0, 0) \in \Gamma_{-a}\}$. $\breve{\Gamma}_{-a}$ is a branch of the unstable manifold of $\breve{p}_{-a}(0)$ for (9.1)–(9.3), which we denote $\breve{W}^u(\breve{p}_{-a}(0))$. It coincides with a branch of the stable manifold of $(0, u_2^*, 0)$ for (9.1)–(9.3), which we denote $\breve{W}^s(0, u_2^*, 0)$. $\breve{W}^u(0, u_2^*, 0)$ is just the line $\{(v, u_2, \delta) : v = 0, u_2 = u_2^*\}$.

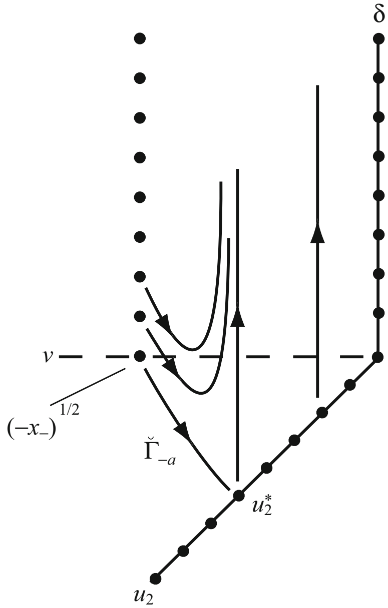

Fig. 6 Phase portrait of (9.1)–(9.3)

Lemma 9.1 *Consider the system (9.1)–(9.3). For $\rho > 0$ and $k \geq 2$, let $U_k = \{(v, u_2, \delta) : |v| < \rho, |u_2 - u_2^*| < \rho, \frac{\rho}{k} < \delta < \rho\}$. If ρ is sufficiently small, then for each $k \geq 2$, as $\delta_0 \to 0+$, $\check{W}^u(\check{p}_{-a}(\delta_0)) \cap U_k$ approaches $\check{W}^u(0, u_2^*, 0) \cap U_k$ in the C^1 topology.*

Proof This is a very simple case of the Corner Lemma in [17]. The statement here is a little different, but the same proof applies. □

With considerable abuse of notation, we let $W^{cu}(0, u_2^*, 0, 0, 0)$ denote the set of points in $vu_2a_3a_4\delta$-space through which the solution of (6.8)–(6.12) approaches $(0, u_2^*, 0, 0, 0)$ as $t \to -\infty$. $W^{cu}(0, u_2^*, 0, 0, 0)$ is just $\{(0, u_2^*, a_3, a_4, \delta) : (a_3, a_4, \delta) \in W^{cu}(0, 0, 0)\}$, where $W^{cu}(0, 0, 0)$ was defined in Sect. 7. Recall that $W^{cu}(0, 0, 0)$ is just $W^{cu}(\hat{q}_-)$ in the chart on M for $\bar{u}_1 < 0$; similarly, $W^{cu}(0, u_2^*, 0, 0, 0)$ is just $W^{cu}(\tilde{q}_-)$ in the chart on blow-up space for $\bar{u}_1 < 0$.

Alternatively, $W^{cu}(0, u_2^*, 0, 0, 0)$ is just the union of the unstable fibers of points in the line $v = 0, u_2 = u_2^*, \delta \geq 0$; this line is part of the normally hyperbolic invariant manifold S_-. Therefore Lemma 9.1 implies:

Lemma 9.2 *Consider the system (6.8)–(6.12). For $\rho > 0$ and $k \geq 2$, let $U_{ka} = \{(v, u_2, a_3, a_4, \delta) : |v| < \rho, |u_2 - u_2^*| < \rho, |a_3| < \rho, |a_4| < \rho, \frac{\rho}{k} < \delta < \rho$. If ρ is sufficiently small, then for each $k \geq 2$, as $\delta_0 \to 0+$, $W^u(p_{-a}(\delta_0)) \cap U_{ka}$ approaches $W^{cu}(0, u_2^*, 0, 0, 0) \cap U_{ka}$ in the C^1 topology.*

In the chart on blow-up space for $\bar{u}_1 > 0$, the system (6.14)–(6.18) has the 2-dimensional manifold of equilibria F_{+c}. The linearization of (6.14)–(6.18) at each point y of F_{+c} has eigenvalues $\pm w^{-1}(g_{yy}(w^2, m(w), 0))^{\frac{1}{2}} = \pm(-1 + 3c_3^2)^{\frac{1}{2}} + \cdots$ with multiplicity one and 0 with multiplicity three. (Note that $c_3 = 1 + \cdots$ on F_{+c}.) As in the chart for $\bar{u}_1 < 0$, given any compact subset K of F_{+c}, there are a 3-dimensional normally hyperbolic invariant manifold S_+ and a number $\gamma_0 > 0$ such that $K \subset S_+$ and S_+ is a smooth graph over $\{(w, u_2, \gamma) : (w, u_2, w^{-1}m(w), 0, 0) \in K$ and $|\gamma| < \gamma_0\}$.

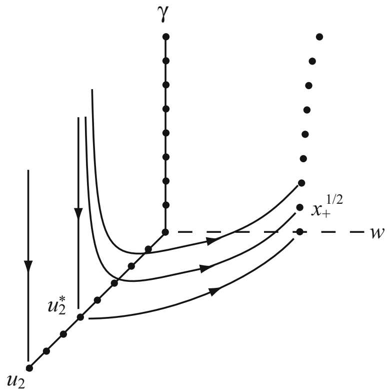

Fig. 7 Phase portrait of (9.4)–(9.6)

The equations for S_+ have the form

$$c_3 = w^{-1}m(w) + \mathcal{O}(\gamma) = 1 + \mathcal{O}(w) + \mathcal{O}(\gamma), \quad c_4 = \mathcal{O}(\gamma).$$

Substituting these formulas into (6.14)–(6.15), (6.18), we obtain the restriction of (6.14)–(6.18) to S_+. Once the system is restricted to S_+, each equation has a factor of γ, so we can divide by γ, obtaining

$$w_t = \frac{1}{2}wu_2, \tag{9.4}$$

$$u_{2t} = w^2 g_x(w^2, m(w) + \mathcal{O}(w\gamma), w^3\gamma), \tag{9.5}$$

$$\gamma_t = -\frac{3}{2}\gamma u_2. \tag{9.6}$$

Note that on the invariant plane $\gamma = 0$, (9.4)–(9.5) with $w > 0$ is just (4.30)–(4.31) with $u_1 > 0$ after making the change of coordinates $u_1 = w^2$ and dividing by w^2.

As we did for the system (9.1)–(9.3), we shall use a $\breve{}$ to designate objects related to (9.4)–(9.6). From assumption (P3), (9.4)–(9.6) has a curve of equilibria $\breve{p}_{+c}(\gamma)$, with $\breve{p}_{+c}(\gamma) = (w, u_2, \gamma)$, $w = x_+^{\frac{1}{2}} + \mathcal{O}(\gamma)$, $u_2 = 0$; this curve consists of equilibria that are normally hyperbolic for (9.4)–(9.6), with one positive eigenvalue and one negative eigenvalue. Of course, the coordinates of $\breve{p}_{+c}(\gamma)$ are just the $wu_2\gamma$-coordinates of $p_{+c}(\gamma)$. Note that $p_{+c}(\gamma) \in S_+$. Also, the line $\{(w, u_2, \gamma) : w = \gamma = 0, u_2 \neq 0\}$ consists of equilibria; by direct calculation, they are also normally hyperbolic for (9.4)–(9.6), with one positive eigenvalue and one negative eigenvalue. See Fig. 7.

Let $W^{cs}(0, u_2^*, 1, 0, 0)$ denote the set of points in $wu_2c_3c_4\gamma$-space through which the solution of (6.14)–(6.18) approaches $(0, u_2^*, 1, 0, 0)$ as $t \to \infty$. $W^{cs}(0, u_2^*, 1, 0, 0)$ is just $W^{cs}(\tilde{q}_+)$ in the chart on blow-up space for $\bar{u}_1 > 0$. An argument similar to that leading to Lemma 9.2 yields:

Lemma 9.3 *Consider the system (6.14)–(6.18). For $\rho > 0$ and $k \geq 2$, let $U_{kc} = \{(w, u_2, c_3, c_4, \gamma) : |w| < \rho, |u_2 - u_2^*| < \rho, |c_3 - 1| < \rho, |c_4| < \rho, \frac{\rho}{k} < \gamma < \rho$. If ρ is sufficiently small, then for each $k \geq 2$, as $\gamma_0 \to 0+$, $W^s(p_{+c}(\gamma_0)) \cap U_{kc}$ approaches $W^{cs}(0, u_2^*, 1, 0, 0) \cap U_{kc}$ in the C^1 topology.*

In the chart on blow-up space for $\bar{u}_1 < 0$, consider the solution of (6.8)–(6.12) that represents $\tilde{\Gamma}_0$. This solution lies in $W^{cu}(0, u_2^*, 0, 0, 0)$. For a fixed large negative value of t, it also lies in U_{ka} for k sufficiently large. Now in Lemma 9.2, $W^{cu}(0, u_2^*, 0, 0, 0)$ is just $W^{cu}(\tilde{q}_-)$ in the chart under consideration, and $W^u(p_{-a}(\delta_0))$ is just $W^u(\tilde{p}_-(\epsilon))$ in the chart under consideration, with $\delta_0 = (-x_- + \mathcal{O}(\epsilon))^{-\frac{3}{2}}\epsilon$. Therefore Lemma 9.2 implies that each compact subset K of $\tilde{\Gamma}_0$ has a neighborhood V such that, as $\epsilon \to 0+$, $W^u(\tilde{p}_-(\epsilon)) \cap V$ approaches $W^{cu}(\tilde{q}_-) \cap V$ in the C^1 topology.

Similarly, Lemma 9.3 implies that as $\epsilon \to 0+$, $W^s(\tilde{p}_+(\epsilon)) \cap V$ approaches $W^{cs}(\tilde{q}_+) \cap V$ in the C^1 topology.

By Proposition 8.1, $W^{cu}(\tilde{q}_-)$ and $W^{cs}(\tilde{q}_+)$ meet transversally within the 3-sphere $\bar{r} = 0$, $u_2 = u_2^*$, which is N_0.

In the part of blow-up space with $\bar{\epsilon} > 0$, N_ϵ corresponds to the set of (b_1, u_2, b_3, b_4, r) such that $H_b = H(p_-(\epsilon))$ and $r = \epsilon^{\frac{1}{3}}$. The functions H_b and r have linearly independent gradients provided $u_2 \neq 0$. Therefore, where $u_2 \neq 0$, the sets $N_\epsilon = N_{r^3}$ depend smoothly on r. Since $W^{cu}(\tilde{q}_-)$ and $W^{cs}(\tilde{q}_+)$ meet transversally within N_0, it follows that $W^u(\tilde{p}_-(\epsilon))$ and $W^s(\tilde{p}_+(\epsilon))$ meet transversally within N_ϵ for ϵ small.

This completes the proof.

Remark 9.4 Let $(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon, u_4^\epsilon)$ denote the solution (modulo translations) of (4.14)–(4.17) corresponding to $\tilde{\Gamma}(\epsilon)$. In [18] it was shown that, after a time translation, $u_3^\epsilon(\sigma) - \epsilon^{\frac{1}{3}} b_3\left(\epsilon^{\frac{1}{3}}\sigma\right) = \mathcal{O}(|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{1}{3}})$, $|\sigma| \leq |\ln\epsilon|\epsilon^{-\frac{1}{3}}$, where $b_3(s)$ was defined in Sect. 7. Here we derive an improved estimate: after a time translation,

$$\text{for each } D \geq 0, \quad u_3^\epsilon(\sigma) - \epsilon^{\frac{1}{3}} b_3\left(\epsilon^{\frac{1}{3}}\sigma\right) = \mathcal{O}\left(\epsilon^{\frac{2}{3}}\right), \quad |\sigma| \leq D\epsilon^{-\frac{1}{3}}. \tag{9.7}$$

To prove (9.7), fix a small $\rho > 0$ and a large integer k such that (1) in $vu_2a_3a_4\delta$-space, the curve corresponding to $\tilde{\Gamma}_0$ meets U_{ka} in a curve $\tilde{\Gamma}_{0a}$ parameterized by δ, $\frac{\rho}{k} \leq \delta \leq \rho$, and (2) in $wu_2c_3c_4\gamma$-space, the curve corresponding to $\tilde{\Gamma}_0$ meets U_{kc} in a curve $\tilde{\Gamma}_{0c}$ parameterized by γ, $\frac{\rho}{k} \leq \gamma \leq \rho$.

For δ_0 small, we may assume that $\breve{W}^u(\breve{p}_{-a}(\delta_0))$ intersects the plane $v = \rho$ when $t = 0$. The coordinates of the point of intersection are $(\rho, u_2(0), \rho^{-3}\epsilon)$. We divide (9.1)–(9.3) by u_2, obtaining

$$v_z = -\frac{1}{2}v, \tag{9.8}$$

$$u_{2z} = u_2^{-1}v^2 g_x(-v^2, e(-v^2) + \mathcal{O}(v\delta), v^3\delta), \tag{9.9}$$

$$\delta_z = \frac{3}{2}\delta. \tag{9.10}$$

Hence, by Lemma 9.1, if ϵ is small, $\breve{W}^u(\breve{p}_{-a}(\delta_0))$ is in U_k for $z \in I = \left[\frac{2}{3}\ln\left(\frac{\rho^4}{k\epsilon}\right), \frac{2}{3}\ln\left(\frac{\rho^4}{\epsilon}\right)\right]$. In the proof of the Corner Lemma in [17] we take $\lambda = -\frac{1}{2}$, $\mu = \frac{3}{2}$, and $\tau \in I$. We deduce that, in U_k, the v component of $\breve{W}^u(\breve{p}_{-a}(\delta_0))$ is $\mathcal{O}(e^{\lambda\tau}) = \mathcal{O}(\epsilon^{\frac{1}{3}})C^1$-close to $v = 0$, and the u_2 component is $\mathcal{O}(e^{-\mu\tau}) + \mathcal{O}(e^{\lambda\tau}) = \mathcal{O}(\epsilon^{\frac{1}{3}})C^1$-close to $u_2 = u_2^*$. It follows that $W^u(p_{-a}(\delta_0))$ is $\mathcal{O}(\epsilon^{\frac{1}{3}})C^1$-close to $W^{cu}(0, u_2^*, 0, 0, 0)$ in U_{ka}. In $b_1u_2b_3b_4r$-coordinates, these two manifolds are $\mathcal{O}\left(\epsilon^{\frac{1}{3}}\right) C^1$-close when $b_1 = -\rho^{-\frac{2}{3}}$ (by the analog of (8.2)).

Let K be the compact portion of $\tilde{\Gamma}_0$ consisting of the parts corresponding to $\tilde{\Gamma}_{0a}$ and $\tilde{\Gamma}_{0c}$, together with the part between them, and let K_b be the corresponding curve in $b_1u_2b_3b_4r$-coordinates. Near K_b, the two manifolds remain $\mathcal{O}\left(\epsilon^{\frac{1}{3}}\right) C^1$-close, because the evolution of

$W^u(p_{-a}(\delta_0))$ is by (6.2)–(6.6) with $r=\epsilon^{\frac{1}{3}}$, that of $W^{cu}(0,u_2^*,0,0,0)$ is by (6.2)–(6.6) with $r=0$, and the manifolds are being followed for a finite time independent of ϵ.

We do the analogous construction for $W^s(p_{+c}(\gamma_0))$. Since the heteroclinic orbit $\tilde{\Gamma}(\epsilon)$ is the intersection of $W^u(p_{-a}(\delta_0))$ and $W^s(p_{+c}(\gamma_0))$ for appropriate δ_0 and γ_0, we see that in $b_1u_2b_3b_4r$-coordinates, $\tilde{\Gamma}(\epsilon)$ is $\mathcal{O}\left(\epsilon^{\frac{1}{3}}\right)$ distance from $\tilde{\Gamma}_0$ for $|b_1|\le\rho^{-\frac{2}{3}}$.

Hence, after a time translation, $b_3^\epsilon(\sigma)=\epsilon^{-\frac{1}{3}}u_3^\epsilon(\sigma)$ and the special solution $b_3(s)$ corresponding to $\tilde{\Gamma}_0$ satisfy

$$b_3^\epsilon(\epsilon^{-\frac{1}{3}}s)-b_3(s)=\mathcal{O}(\epsilon^{\frac{1}{3}}),\quad |s|\le C\rho^{-\frac{2}{3}}.$$

Hence, by (6.1),

$$u_3^\epsilon(\sigma)-\epsilon^{\frac{1}{3}}b_3\left(\epsilon^{\frac{1}{3}}\sigma\right)=\epsilon^{\frac{1}{3}}\left(b_3^\epsilon(\sigma)-b_3(\epsilon^{\frac{1}{3}}\sigma)\right)=\mathcal{O}(\epsilon^{\frac{2}{3}}),\quad |\sigma|\le C\rho^{-\frac{2}{3}}\epsilon^{-\frac{1}{3}}.$$

Similarly we can estimate $u_1^\epsilon, u_2^\epsilon, u_4^\epsilon$.

10 Transcritical Bifurcation

We again consider the system (4.6)–(4.7). We assume:

$$\text{At } (x,y,\epsilon)=(0,0,0),\quad k=k_x=k_y=k_\epsilon=0. \tag{10.1}$$

Hence

$$k(x,y,0)=\lambda x^2+\mu xy+\frac{\nu}{3}y^2+\cdots. \tag{10.2}$$

Therefore

$$g_x(x,y,0)=h_x(x,0)+yk_x(x,y,0)=h_x(x,0)+y(2\lambda x+\mu y+\cdots), \tag{10.3}$$
$$g_y(x,y,0)=k(x,y,0)+yk_y(x,y,0)=\lambda x^2+2\mu xy+\nu y^2+\cdots; \tag{10.4}$$

We assume:

(T1) $\mu^2-\lambda\nu>0$ and $\nu\ne 0$ (without loss of generality we take $\nu>0$).

Let $m_1<m_2$ be the roots of $\lambda+2\mu m+\nu m^2=0$, which are real. Then the equation $g_y(x,y,0)=0$ has two smooth solutions near $(0,0)$, given by $y=f_i(x)=m_ix+\cdots$, $i=1,2$.

The slow system, fast system, first integral, and fast limit system are given by (4.10)–(4.13), (4.14)–(4.17), (4.18), and (4.19)–(4.22) respectively.

From (T1), there exists $u_f>0$ such that the fast limit system (4.19)–(4.22) has two 2-dimensional manifolds of equilibria E_i, $i=1,2$:

$$E_i=\{(u_1,u_2,u_3,0):|u_1|<u_f,\ u_2 \text{ arbitrary},\ u_3=f_i(u_1)=m_iu_1+\cdots\}.$$

E_1 and E_2 meet along the u_2-axis. We have $g_{yy}(x,y,0)=2\mu x+2\nu y+\ldots$. Therefore, on E_i,

$$g_{yy}(u_1,f_i(u_1),0)=2(\mu+\nu m_i)u_1+\ldots=(-1)^i2(\mu^2-\lambda\nu)^{\frac{1}{2}}u_1+\ldots.$$

Hence (4.19)–(4.22) has two manifolds of normally hyperbolic equilibria:

$$E_{1-}=\{(u_1,u_2,u_3,0):-u_f<u_1<0 \text{ and } u_3=f_1(u_1)\},$$
$$E_{2+}=\{(u_1,u_2,u_3,0):0<u_1<u_f \text{ and } u_3=f_2(u_1)\}.$$

Setting $\epsilon = 0$ in (4.10)–(4.13), we obtain the slow limit system (4.24)–(4.27). E_{1-} and E_{2+} are manifolds of solutions of the system of Eqs. 4.26–4.27. Equations 4.24–4.25 then give the slow system on these manifolds:

$$u_{1\tau} = u_2, \tag{10.5}$$
$$u_{2\tau} = g_x(u_1, f_i(u_1), 0), \tag{10.6}$$

with $i = 1$ for E_{1-} and $i = 2$ for E_{2+}.

We assume:

(T2) The system (10.5), (10.6) with $i = 1$ has a hyperbolic saddle equilibrium $(x_-, 0)$ with $-u_f < x_- < 0$. One branch of its unstable manifold arrives at a point $(0, u_2^*)$ on the u_2-axis with $u_2^* > 0$.
(T3) The system (10.5), (10.6) with $i = 2$ has a hyperbolic saddle equilibrium $(x_+, 0)$ with $0 < x_+ < u_f$. One branch of its stable manifold arrives at the same point $(0, u_2^*)$ on the u_2-axis.

It follows that:

(1) System (4.14)–(4.17) has the smooth families of equilibria $p_-(\epsilon)$ and $p_+(\epsilon)$, with $p_-(0) = (x_-, 0, f_1(x_-), 0)$ and $p_+(0) = (x_+, 0, f_2(x_+), 0)$. For $\epsilon > 0$ these equilibria are hyperbolic, with two eigenvalues with positive real part and two with negative real part.
(2) $H(p_-(0)) = H(p_+(0))$.

We assume:

(T4) $H(p_-(\epsilon)) = H(p_+(\epsilon))$ for $\epsilon \geq 0$.

Let

- Γ_- denote the set of $(u_1, u_2, f_1(u_1), 0)$ such that $(u_1, u_2) \in B_-$;
- Γ_+ denote the set of $(u_1, u_2, f_2(u_1), 0)$ such that $(u_1, u_2) \in B_+$.

Theorem 10.1 *Assume (T1)–(T4). Then for small $\epsilon > 0$, there is a heteroclinic solution of (4.14)–(4.17) from $p_-(\epsilon)$ to $p_+(\epsilon)$ that is close to $\Gamma_- \cup \{(0, u_2^*, 0, 0)\} \cup \Gamma_+$.*

The proof is similar to that of Theorem 4.2. We give only a few key steps.

The blowup transformation is

$$u_1 = \bar{r}^2 \bar{u}_1, \quad u_2 = u_2, \quad u_3 = \bar{r}^2 \bar{u}_3, \quad u_4 = \bar{r}^3 \bar{u}_4, \quad \epsilon = \bar{r}^3 \bar{\epsilon}. \tag{10.7}$$

Chart for $\bar{\epsilon} > 0$:

$$u_1 = r^2 b_1, \quad u_2 = u_2, \quad u_3 = r^2 b_3, \quad u_4 = r^3 b_4, \quad \epsilon = r^3, \tag{10.8}$$

with $r \geq 0$. After division by r, the system (5.1)–(5.5) becomes

$$b_{1s} = u_2, \tag{10.9}$$
$$u_{2s} = r^2 g_x(r^2 b_1, r^2 b_3, r^3), \tag{10.10}$$
$$b_{3s} = b_4, \tag{10.11}$$
$$\begin{aligned} b_{4s} &= r^{-4} g_y(r^2 b_1, r^2 b_3, r^3) = \lambda b_1^2 + 2\mu b_1 b_3 + \nu b_3^2 + \mathcal{O}(r) \\ &= \nu(b_3 - m_1 b_1)(b_3 - m_2 b_1) + \mathcal{O}(r), \end{aligned} \tag{10.12}$$
$$r_s = 0. \tag{10.13}$$

In calculating b_{4s} we have used the assumption $k_\epsilon(0, 0, 0) = 0$; without this assumption, b_{4s} would include an $\mathcal{O}(r^{-1})$ term.

To establish the analog of Proposition 5.1 we make use of

Proposition 10.2 *The equation*

$$b_{ss} = \nu(b - m_1 u_2^* s)(b - m_2 u_2^* s) \tag{10.14}$$

has a solution $b(s)$ such that for all $s \in \mathbb{R}$,

$$b(s) > \check{b}(s) := \begin{cases} m_2 u_2^* s, & s > 0, \\ m_1 u_2^* s, & s \le 0, \end{cases}$$

and

$$b(s) - \check{b}(s) \to 0 \quad as \ |s| \to \infty.$$

Proof Let $\underline{b}(s) = \check{b}(s),\ s \in \mathbb{R}$. Then, $\underline{b}'(0^-) < \underline{b}'(0^+)$ and

$$-\underline{b}_{ss} + \nu(\underline{b} - m_1 u_2^* s)(\underline{b} - m_2 u_2^* s) = 0 \le 0, \quad s \ne 0,$$

i.e., $\underline{b}$ is a weak sub-solution of (10.14) in $\mathbb{R}$.

Fix a continuous function $\theta(\cdot) \ge 0$ such that $\check{b} + \theta \in C^2(\mathbb{R})$ and $\theta(s) = 0$ if $|s| \ge 1$. Let $\mu_1 > 0$, $\psi_1 > 0$ denote the principal eigenvalue and the corresponding L^∞-normalized eigenfunction of

$$-\psi_{ss} + \nu(m_2 - m_1)u_2^*|s|\psi = \mu\psi, \quad \psi \in L^2(\mathbb{R}).$$

Such $\mu_1,\ \psi_1$ exist since $(m_2 - m_1)u_2^*|s| \to \infty$ as $|s| \to \infty$, and

$$0 < \psi_1(s) \le C|s|^{-\frac{1}{4}} e^{-\frac{2}{3}(\nu(m_2 - m_1)u_2^*)^{\frac{1}{2}}|s|^{\frac{3}{2}}}, \quad s \in \mathbb{R} \ \text{(see [3, p. 100])},$$

where $C > 0$ is a generic constant. Let $\bar{b}(s) = \check{b}(s) + \theta(s) + M\psi_1(s),\ s \in \mathbb{R}$, with $M > 1$ a large constant to be chosen. In $[-1, 0]$,

$$\begin{aligned} -\bar{b}_{ss} + \nu(\bar{b} - m_1 u_2^* s)(\bar{b} - m_2 u_2^* s) &\ge -CM + \nu(\theta + M\psi_1)(m_1 u_2^* s + \theta + M\psi_1 - m_2 u_2^* s) \\ &\ge -CM + \nu M^2 \psi_1^2 \ge -CM + \nu M^2 c > 0, \end{aligned}$$

provided $M > 0$ is sufficiently large ($C, c > 0$ are independent of M). We fix such an $M > 0$. In $(-\infty, -1]$,

$$\begin{aligned} -\bar{b}_{ss} + \nu(\bar{b} - m_1 u_2^* s)(\bar{b} - m_2 u_2^* s) &= -M\psi_{1ss} + \nu M\psi_1(m_1 u_2^* s + M\psi_1 - m_2 u_2^* s) \\ &\ge -M\psi_{1ss} + \nu M\psi_1(m_2 - m_1)u_2^*|s| = \mu_1 M\psi_1 > 0. \end{aligned}$$

Analogous calculations also hold in $[0, \infty)$. Hence $\bar{b}$ is a super-solution of (10.14) and $\underline{b} < \bar{b}$ in $\mathbb{R}$.

By a well-known theorem (see [16] for example) it follows that there exists a solution $b \in C^2(\mathbb{R})$ of (10.14) such that $\underline{b}(s) < b(s) < \bar{b}(s)$, $s \in \mathbb{R}$. The assertions of the Proposition follow immediately since $\underline{b} \equiv \check{b}$ and $\bar{b}(s) = \check{b}(s) + M\psi_1(s)$ if $|s| \ge 1$. □

Remark 10.3 Independently, Proposition 10.2 has been announced in [12].

The proof of the analog of Proposition 8.1 uses

Proposition 10.4 *For the function $b(s)$ given by Proposition 10.2, the linear equation*

$$-B_{ss}+\nu(2b(s)-m_1u_2^*s-m_2u_2^*s)B=0 \tag{10.15}$$

has no nontrivial solutions in $L^\infty(\mathbb{R})$.

Proof By Proposition 10.2 we have

$$2b(s)-m_1u_2^*s-m_2u_2^*s>(m_2-m_1)u_2^*|s|,\quad s\in\mathbb{R}. \tag{10.16}$$

So it is easy to see that any bounded solution B of (10.15) satisfies $B(s)\to 0$ super-exponentially as $|s|\to\infty$ (a similar estimate also holds for B_s, B_{ss}).

Hence

$$\int_{-\infty}^{\infty}B_s^2(s)ds+\nu\int_{-\infty}^{\infty}(2b(s)-m_1u_2^*s-m_2u_2^*s)B^2(s)ds=0.$$

In view of (10.16) we obtain that $B\equiv 0$ and the proof is concluded. □

Remark 10.5 Boundary value problems for pairs of second-order equations with transcritical slow-manifold bifurcation have been studied in [5] by constructing suitable sub- and super-solutions (see also the review [6]). (This approach can only be applied to systems with a special monotonicity property.) It is assumed that the reduced boundary value problem has a nondegenerate solution. We expect that the approach of the present paper can also be applied by adjoining time as a dependent variable (see [19]). The nondegeneracy assumption is translated geometrically into the transversality required for establishing the analog of Lemma 9.1 (see [17]).

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